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Some Grüss Type Inequalities for Vector-Valued Functions in Banach Spaces and Applications*

N. S. Barnett†, P. Cerone‡, and S. S. Dragomir§

School of Computer Science and Mathematics
Victoria University of Technology PO Box 14428,
MCMC 8001, Victoria, Australia.

and

C. Buşe¶

Department of Mathematics West University of Timişoara
Timişoara, 1900, Bd. V. Pârvan. Nr. 4 România

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Abstract

Some Grüss type inequalities for the Bochner integral of vector-valued functions in real or complex Banach spaces are given. Applications in connection to the Heisenberg inequality for functions with values in Hilbert spaces are also pointed out.

Keywords and Phrases: *Grüss inequality, Bochner integral, Banach spaces, Hilbert spaces.*

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†E-mail: neil@csm.vu.edu.au
‡E-mail: pc@csm.vu.edu.au
§E-mail: sever.dragomir@vu.edu.au
¶E-mail: buse@math.uvt.ro

1. Introduction

In 1934, G. Grüss [5] proved the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{4}(M-m)(N-n), \quad (1.1)$$

provided

$$-\infty<m\leq f(t)\leq M<\infty, \quad -\infty<n\leq g(t)\leq N<\infty$$

for a.e. $t\in[a,b]$; and showed that the constant $\frac{1}{4}$ is the best possible.

An extension of the above result to vector-valued functions in Hilbert spaces was obtained in 2001 by S.S. Dragomir [3]:

Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$, $\Omega\subset\mathbb{R}^n$ a measurable set, $f,g:\Omega\to H$ Bochner measurable functions on Ω and $f,g\in L_{2,\rho}(\Omega,H)$, where

$$L_{2,\rho}(\Omega,H):=\left\{f:\Omega\to H;\ \int_\Omega\rho(t)\|f(t)\|^2\,dt<\infty\right\}$$

and $\rho:\Omega\to[0,\infty)$ is a Lebesgue integrable function with $\int_\Omega\rho(x)\,dx=1$. If there exist vectors $x,X,y,Y\in H$ such that either

$$\int_\Omega\rho(t)\,Re\langle X-f(t),f(t)-x\rangle\,dt\geq 0, \quad \text{and} \quad (1.2)$$
$$\int_\Omega\rho(t)\,Re\langle Y-g(t),g(t)-y\rangle\,dt\geq 0,$$

or, equivalently, [1], either,

$$\int_\Omega\rho(t)\left\|f(t)-\frac{x+X}{2}\right\|^2dt\leq\frac{1}{4}\|X-x\|^2, \quad \text{and} \quad (1.3)$$
$$\int_\Omega\rho(t)\left\|g(t)-\frac{y+Y}{2}\right\|^2dt\leq\frac{1}{4}\|Y-y\|^2$$

then

$$\left|\int_\Omega\rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega\rho(t)f(t)\,dt,\int_\Omega\rho(t)g(t)\,dt\right\rangle\right| \leq\frac{1}{4}\|X-x\|\,\|Y-y\|. \quad (1.4)$$

The constant $\frac{1}{4}$ in (1.4) is again the best possible.

This result was improved in [1], where the authors, on using a finer argument, proved that

$$\begin{aligned}
&\left|\int_{\Omega}\rho(t)\langle f(t),g(t)\rangle dt-\left\langle\int_{\Omega}\rho(t)f(t)\,dt,\int_{\Omega}\rho(t)g(t)\,dt\right\rangle\right| \qquad (1.5)\\
&\leq\frac{1}{4}\|X-x\|\,\|Y-y\|\\
&\quad-\left[\int_{\Omega}\rho(t)\,Re\langle X-f(t),f(t)-x\rangle\,dt\right.\\
&\qquad\qquad\left.\times\int_{\Omega}\rho(t)\,Re\langle Y-g(t),g(t)-y\rangle\,dt\right]^{\frac{1}{2}}\\
&\leq\frac{1}{4}\|X-x\|\,\|Y-y\|,
\end{aligned}$$

provided f and g satisfy either (1.2) or, equivalently, (1.3).

Under the same type of hypothesis, the authors of [1] also established the following result:

$$\begin{aligned}
&\left\|\int_{\Omega}\rho(t)\alpha(t)f(t)\,dt-\int_{\Omega}\rho(t)\alpha(t)\,dt\int_{\Omega}\rho(t)f(t)\,dt\right\| \qquad (1.6)\\
&\leq\frac{1}{4}|A-a|\,\|X-x\|\\
&\quad-\left(\int_{\Omega}\rho(t)\,Re\left[(A-\alpha(t))\left(\overline{\alpha(t)}-\overline{a}\right)\right]dt\right.\\
&\qquad\qquad\left.\times\int_{\Omega}\rho(t)\,Re\langle X-f(t),f(t)-x\rangle\,dt\right)^{\frac{1}{2}}\\
&\leq\frac{1}{4}|A-a|\,\|X-x\|,
\end{aligned}$$

provided f satisfies either (1.2) or (1.3) and the scalar function $\alpha:\Omega\to\mathbb{K}$ satisfies the equivalent conditions:

$$Re\left[(A-\alpha(t))\left(\overline{\alpha(t)}-\overline{a}\right)\right]\geq 0$$

and

$$\left|\alpha(t)-\frac{A+a}{2}\right|\leq\frac{1}{2}|A-a|,$$

for a.e. $t \in \Omega$, where $A, a \in \mathbb{K}$ are given constants.

Note that in both inequalities (1.5) and (1.6) the quantity $\frac{1}{4}$ is again the best possible.

The main aim of this paper is to establish some Grüss type inequalities for Bochner integrable functions taking values in a Banach space. Applications for the case of Hilbert spaces and in connection with the Heisenberg inequality are also given.

2. Inequalities in Banach Spaces

Theorem 1. *Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ a measurable set and $\rho : \Omega \to [0, \infty)$ a Lebesgue integrable function with $\int_\Omega \rho(x)\,dx = 1$. If $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function such that there exists $\gamma, \Gamma \in \mathbb{K}$ with*

$$\left|\alpha(x) - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \tag{2.1}$$

or, equivalently,

$$Re\left[(\Gamma - \alpha(x))\left(\overline{\alpha(x)} - \overline{\gamma}\right)\right] \geq 0 \tag{2.2}$$

for a.e. $x \in \Omega$, and $f : \Omega \to X$ is a Bochner measurable function such that $\rho\alpha f$ and ρf are Bochner integrable on Ω, then,

$$\begin{aligned}\left\|\int_\Omega \rho(x)\alpha(x) f(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx \cdot \int_\Omega \rho(x) f(x)\,dx\right\| \\ \leq \frac{1}{2}|\Gamma - \gamma| \int_\Omega \rho(x)\left\|f(x) - \int_\Omega \rho(y) f(y)\,dy\right\| dx. \end{aligned}\tag{2.3}$$

The constant $\frac{1}{2}$ in (2.3) is the best possible.

Proof. The following Sonin type identity for the Bochner integral holds:

$$\begin{aligned}\int_\Omega \rho(x)\alpha(x) f(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx \cdot \int_\Omega \rho(x) f(x)\,dx \\ = \int_\Omega \rho(x)\left(\alpha(x) - \frac{\gamma + \Gamma}{2}\right)\left(f(x) - \int_\Omega \rho(y) f(y)\,dy\right) dx. \end{aligned}\tag{2.4}$$

(for the scalar case, see [6, p. 246]). Taking the norm in (2.4), we deduce

$$\begin{aligned}&\left\|\int_{\Omega}\rho(x)\alpha(x)f(x)\,dx-\int_{\Omega}\rho(x)\alpha(x)\,dx\cdot\int_{\Omega}\rho(x)f(x)\,dx\right\|\\&\leq\int_{\Omega}\rho(x)\left|\alpha(x)-\frac{\gamma+\Gamma}{2}\right|\left\|f(x)-\int_{\Omega}\rho(y)f(y)\,dy\right\|dx\\&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\int_{\Omega}\rho(x)\left\|f(x)-\int_{\Omega}\rho(y)f(y)\,dy\right\|dx.\end{aligned}$$

and the inequality (2.3) is obtained.

Now, to prove the sharpness of the constant $\frac{1}{2}$, assume that (2.3) holds for $\Omega=[a,b]$, $X=\mathbb{R}$, $\rho\equiv\frac{1}{b-a}$, with a constant $c>0$. That is:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\alpha(t)f(t)\,dt-\frac{1}{b-a}\int_a^b\alpha(t)\,dt\cdot\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\qquad\leq c(\Gamma-\gamma)\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt,\end{aligned}\tag{2.5}$$

where $-\infty<\gamma\leq\alpha(t)\leq\Gamma<\infty$ for a.e. $t\in[a,b]$, and $\int_a^b$ is the usual Lebesgue integral on $[a,b]$.

If we choose, in (2.5), $\alpha=f$ and $f:[a,b]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}-1 & \text{if } x\in\left[a,\frac{a+b}{2}\right],\\[2ex] 1 & \text{if } x\in\left(\frac{a+b}{2},b\right],\end{cases}$$

then, obviously $\gamma=-1$, $\Gamma=1$,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2&=1,\\\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt&=1,\end{aligned}$$

and by (2.5) we get $c\geq\frac{1}{2}$.

Remark 1. *If α takes real values and there exist constants m, M such that $-\infty < m \leq \alpha \leq M < \infty$ for a.e. $x \in \Omega$, then (2.3) becomes:*

$$\left\| \int_\Omega \rho(x) \alpha(x) f(x) \, dx - \int_\Omega \rho(x) \alpha(x) \, dx \cdot \int_\Omega \rho(x) f(x) \, dx \right\| \leq \frac{1}{2}(M - m) \int_\Omega \rho(x) \left\| f(x) - \int_\Omega \rho(y) f(y) \, dy \right\| dx.$$

Note that a scalar version of this inequality has been obtained previously by Cerone and Dragomir in [2], using a different technique.

Remark 2. *A slightly more general result for $\alpha(t) \in \bar{B}(c, r) := \{z \in \mathbb{C} | \, |z - c| \leq r\}$ for a.e. $x \in \Omega$, is:*

$$\left\| \int_\Omega \rho(x) \alpha(x) f(x) \, dx - \int_\Omega \rho(x) \alpha(x) \, dx \cdot \int_\Omega \rho(x) f(x) \, dx \right\| \leq r \int_\Omega \rho(x) \left\| f(x) - \int_\Omega \rho(y) f(y) \, dy \right\| dx. \quad (2.6)$$

Here the inequality (2.6) is also sharp.

The following dual result may be stated as well.

Theorem 2. *Let $(X, \|\cdot\|)$ and Ω, ρ be as above. If $f : \Omega \to X$ is Bochner measurable on Ω and there exist vector $v \in X$ and $r > 0$ such that $f(x) \in \bar{B}(v, r) := \{y \in X | \, \|y - v\| \leq r\}$ for a.e. $x \in \Omega$ and $\alpha : \Omega \to \mathbb{K}$ a Lebesgue integrable function with $\rho\alpha f$, ρf Bochner integrable functions on Ω, then we have the sharp inequalities*

$$\begin{aligned} &\left\| \int_\Omega \rho(x) \alpha(x) f(x) \, dx - \int_\Omega \rho(x) \alpha(x) \, dx \cdot \int_\Omega \rho(x) f(x) \, dx \right\| \quad (2.7) \\ &\leq r \int_\Omega \rho(x) \left| \alpha(x) - \int_\Omega \rho(y) \alpha(y) \, dy \right| dx \\ &\leq r \left[\int_\Omega \rho(x) |\alpha(x)|^2 \, dx - \left| \int_\Omega \rho(x) \alpha(x) \, dx \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Proof. The first inequality in (2.7) is obvious from the Sonin type identity:

$$\int_{\Omega}\rho(x)\alpha(x)f(x)\,dx-\int_{\Omega}\rho(x)\alpha(x)\,dx\cdot\int_{\Omega}\rho(x)f(x)\,dx$$
$$=\int_{\Omega}\rho(x)\left(\alpha(x)-\int_{\Omega}\rho(y)\alpha(y)\,dy\right)(f(x)-v)\,dx.$$

The second inequality follows by Schwarz's integral inequality:

$$\int_{\Omega}\rho(x)\left|\alpha(x)-\int_{\Omega}\rho(y)\alpha(y)\,dy\right|dx\leq\left[\int_{\Omega}\rho(x)\left|\alpha(x)-\int_{\Omega}\rho(y)\alpha(y)\,dy\right|^{2}dx\right]^{\frac{1}{2}}$$
$$=\left[\int_{\Omega}\rho(x)\left|\alpha(x)\right|^{2}dx-\left|\int_{\Omega}\rho(x)\alpha(x)\,dx\right|^{2}\right]^{\frac{1}{2}}.$$

The details are omitted.

The following particular case holding for Hilbert spaces may be useful for applications.

Corollary 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field and Ω,ρ and α as in Theorem 2. If there exist vectors $v,V\in H$ such that for the Bochner measurable function $\rho:\Omega\to H$ either*

$$Re\left\langle V-f(x),f(x)-v\right\rangle\geq 0, \tag{2.8}$$

or, equivalently,

$$\left\|f(x)-\frac{v+V}{2}\right\|\leq\frac{1}{2}\left\|V-v\right\| \tag{2.9}$$

for a.e. $x\in\Omega$ and $\rho\alpha f$, ρf Bochner integrable on Ω, then,

$$\left\|\int_{\Omega}\rho(x)\alpha(x)f(x)\,dx-\int_{\Omega}\rho(x)\alpha(x)\,dx\cdot\int_{\Omega}\rho(x)f(x)\,dx\right\| \tag{2.10}$$
$$\leq\frac{1}{2}\left\|V-v\right\|\int_{\Omega}\rho(x)\left|\alpha(x)-\int_{\Omega}\rho(y)\alpha(y)\,dy\right|dx$$
$$\leq\frac{1}{2}\left\|V-v\right\|\left[\int_{\Omega}\rho(x)\left|\alpha(x)-\int_{\Omega}\rho(y)\alpha(y)\,dy\right|^{2}dx\right]^{\frac{1}{2}}.$$

The quantity $\frac{1}{2}$ is the best possible in both inequalities in (2.10).

Proof. The proof is obvious by Theorem 2 on taking into account that in the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ the following two statements are equivalent

(i) $\left\|y-\frac{V+v}{2}\right\|\leq\frac{1}{2}\|V-v\|$

(ii) $Re\langle V-y,y-v\rangle\geq 0,$

where $y,v,V\in H$.

The following result is similar to (1.5).

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field and $f,g:\Omega\to H$ Bochner measurable on Ω while $\rho:\Omega\to[0,\infty)$ is Lebesgue integrable and $\int_\Omega\rho(x)\,dx=1$. If there exist vectors $v,V\in H$ such that either (2.8) or, equivalently, (2.9) hold for a.e. $x\in\Omega$ and αf, ρg are Bochner integrable on Ω, then,*

$$\left|\int_\Omega\rho(x)\langle f(x),g(x)\rangle\,dx-\left\langle\int_\Omega\rho(x)f(x)\,dx,\int_\Omega\rho(x)g(x)\,dx\right\rangle\right| \tag{2.11}$$
$$\leq\frac{1}{2}\|V-v\|\int_\Omega\rho(x)\left\|g(x)-\int_\Omega\rho(y)g(y)\,dy\right\|dx$$
$$\leq\frac{1}{2}\|V-v\|\left[\int_\Omega\rho(x)\|g(x)\|^2-\left\|\int_\Omega\rho(y)g(y)\,dy\right\|^2dx\right]^{\frac{1}{2}}$$

(provided $g\in L_{2,\rho}(\Omega,H)$).

Again, the constant $\frac{1}{2}$ is the best possible.

Proof. The following Sonin type identity may be stated as well.

$$\int_\Omega\rho(x)\langle f(x),g(x)\rangle\,dx-\left\langle\int_\Omega\rho(x)f(x)\,dx,\int_\Omega\rho(x)g(x)\,dx\right\rangle$$
$$=\int_\Omega\rho(x)\left\langle f(x)-\frac{V+v}{2},g(x)-\int_\Omega\rho(y)g(y)\,dy\right\rangle dx. \tag{2.12}$$

Taking the modulus, using the hypothesis and the Schwarz inequality in $(H;\langle\cdot,\cdot\rangle)$, we have,

$$\begin{aligned}
&\left|\int_\Omega \rho(x)\langle f(x), g(x)\rangle dx - \left\langle \int_\Omega \rho(x) f(x)\, dx, \int_\Omega \rho(x) g(x)\, dx\right\rangle\right| \\
&\leq \int_\Omega \rho(x)\left|\left\langle f(x) - \frac{V+v}{2}, g(x) - \int_\Omega \rho(y) g(y)\, dy\right\rangle\right| dx \\
&\leq \int_\Omega \rho(x)\left\| f(x) - \frac{V+v}{2}\right\| \left\| g(x) - \int_\Omega \rho(y) g(y)\, dy\right\| dx \\
&\leq \frac{1}{2}\|V-v\| \int_\Omega \rho(x)\left\| g(x) - \int_\Omega \rho(y) g(y)\, dy\right\| dx \\
&\leq \frac{1}{2}\|V-v\| \left[\int_\Omega \rho(x)\left\| g(x) - \int_\Omega \rho(y) g(y)\, dy\right\|^2 dx\right]^{\frac{1}{2}} \\
&= \frac{1}{2}\|V-v\| \left[\int_\Omega \rho(x)\|g(x)\|^2 - \left\|\int_\Omega \rho(y) g(y)\, dy\right\|^2 dx\right]^{\frac{1}{2}},
\end{aligned}$$

provided $g \in L_{2,\rho}(\Omega, H)$.

Remark 3. *Assume that for the Lebesgue integrable function $\alpha : \Omega \to \mathbb{K}$ there exist $\gamma, \Gamma \in \mathbb{K}$ such that either (2.1) or, equivalently, (2.2) hold, then,*

$$\begin{aligned}
0 &\leq \int_\Omega \rho(x)|\alpha(x)|^2 dx - \left|\int_\Omega \rho(x)\alpha(x)\, dx\right|^2 \\
&\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \rho(x)\left|\alpha(x) - \int_\Omega \rho(y)\alpha(y)\, dy\right| dx,
\end{aligned} \tag{2.13}$$

and [1]

$$\begin{aligned}
0 &\leq \left|\int_\Omega \rho(x)\alpha^2(x)\, dx - \left(\int_\Omega \rho(x)\alpha(x)\, dx\right)^2\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \rho(x)\left|\alpha(x) - \int_\Omega \rho(y)\alpha(y)\, dy\right| dx.
\end{aligned} \tag{2.14}$$

The quantity $\frac{1}{2}$ is sharp in both instances.

3. Applications for Some Integral Inequalities of the Heisenberg Type

In the following we use the Grüss type inequality

$$\left|\int_{\Omega}\rho(t)\,Re\left\langle f(t),g(t)\right\rangle dt-Re\left\langle\int_{\Omega}\rho(t)\,f(t)\,dt,\int_{\Omega}\rho(t)\,g(t)\,dt\right\rangle\right|$$
$$\leq\frac{1}{2}\left\|V-v\right\|\int_{a}^{b}\rho(t)\left\|g(t)-\int_{a}^{b}\rho(s)\,g(s)\,ds\right\|dt,\quad(3.1)$$

provided $\rho\in L([a,b])$, $\int_a^b\rho(t)\,dt=1$, $\rho f,\rho g\in L([a,b],H)$, $(H,\langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f:[a,b]\to H$ is Bochner measurable and such that either

$$Re\left\langle V-f(t),f(t)-v\right\rangle\geq 0\ \text{ for a.e. }t\in[a,b],\quad(3.2)$$

or, equivalently,

$$\left\|f(t)-\frac{v+V}{2}\right\|\leq\frac{1}{2}\left\|V-v\right\|\ \text{ for a.e. }t\in[a,b].$$

Notice that the inequality (3.1) follows by (2.10) on taking into account that, for complex numbers $z\in\mathbb{C}$, $|Re z|\leq|z|$.

It is well known that if $(H;\langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f:[a,b]\subset\mathbb{R}\to H$ is an *absolutely continuous vector-valued* function, then f is differentiable almost everywhere on $[a,b]$, the derivative $f':[a,b]\to H$ is Bochner integrable on $[a,b]$ and

$$f(t)=\int_{a}^{t}f'(s)\,ds\ \text{ for any }t\in[a,b].\quad(3.3)$$

The following theorem provides a version of the Heisenberg inequality in the general setting of Hilbert spaces and has been obtained by S.S. Dragomir in [4].

Theorem 4. *Let $\varphi:[a,b]\to H$ be an absolutely continuous function with the property that $b\left\|\varphi(b)\right\|^2=a\left\|\varphi(a)\right\|^2$, then,*

$$\int_{a}^{b}\left\|\varphi(t)\right\|^2dt\leq 2\left[\int_{a}^{b}\left\|\varphi'(t)\right\|^2dt\cdot\int_{a}^{b}t^2\left\|\varphi(t)\right\|^2dt\right]^{\frac{1}{2}}.\quad(3.4)$$

The constant 2 *is the best possible.*

Remark 4. *It is obvious that a sufficient condition for (3.4) to hold is that* $\varphi(a) = \varphi(b) = 0$.

In the following we point out different upper bounds from (3.4), for the integral $\int_a^b \|\varphi(t)\|^2 dt$.

Proposition 1. *Let* $\varphi : [a,b] \to H$ *be an absolutely continuous function with the property that* $\varphi(a) = \varphi(b) = 0$. *If there exist vectors* $v, V \in H$ *such that either*

$$\left\| \varphi'(t) - \frac{v+V}{2} \right\| \leq \frac{1}{2} \|V - v\| \quad \text{for a.e. } t \in [a,b] \tag{3.5}$$

or, equivalently,

$$Re \langle V - \varphi'(t), \varphi'(t) - v \rangle \geq 0 \quad \text{for a.e. } t \in [a,b], \tag{3.6}$$

then,

$$\int_a^b \|\varphi(t)\|^2 dt \leq \|V - v\| \int_a^b \left\| t\varphi(t) - \frac{1}{b-a} \int_a^b s\varphi(s)\, ds \right\| dt. \tag{3.7}$$

Proof. Applying the inequality (3.1) for $\rho(t) = \frac{1}{b-a}$, $f(t) = \varphi'(t)$ and $g(t) = t\varphi(t)$, $t \in [a,b]$, we can write:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b t Re \langle \varphi'(t), \varphi(t) \rangle dt \right. \\ &\qquad \left. - Re \left\langle \frac{1}{b-a} \int_a^b \varphi'(t)\, dt, \frac{1}{b-a} \int_a^b t\varphi(t)\, dt \right\rangle \right| \\ &\qquad \leq \frac{1}{2} \|V - v\| \frac{1}{b-a} \int_a^b \left\| t\varphi(t) - \frac{1}{b-a} \int_a^b s\varphi(s)\, ds \right\| dt. \end{aligned} \tag{3.8}$$

Since $\varphi(a) = \varphi(b) = 0$, hence

$$\int_a^b \varphi'(t)\, dt = 0, \tag{3.9}$$

$$\int_a^b tRe\left\langle \varphi'(t), \varphi(t)\right\rangle dt = -\frac{1}{2}\cdot\int_a^b \|\varphi(t)\|^2 dt, \tag{3.10}$$

where, for the last equality we have used an identity obtained in [4] (see the Eq. (5.3) from [4]) under the more general assumption, i.e., $b\|\varphi(b)\|^2 = a\|\varphi(a)\|^2$. Making use of (3.9), (3.10) and (3.8), we conclude that (3.7) holds true and the proposition is proven.

Proposition 2. *Let $\varphi : [a,b] \to H$ be an absolutely continuous function with the property that $\varphi(a) = \varphi(b) = 0$. If there exist vectors $w, W \in H$ so that either*

$$\left\| t\varphi'(t) - \frac{w+W}{2}\right\| \le \frac{1}{2}\|W-w\| \quad \textit{for a.e. } t\in[a,b], \tag{3.11}$$

or, equivalently,

$$Re\left\langle W - t\varphi'(t), t\varphi'(t) - w\right\rangle \ge 0 \quad \textit{for a.e. } t\in[a,b], \tag{3.12}$$

then

$$\left| \left\| \int_a^b \varphi(t)\,dt\right\|^2 - \frac{1}{2}(b-a)\int_a^b \|\varphi(t)\|^2 dt\right| \le \frac{1}{2}\|W-w\|\int_a^b \left\| \varphi(t) - \frac{1}{b-a}\int_a^b \varphi(s)\,ds\right\| dt. \tag{3.13}$$

Proof. Applying the inequality (3.1) for $\rho(t) = \frac{1}{b-a}$, $f(t) = t\varphi'(t)$ and $g(t) = \varphi(t)$, $t\in[a,b]$, we can write:

$$\left| \frac{1}{b-a}\int_a^b tRe\left\langle \varphi'(t), \varphi(t)\right\rangle dt - Re\left\langle \frac{1}{b-a}\int_a^b t\varphi'(t)\,dt, \frac{1}{b-a}\int_a^b \varphi(t)\,dt \right\rangle\right| \le \frac{1}{2}\|W-w\|\int_a^b \left\| \varphi(t) - \frac{1}{b-a}\int_a^b \varphi(s)\,ds\right\| dt. \tag{3.14}$$

Since $\varphi(a) = \varphi(b) = 0$, hence

$$\int_a^b t\varphi'(t)\,dt = -\int_a^b \varphi(t)\,dt. \tag{3.15}$$

Therefore, by (3.10), (3.15) and (3.14), we deduce

$$\left|-\frac{1}{2(b-a)}\int_a^b \|\varphi(t)\|^2\,dt + Re\left\langle \frac{1}{b-a}\int_a^b \varphi(t)\,dt, \frac{1}{b-a}\int_a^b \varphi(t)\,dt\right\rangle\right| \\ \leq \frac{1}{2}\|W-w\|\cdot\int_a^b \left\|\varphi(t)-\frac{1}{b-a}\int_a^b \varphi(s)\,ds\right\|dt,$$

which is clearly equivalent to (3.13).

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