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A Generalisation of Cerone's Identity and Applications*

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Abstract

An identity due to P. Cerone for the Čebyšev functional is extended for Stieltjes integrals. A sharp inequality and its application in approximating Stieltjes integrals are also given.

Keywords and Phrases: *Integral inequalities, Stieltjes integrals.*

1. Introduction

In 2001, P. Cerone [1] established the following identity for the Čebyšev functional:

$$T(f,g;p) := \frac{1}{\int_a^b p(s)\,ds}\int_a^b p(t)\,f(t)\,g(t)\,dt - \frac{1}{\int_a^b p(s)\,ds}\int_a^b p(t)\,f(t)\,dt \cdot \frac{1}{\int_a^b p(s)\,ds}\int_a^b p(t)\,g(t)\,dt \tag{1.1}$$

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$$= \frac{1}{\left(\int_a^b p(s)\,ds\right)^2} \int_a^b \left[\int_a^t p(s)\,ds \int_t^b p(s)\,g(s)\,ds - \int_t^b p(s)\,ds \int_a^t p(s)\,g(s)\,ds \right] df(t),$$

provided f is of bounded variation on $[a,b]$ and g is continuous on $[a,b]$. He proved (1) on utilising the auxiliary function $\Psi : [a,b] \to \mathbb{R}$,

$$\Psi(t) := (t-a) \int_t^b g(s)\,ds - (b-t) \int_a^t g(s)\,ds \tag{1.2}$$

and integrating by parts in the Stieltjes integral $\int_a^b \Psi(t)\,df(t)$, which exists, since f is of bounded variation and Ψ is differentiable on (a,b).

One may observe that the result remains valid if one assumes that g is Lebesgue integrable on $[a,b]$ and f is of bounded variation. This follows by the fact that, in this case Ψ becomes absolutely continuous on $[a,b]$, the Stieltjes integral $\int_a^b \Psi(t)\,df(t)$ still exists and the argument will follow as in [1].

The weighted version of this inequality has been obtained in the same paper [1] and can be stated as:

$$T(f,g;p) := \frac{1}{\int_a^b p(s)\,ds} \int_a^b p(t)\,f(t)\,g(t)\,dt - \frac{1}{\int_a^b p(s)\,ds} \int_a^b p(t)\,f(t)\,dt \cdot \frac{1}{\int_a^b p(s)\,ds} \int_a^b p(t)\,g(t)\,dt \tag{1.3}$$

$$= \frac{1}{\left(\int_a^b p(s)\,ds\right)^2} \int_a^b \left[\int_a^t p(s)\,ds \int_t^b p(s)\,g(s)\,ds - \int_t^b p(s)\,ds \int_a^t p(s)\,g(s)\,ds \right] df(t),$$

provided f is of bounded variation on $[a,b]$ and p,g are continuous on $[a,b]$ with $\int_a^b p(s)\,ds > 0$. The same remark for the extension of the identity in the case that p, g are Lebesgue integrable on $[a,b]$ so that pg is also integrable, may apply.

The above two identities have been applied in [1] to obtain some interesting new bounds for the Čebyšev functionals $T(f,g)$ and $T(f,g;p)$ from which we only mention the following:

$$|T(f,g)| \leq \frac{1}{(b-a)^2} \times \begin{cases} \sup\limits_{t\in[a,b]} |\Psi(t)| \bigvee_a^b (f); \\ L\int_a^b |\Psi(t)|\,dt & \text{for } f \; L-\text{Lipschitzian;} \\ \int_a^b |\Psi(t)|\,df(t) & \text{for } f \text{ monotonic nondecreasing,} \end{cases} \tag{1.4}$$

where $\bigvee_a^b (f)$ is the total variation of f on $[a,b]$, $\Psi(t)$ is given by (1), and

$$|T(f,g;p)| \leq \frac{1}{\left(\int_a^b p(s)\,ds\right)^2} \times \begin{cases} \sup\limits_{t\in[a,b]} |\Psi_p(t)| \bigvee_a^b (f); \\ L\int_a^b |\Psi_p(t)|\,dt & \text{if } f \text{ is } L-\text{Lipschitzian;} \\ \int_a^b |\Psi_p(t)|\,df(t) & \text{for } f \text{ monotonically nondecreasing,} \end{cases} \tag{1.5}$$

where in this case the wighted auxiliary mapping Ψ_p is defined as $\Psi_p : [a,b] \to \mathbb{R}$,

$$\Psi_p(t) := \int_a^t p(s)\,ds \int_t^b p(s)\,g(s)\,ds - \int_t^b p(s)\,ds \int_a^t p(s)\,g(s)\,ds.$$

For other inequalities and applications for moments, see [1].

For further results, see the follow up paper [2] where various lower and other upper bounds were established.

2. A Related Functional

In [4], the authors have considered the following functional

$$D(f;u) := \int_a^b f(x)\,du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t)\,dt, \tag{2.1}$$

provided that the Stieltjes integral $\int_a^b f(x)\,du(x)$ exists.

This functional palys an important role in approximating the Stieltjes integral $\int_a^b f(x)\,du(x)$ in terms of the Riemann integral $\int_a^b f(t)\,dt$ and the divided diference of the integrator u. Therefore, further bounds on $D(f;u)$ will generate a flow of different error estimates for the approximation of the Stieltjes integral that plays an important role in various fields of Analysis, Numerical Analysis, Integral Operator Theory, Probability & Statistics and other fields of Modern Mathematics.

In [4], the following result in estimating the above functional $D(f;u)$ has been obtained:

$$|D(f;u)| \leq \frac{1}{2} L(M-m)(b-a), \tag{2.2}$$

provided u is $L-$*Lipschitzian* and f is *Riemann integrable* and with the property that there exists the constants $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M \quad \text{for any} \quad x \in [a,b]. \tag{2.3}$$

The constant $\frac{1}{2}$ is best possible in (3) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that u is of *bounded variation* and f is $K-$*Lipschitzian*, then $D(f,u)$ satisfies the inequality [5]

$$|D(f;u)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u). \tag{2.4}$$

Here the constant $\frac{1}{2}$ is also best possible.

The above inequalities have been used in [4] and [5] for obtaining inequalities between special means and on estimating the error in approximating the Stieltjes integral $\int_a^b f(x)\,du(x)$ in terms of the Riemann integral for the function f and the divided difference of u.

Now, for the function $u:[a,b]\to R$, consider the following auxiliary mappings Φ,Γ and Δ [3]:

$$\Phi(t) := \frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t), \qquad t\in[a,b],$$
$$\Gamma(t) := (t-a)\,[u(b)-u(t)] - (b-t)\,[u(t)-u(a)], \qquad t\in[a,b],$$
$$\Delta(t) := [u;b,t] - [u;t,a], \qquad t\in(a,b),$$

where $[u;\alpha,\beta]$ is the divided difference of u in α,β, i.e.,

$$[u;\alpha,\beta] := \frac{u(\alpha)-u(\beta)}{\alpha-\beta}.$$

The following representation of $D(f,u)$ may be stated.

Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that the Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. Then*

$$D(f,u) = \int_a^b \Phi(t)\,df(t) = \frac{1}{b-a}\int_a^b \Gamma(t)\,df(t) \tag{2.5}$$
$$= \frac{1}{b-a}\int_a^b (t-a)(b-t)\,\Delta(t)\,df(t).$$

Proof. Since $\int_a^b f(t)\,du(t)$ exists, hence $\int_a^b \Phi(t)\,df(t)$ also exists, and the integration by parts formula for Stieltjes integrals gives that

$$\int_a^b \Phi(t)\,df(t) = \int_a^b \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right] df(t)$$
$$= \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right] f(t)\Big|_a^b$$
$$- \int_a^b f(t)\,d\left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right]$$
$$= -\int_a^b f(t)\left[\frac{u(b)-u(a)}{b-a}dt - du(t)\right] = D(f,u),$$

proving the required identity.

Remark 1. *The identity (1) has been established in [3]. There were some typographical errors in [3] that have been corrected above.*

Remark 2. *If u is an integral, i.e., $u(t) = \int_a^t g(s)\,ds$, $t \in [a,b]$, then*

$$\Phi(t) = \frac{t-a}{b-a}\int_a^b g(s)\,ds - \int_a^t g(s)\,ds,$$
$$\Gamma(t) = (t-a)\int_t^b g(s)\,ds - (b-t)\int_a^t g(s)\,ds,$$
$$\Delta(t) = \frac{\int_t^b g(s)\,ds}{b-t} - \frac{\int_a^t g(s)\,ds}{t-a},$$

and then, from (1), one may recapture Cerone's identity (1) for the Čebyšev functional $T(f,g)$.

Since it well known that u is an integral if and only if u is absolutely continuous, and in this case $g(s) = u'(s)$ for $s \in [a,b]$, hence (1) is indeed a proper generalisation of (1) holding for larger classes of functions than the absolutely continuous functions.

Remark 3. *If one chooses $u : [a,b] \to \mathbb{R}$,*

$$u(t) = \frac{\int_a^t p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds}, \qquad t \in [a,b],$$

where p, g are Lebesgue integrable with pg is also integrable and $\int_a^b p(s)\,ds \neq 0$, then the identity (1) produces the representation:

$$E(f,g;p) := \frac{\int_a^b p(s)\,f(s)\,g(s)\,ds}{\int_a^b p(s)\,ds} - \frac{\int_a^b p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds}\cdot\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.6}$$
$$= \int_a^b \Phi_p(t)\,df(t) = \frac{1}{b-a}\int_a^b \Gamma_p(t)\,df(t)$$
$$= \frac{1}{b-a}\int_a^b (t-a)(b-t)\,\Delta_p(t)\,df(t),$$

where

$$\Phi_p(t) := \frac{t-a}{b-a} \cdot \frac{\int_a^b p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds} - \frac{\int_a^t p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds},$$

$$\Gamma_p(t) := (t-a)\,\frac{\int_t^b p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds} - (b-t)\,\frac{\int_a^t p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds}$$

and

$$\Delta_p(t) := \frac{\int_t^b p(s)\,g(s)\,ds}{(b-t)\int_a^b p(s)\,ds} - \frac{\int_a^t p(s)\,g(s)\,ds}{(t-a)\int_a^b p(s)\,ds}.$$

One must observe that the identity (3) is not the same as Cerone's identity for weighted integrals (1).

For recent inequalities related to $D(f;u)$ for various pairs of functions (f,u), see [3, pp. 112-118].

3. A Bound for f of Bounded Variation and u Continuous

It is known that if u is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. This integral may exists even for larger clases of integrators f, for instance, piecewise continuous functions for which the discontinuities of the integrand f do not overlap with those of the integrator u.

The following result may be stated:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ such that there exist the constants $\gamma, \Gamma \in \mathbb{R}$ with:*

$$\gamma \le u(t) \le \Gamma \quad \text{for any} \quad t \in [a,b] \tag{3.1}$$

and the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. Then

$$|D(f;u)| \le (\Gamma - \gamma) \bigvee_a^b (f). \tag{3.2}$$

The multiplicative constant 1 in front of $\Gamma - \gamma$ cannot be replaced by a smaller quantity.

Proof. By (1), we obviously have:

$$\gamma(b-t) \le (b-t)\,u(a) \le (b-t)\,\Gamma,$$
$$\gamma(t-a) \le (t-a)\,u(b) \le (t-a)\,\Gamma,$$
$$-(b-a)\,\Gamma \le -(b-a)\,u(t) \le -(b-a)\,\gamma,$$

which gives by addition and division with $b-a$ that

$$-(\Gamma-\gamma) \le \frac{(b-t)\,u(a)+(t-a)\,u(b)}{b-a} - u(t) \le \Gamma-\gamma,$$

showing that $|\Phi(t)| \le \Gamma-\gamma$ for any $t \in [a,b]$.

Taking into account that for φ bounded and ψ of bounded variation on $[a,b]$ one has

$$\left|\int_a^b \varphi(t)\,d\psi(t)\right| \le \sup_{t\in[a,b]} |\varphi(t)| \bigvee_a^b(\psi),$$

provided the Stieltjes integral exists, we have by (1) that

$$|D(f;u)| \le \sup_{t\in[a,b]} |\phi(t)| \bigvee_a^b(f) \le (\Gamma-\gamma)\bigvee_a^b(f),$$

proving the required inequality (7).

Now, for the sharpness of the inequality.

Assume that there exists a $c>0$ such that

$$|D(f;u)| \le c\,(\Gamma-\gamma)\bigvee_a^b(f), \tag{3.3}$$

where u and f are as in the hypothesis of the theorem.

Consider $u,f:[a,b]\to\mathbb{R}$ with

u$(t)=\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2$, $f(t)=sgn\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$.Then u is continuous, f is of bounded variation, the integral $\int_a^b f(t)\,du(t)$ exists and

$$\bigvee_a^b(f)=2, \qquad \int_a^b f(t)\,dt=0,$$

$$\Gamma=\sup_{t\in[a,b]} u(t)=\frac{(b-a)^2}{8}, \qquad \gamma=\inf_{t\in[a,b]} u(t)=0,$$

$$\int_a^b f(t)\,du(t) = \int_a^b sgn\left(t - \frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right) dt$$
$$= \int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{(b-a)^2}{4}.$$

Substituting into (8) we get $\frac{(b-a)^2}{4} \leq \frac{c(b-a)^2}{4}$, which implies that $c \geq 1$.

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation and $u : [a,b] \to \mathbb{R}$ continuous on $[a,b]$. Then:*

$$|D(f;u)| \leq \left[\max_{t\in[a,b]} u(t) - \min_{t\in[a,b]} u(t)\right] \bigvee_a^b (f). \tag{3.4}$$

The inequality (9) is sharp.

If we consider the Čebyšev functional $T(f,g)$, then we can state the following corollary as well:

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation and $g : [a,b] \to \mathbb{R}$ a Lebesgue integrable function such that there exists the constants m and M with*

$$m \leq g(s) \leq M \quad \textit{for a.e.} \quad s \in [a,b]. \tag{3.5}$$

Then

$$|T(f,g)| \leq (b-a)(M-m) \bigvee_a^b (f). \tag{3.6}$$

Proof. We choose $u(t) := \int_a^t g(s)\,ds$ which is continuous on $[a,b]$ and satisfies the inequality (6) with $\gamma = (b-a)m$ and $\Gamma = (b-a)M$ and apply Theorem 2.

Remark 4. *If we assume that for the Lebesgue integrable function g, $\int_a^{\cdot} g(s)\,ds$ satisfies the condition*

$$\gamma \leq \int_a^t g(s)\,ds \leq \Gamma \quad \textit{for any} \quad t \in [a,b],$$

then

$$|T(f,g)| \le (\Gamma-\gamma)\bigvee_a^b(f)$$

and the inequality is sharp. The equality case is realised for $g(t) = t - \frac{a+b}{2}$ *and* $f(t) = sgn\left(t - \frac{a+b}{2}\right), \quad t \in [a,b]$.

It is an open problem wether or not the bound in (11) *is sharp.*

Remark 5. *If* $p, g \in L[a,b]$ *so that* $pg \in L[a,b]$ *and* $\int_a^b p(s)\,ds \neq 0$ *and there exists the constants* δ, Δ *so that*

$$\delta \le \frac{\int_a^t p(s)\,g(s)\,ds}{\int_a^b p(s)\,ds} \le \Delta$$

for any $t \in [a,b]$, *then*

$$|E(f,g;p)| \le (\Delta-\delta)\bigvee_a^b(f).$$

The last inequality is sharp.

4. Application for Approximating the Stieltjes Integral

Let us consider the partition of the interval $[a,b]$ given by

$$I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Denote $v(I_n) := \max\{h_i | i = 0,\ldots,n-1\}$, where $h_i := t_{i+1} - t_i$, $i = 0,\ldots,n-1$. If $u : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and if we define

$$M_i := \sup_{t\in[t_i,t_{i+1}]} u(t), \qquad m_i := \inf_{t\in[t_i,t_{i+1}]} u(t)$$

and

$$v(u,I_n) := \max_{0\le i\le n-1}(M_i - m_i),$$

then, obviously, by the continuity of u on $[a,b]$, for any $\varepsilon \ge 0$, there exists a $\delta > 0$ and a division I_n with norm $v(I_n) < \delta$ such that $v(u,I_n) < \varepsilon$.

Consider now the quadrature rule

$$S_n(f,u,I_n) := \sum_{i=0}^{n-1} \frac{[u(t_{i+1}) - u(t_i)]}{t_{i+1} - t_i} \cdot \int_{t_i}^{t_{i+1}} f(t)\,dt, \tag{4.1}$$

provided u is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$.

We may state the following result in approximating the Stieltjes integral:

Theorem 3. *Let $f,u:[a,b]\to\mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and u is continuous on $[a,b]$. Then for any division I_n as above,*

$$\int_a^b f(t)\,du(t) = S_n(f,u,I_n) + R_n(f,u,I_n), \tag{4.2}$$

where the remainder $R_n(f,u,I_n)$ satisfies the estimate:

$$|R_n(f,u,I_n)| \le v(u,I_n) \bigvee_a^b (f). \tag{4.3}$$

Proof. Applying Theorem 2 on the intervals $[t_i,t_{i+1}]$, $i=0,\dots,n-1$, we have sucessively:

$$\begin{aligned}
|R_n(f,u,I_n)| &= \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\,du(t) - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(t)\,dt \right| \\
&\le \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} f(t)\,du(t) - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(t)\,dt \right| \\
&\le \sum_{i=0}^{n-1} (M_i - m_i) \bigvee_{t_i}^{t_{i+1}} (f) \le v(u,I_n) \bigvee_a^b (f)
\end{aligned}$$

and the estimate (14) is obtained.

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