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Some Properties of Certain Analytic and Univalent Functions*

B. A. Frasin†

Department of Mathematics, Al al-Bayt University

P.O. Box: 130095, Mafraq, Jordan.

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Abstract

In [2], Frasin and Jahangiri introduced the class $\mathcal{B}(\mu, \alpha)$ of analytic and univalent functions to give some properties for this class. The aim of this paper is to obtain new additional results for functions belonging to this class.

Keywords and Phrases: *Analytic functions, Univalent functions, Starlike and convex functions.*

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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†E-mail: bafrasin@yahoo.com.

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$. A function $f \in \mathcal{S}$ *is* said to be in the class $\mathcal{P}(\alpha)$ iff

$$\mathrm{Re} f'(z) > \alpha \qquad (z \in \mathcal{U}). \tag{1.4}$$

If $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}) \tag{1.5}$$

for some $0 < \alpha \leq 1$, then $f(z)$ said to be strongly starlike function of order α in $\mathcal{U}$, and this class denoted by $\overline{\mathcal{S}}^*(\alpha)$. Further, if $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}) \tag{1.6}$$

for some $0 < \alpha \leq 1$, then we say that $f(z)$ is strongly convex function of order α in $\mathcal{U}$, and we denote by $\overline{\mathcal{C}}(\alpha)$ the class of all such functions.

In [2], Frasin and Jahangiri introduced the class $\mathcal{B}(\mu, \alpha)$ defined as follows.

Definition. A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu, \alpha)$ if and only if

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \alpha \qquad (z \in \mathcal{U}) \tag{1.7}$$

for some $\alpha(0 \leq \alpha < 1)$ and $\mu \geq 0$.

Note that the condition (1.6) is equivalent to

$$\operatorname{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right\} > \alpha \qquad (0 \leq \alpha < 1, \mu \geq 0,\ z \in \mathcal{U}). \tag{1.8}$$

Clearly , $\mathcal{B}(1,\alpha) \equiv \mathcal{S}^*(\alpha)$, $\mathcal{B}(0,\alpha) \equiv \mathcal{P}(\alpha)$ and $\mathcal{B}(2,\alpha) \equiv \mathcal{B}(\alpha)$ the class which has been introduced and studied by Frasin and Darus [3]. (See also [1]).

In this paper, we shall give some additional properties for functions belonging to the class $\mathcal{B}(\mu,\alpha)$.

2. Main results

In order to prove our main results, we recall the following lemmas:

Lemma 1 ([4]). *Let $w(z)$ be analytic in $\mathcal{U}$ and such that $w(0) = 0$. Then if $|w(z)|$attains its maximum value on circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, we have*

$$z_0 w'(z_0) = k w(z_0) \tag{2.1}$$

where $k \geq 1$ is a real number.

Lemma 2 ([7]). *Let $f(z) \in \mathcal{A}$ satisfy the condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 \qquad (z \in \mathcal{U}) \tag{2.2}$$

then $f(z)$ is univalent in $\mathcal{U}$.

Lemma 3 ([5]). *Let $\phi(u,v)$ be a complex valued function,*

$$\phi : D \to C, \qquad (D \subset C \times C;\ C \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$ *and* $v = v_1 + iv_2$. *Suppose that the function* $\phi(u, v)$ *satisfies*

(i) $\phi(u, v)$ *is continuous in* D;

(ii) $(1, 0) \in D$ *and* $\mathrm{Re}(\phi(1, 0)) > 0$;

(iii) $\mathrm{Re}(\phi(iu_2, v_1)) \leq 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \leq -(1+u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *be regular in* $\mathcal{U}$ *such that* $(p(z), zp'(z)) \in D$ *for all* $z \in \mathcal{U}$. *If* $\mathrm{Re}(\phi(p(z), zp'(z))) > 0\,(z \in \mathcal{U})$, *then* $\mathrm{Re}(p(z)) > 0\,(z \in \mathcal{U})$.

Lemma 4([6]). Let a function $p(z)$ be analytic in $\mathcal{U}$, $p(0) = 1$, and $p(z) \neq 0$ $(z \in \mathcal{U})$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \qquad \text{for } |z| < |z_0| \tag{2.3}$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

with $0 < \alpha \leq 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \geq 1 \qquad \text{when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

$$k \leq -\frac{1}{2}(a + \frac{1}{a}) \leq -1 \qquad \text{when } \arg p(z_0) = -\frac{\pi}{2}\alpha$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm ai, \quad (a > 0).$$

Making use of Lemma 1, we first prove

Theorem 1. *Suppose that*

$$\left|\frac{zf''(z)}{f'(z)}+\mu\left(1-\frac{zf'(z)}{f(z)}\right)\right|<\frac{1-\alpha}{2\alpha}\qquad(z\in\mathcal{U}),\tag{2.4}$$

where $1/2\leq\alpha<1$ and $\mu\geq0$ *then* $f(z)\in\mathcal{B}(\mu,\alpha)$.

Proof. Let $w(z)$ be defined by

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu}=\frac{1+(1-2\alpha)w(z)}{1-w(z)},\qquad(w(z)\neq1).\tag{2.5}$$

Then $w(z)$ is regular in $\mathcal{U}$ and $w(0)=0$. Differentiating both sides of (2.5) logarithmically, we obtain

$$\frac{zf''(z)}{f'(z)}+\mu\left(1-\frac{zf'(z)}{f(z)}\right)=\frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)}+\frac{zw'(z)}{1-w(z)}\tag{2.6}$$

or, equivalently

$$\frac{zf''(z)}{f'(z)}+\mu\left(1-\frac{zf'(z)}{f(z)}\right)=\frac{2(1-\alpha)w(z)}{1-w(z)}\left\{\frac{zw'(z)}{[1+(1-2\alpha)w(z)]w(z)}\right\}\tag{2.7}$$

Suppose that there exists a point $z_0\in\mathcal{U}$ such that

$$\max_{|z|<|z_0|}|w(z)|=|w(z_0)|=1\qquad(w(z_0)\neq1).\tag{2.8}$$

Then, using Lemma 1, we have $z_0w'(z_0)=kw(z_0)$ $(k\geq1)$. Therefore, letting $w(z_0)=e^{i\theta}(\theta\neq0)$,we obtain that

$$\begin{aligned}&\left|\frac{z_0f''(z_0)}{f'(z_0)}+\mu\left(1-\frac{z_0f'(z_0)}{f(z_0)}\right)\right|\\ =\;&2(1-\alpha)k\left\{\frac{1}{2(1-\cos\theta)(1+2(1-2\alpha)\cos\theta+(1-2\alpha)^2}\right\}^{1/2}\geq\frac{1-\alpha}{2\alpha}\end{aligned}$$

which contradicts the condition (2.4), we have $|w(z)|<1$ for all $z\in\mathcal{U}$. Consequently, we conclude that $f(z)\in\mathcal{B}(\mu,\alpha)$.

Letting $\mu = 0$ in Theorem 1, we have

Corollary 1. *Suppose that*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1-\alpha}{2\alpha} \qquad (z \in \mathcal{U}), \tag{2.9}$$

where $1/2 \leq \alpha < 1$, *then* $f(z) \in \mathcal{P}(\alpha)$.

Letting $\mu = 1$ in Theorem 1, we have

Corollary 2. *Suppose that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{1-\alpha}{2\alpha} \qquad (z \in \mathcal{U}), \tag{2.10}$$

where $1/2 \leq \alpha < 1$, *then* $f(z) \in \mathcal{S}^*(\alpha)$.

Next, we prove

Theorem 2. *Let* $f(z) \in \mathcal{A}$, *If*

$$\mathrm{Re}\left\{\beta\frac{z^2 f'(z)}{f^2(z)} - (1-\beta)z^2\left(\frac{z}{f(z)}\right)''\right\} > \gamma \qquad (z \in \mathcal{U}) \tag{2.11}$$

for some $\gamma(\gamma < \beta)$, $0 \leq \beta \leq 1$, *then* $f(z) \in \mathcal{B}(2, \delta)$, *where* $\delta = \frac{2\gamma+1-\beta}{\beta+1} < 1$.

Proof. Define the function $p(z)$ by

$$\frac{z^2 f'(z)}{f^2(z)} = \delta + (1-\delta)p(z). \tag{2.12}$$

Then, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in $\mathcal{U}$. Since $\left(\frac{z^2 f'(z)}{f^2(z)}\right)' = -z\left(\frac{z}{f(z)}\right)''$, it follows from (2.11)

$$\beta\frac{z^2 f'(z)}{f^2(z)} - (1-\beta)z^2\left(\frac{z}{f}\right)'' = \beta\delta + \beta(1-\delta)p(z) + (1-\beta)(1-\delta)zp'(z) \tag{2.13}$$

and hence

$$\operatorname{Re}\left\{\beta\frac{z^2f'(z)}{f^2(z)}-(1-\beta)z^2\left(\frac{z}{f(z)}\right)''-\gamma\right\}$$

$$=\operatorname{Re}\left\{\beta\delta-\gamma+\beta(1-\delta)p(z)+(1-\beta)(1-\delta)zp'(z)\right\}>0. \tag{2.14}$$

If we define the function $\Phi(u,v)$ by

$$\Phi(u,v)=\beta\delta-\gamma+\beta(1-\delta)u+(1-\beta)(1-\delta)v \tag{2.15}$$

with $u=u_1+iu_2$ and $v=v_1+iv_{2,}$ then

(i) $\phi(u,v)$ is continuous in $D=C^2$;

(ii) $(1,0)\in D$ and $\operatorname{Re}(\phi(1,0))=\beta-\gamma>0$;

(iii) For all $(iu_2,v_1)\in D$ and such that $v_1\le -(1+u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2,v_1)\} &= \beta\delta-\gamma+(1-\beta)(1-\delta)v_1\\ &\le \beta\delta-\gamma-(1-\beta)(1-\delta)\frac{(1+u_2^2)}{2}\\ &= -\frac{1}{2}(1-\beta)(1-\delta)u_2^2\\ &\le 0\end{aligned}$$

Therefore, $\phi(u,v)$ satisfies the conditions of Lemma 3. This show that $\operatorname{Re}(p(z))>0$ $(z\in\mathcal{U})$, so that

$$\operatorname{Re}\left\{\frac{z^2f'(z)}{f^2(z)}\right\}>\delta=\frac{2\gamma+1-\beta}{\beta+1}\qquad(z\in\mathcal{U}).$$

Letting $\beta=0$ in Theorem 2, we have

Corollary 3. *Let $f(z)\in\mathcal{A}$, If*

$$\operatorname{Re}\left\{-z^2\left(\frac{z}{f(z)}\right)''\right\}>\gamma\qquad(z\in\mathcal{U}) \tag{2.16}$$

then $f(z) \in \mathcal{B}(2\gamma+1)$.

Letting $\gamma = -1/2$ in Corollary 3 and using Lemma 2, we have

Corollary 4. *Let* $f(z) \in \mathcal{A}$, *If*

$$\operatorname{Re}\left\{z^2\left(\frac{z}{f(z)}\right)''\right\} < \frac{1}{2} \qquad (z \in \mathcal{U}) \tag{2.17}$$

then $f(z)$ *is univalent in* $\mathcal{U}$.

From Corollary 4, we easily have

Corollary 5. *Let* $f(z) \in \mathcal{A}$, *If*

$$\left|z^2\left(\frac{z}{f(z)}\right)''\right| < \frac{1}{2} \qquad (z \in \mathcal{U}) \tag{2.18}$$

then $f(z)$ *is univalent in* $\mathcal{U}$.

Finally, we prove the following theorem

Theorem 3. *Let* $p(z)$ *be analytic in* $\mathcal{U}$ *with* $p(0) = 1$ *and* $p(z) \neq 0$ in $\mathcal{U}$ *and suppose that*

$$\left|\arg\left(p(z) + \frac{z^{\mu+1} f'(z)}{f^{\mu}(z)} p'(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}), \tag{2.19}$$

where $0 < \alpha < 1$, $\mu \geq 0$ *and* $f(z) \in \mathcal{B}(\mu, \alpha)$, *then we have*

$$|\arg(p(z))| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}). \tag{2.20}$$

Proof. Suppose that there exists point $z_0 \in \mathcal{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \qquad \text{for } |z| < |z_0|, \qquad |\arg p(z_0)| = \frac{\pi}{2}\alpha. \tag{2.21}$$

Then applying Lemma 4, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \geq 1 \qquad \text{when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

$$k \leq -\frac{1}{2}(a + \frac{1}{a}) \leq -1 \qquad \text{when } \arg p(z_0) = -\frac{\pi}{2}\alpha$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm ai, \quad (a > 0).$$

Then it follows that

$$\begin{aligned} \arg\left(p(z_0) + \frac{z_o^{\mu+1} f'(z_0)}{f^{\mu}(z_0)} p'(z_0)\right) &= \arg\left(p(z_0)\left(1 + \frac{z_o^{\mu} f'(z_0)}{f^{\mu}(z_0)} \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\ &= \arg\left(p(z_0)\left(1 + i\frac{z_o^{\mu} f'(z_0)}{f^{\mu}(z_0)}\alpha k\right)\right). \quad (2.22) \end{aligned}$$

When $\arg p(z_0) = \pi\alpha/2$, we have

$$\arg\left(p(z_0) + \frac{z_o^{\mu+1} f'(z_0)}{f^{\mu}(z_0)} p'(z_0)\right) = \arg(p(z_0)) + \arg\left(1 + i\frac{z_o^{\mu} f'(z_0)}{f^{\mu}(z_0)}\alpha k\right) > \frac{\pi}{2}\alpha, \quad (2.23)$$

because $\mathrm{Re}\frac{z_o^{\mu} f'(z_0)}{f^{\mu}(z_0)}\alpha k > 0$.

Similarly, if $\arg p(z_0) = -\pi\alpha/2$, then we obtain that

$$\arg\left(p(z_0) + \frac{z_o^{\mu+1} f'(z_0)}{f^{\mu}(z_0)} p'(z_0)\right) = \arg(p(z_0)) + \arg\left(1 + i\frac{z_o^{\mu} f'(z_0)}{f^{\mu}(z_0)}\alpha k\right) < -\frac{\pi}{2}\alpha, \quad (2.24)$$

because $\mathrm{Re}\frac{z_o^{\mu} f'(z_0)}{f^{\mu}(z_0)}\alpha k < 0$. Thus we see that (2.22) and (2.23) contradicts the condition (2.18). Consequently, we conclude that

$$|\arg(p(z))| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}). \quad (2.25)$$

Taking $p(z) = zf'(z)/f(z)$ in Theorem 3, we have

Corollary 6. *If* $f(z) \in \mathcal{A}$ *satisfying*

$$\left|\arg\left[\frac{zf'(z)}{f(z)} + f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}\left((zf'(z))' - \frac{z(f'(z))^2}{f(z)}\right)\right]\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}), \tag{2.26}$$

where $0 < \alpha < 1$, $\mu \geq 0$ *and* $f(z) \in \mathcal{B}(\mu, \alpha)$, *then* $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Taking $p(z) = 1 + zf''(z)/f'(z)$ in Theorem 3, we have

Corollary 7. *If* $f(z) \in \mathcal{A}$ *satisfying*

$$\left|\arg\left[\frac{(zf'(z))'}{f'(z)} + \left(\frac{z}{f(z)}\right)^{\mu+1}\left((zf''(z))' - \frac{z(f''(z))^2}{f'(z)}\right)\right]\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}), \tag{2.27}$$

where $0 < \alpha < 1$, $\mu \geq 0$ *and* $f(z) \in \mathcal{B}(\mu, \alpha)$, *then* $f(z) \in \overline{\mathcal{C}}(\alpha)$.

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