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Positive Solutions for Singular Boundary Value Problem with p-Laplacian*

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Abstract

In this paper, five functionals fixed point theorem is extended and applied to singular boundary value problem with p-Laplacian. The existence of at least three positive solutions is obtained.

Keywords and Phrases: *Positive solutions, Cone, Fixed point theorem.*

1. Introduction

Nonlinear multi-point boundary value problems have been studied extensively in the literature (see [?], [?], and [?], and the references cited therein). Singular differential boundary value problems arise in many nonlinear complex phenomena in the science, engineering and technology and have been studied

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extensively (see [1-4, 8, 9]).

Agarwal *et al.* [**?**] consider the following one-dimensional singular equation

$$\begin{cases} (\Phi_p y')' + q(t)f(t,y) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases}$$

The existence of one positive solution have been obtained by using upper-lower solution method.

Xian Xu [**?**] consider the following boundary value problems

$$\begin{cases} y'' + \lambda f(t,y) = 0, & 0 < t < 1, \\ y(0) = 0, y(1) = \sum_{i=0}^{m-2} \alpha_i y(\eta_i), \end{cases}$$

give some existence results for at least one positive solution by using Leray-Schauder continuation theorem.

Motivated by the results mentioned, we study the existence of three positive solutions for the following BVP

$$\begin{cases} (\Phi_p(u'))' + e(t)f(t,u) = 0, & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases} \tag{1}$$

where $\Phi_p v := |v|^{p-2}v, p > 1, 0 \leq \alpha < 1, f(t,u) : [0,1] \times (0,\infty) \to [0,\infty)$, f may have singularity at $u = 0$. $e(t)$ is a nonnegative measurable function defined on $(0,1)$, and $e(t)$ is not identically zero on any compact subinterval of $(0,1)$. Furthermore $e(t)$ satisfies $0 < \int_0^1 e(t)dt < +\infty$. When $\alpha = 0$, (1.1) becomes the Dirichlet problem.

As far as we know, there were not any papers to consider singular 3-point boundary value problem with p-laplacian operator by means of five functionals fixed point theorem[19]. The main reason is that $A : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$ is difficult to verified. To implying the theorem we improve the condition of five functionals fixed point theorem such that we only need to verify $A : \partial P(\gamma,c) \to \overline{P(\gamma,c)}$. At least three positive solutions for singular p-laplacian equation(1.1) are obtained.

By a positive solution of problem (1.1), we mean a function $y \in C^1[0,1]$, $\Phi_p(y') \in C^1(0,1]$ satisfying the boundary value problem (1.1) and that $y(t) > 0$ for $t \in (0,1]$.

The paper is organized as follows. Section 2 present some background definitions and five functionals fixed point theorem. Furthermore, we extend the five functionals fixed point theorem. The main result about the existence of solutions to BVP(1.1) are given in Section 3. Section 4 present the proof of existence theorem.

2. Background Knowledge and Results

Let γ, β, θ be nonnegative continuous convex functionals on K and let α, ψ be nonnegative continuous concave functionals on K . Then for nonnegative numbers h, a, b, d, and c, we define the following convex sets:

$$P(\gamma, c) = \{x \in K | \gamma(x) < c\}, \quad P(\gamma, \alpha, a, c) = \{x \in K | a \leq \alpha(x), \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, d, c) = \{x \in K | \beta(x) \leq d, \gamma(x) \leq c\},$$

$$P(\gamma, \theta, \alpha, a, b, c) = \{x \in K | a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in K | h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}.$$

Theorem 2.1[19]. *Let K be a cone in a real Banach space E. Let α and ψ be nonnegative continuous concave functionals on K and γ, β, and θ are nonnegative continuous convex functionals on K such that for some positive numbers c and m,*

$$\alpha(x) \leq \beta(x) \ \ and \ \ \|x\| \leq m\gamma(x) \ \ for\ all \ \ x \in \overline{P(\gamma, c)}.$$

Suppose further that $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist $h, d, a, b \geq 0$ with $0 < d < a$ such that each of the following is satisfied:

$(A_1)\{x \in P(\gamma, \theta, \alpha, a, b, c) | \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\gamma, \theta, \alpha, a, b, c)$,

$(A_2)\{x \in Q(\gamma, \beta, \psi, h, d, c) | \beta(x) < d\} \neq \emptyset$ and $\beta(Ax) < d$ for $x \in Q(\gamma, \beta, \psi, h, d, c)$,

$(A_3)\alpha(Ax) > a$ provided $x \in P(\gamma, \alpha, a, c)$ with $\theta(Ax) > b$,

$(A_4)\beta(Ax) < d$ provided $x \in Q(\gamma, \beta, d, c)$ with $\psi(Ax) < h$.

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < d, a < \alpha(x_2), \ \ and \ \ d < \beta(x_3) \ \ with \ \ \alpha(x_3) < a.$$

Theorem 2.2. *Let K be a cone in a real Banach space E. Let α and ψ be nonnegative continuous concave functionals on K and γ, β, and θ are nonnegative continuous convex functionals on K such that for some positive numbers c and m,*

$$\alpha(x) \le \beta(x) \ \text{ and } \ \|x\| \le m\gamma(x) \ \text{ for all } \ x \in \overline{P(\gamma, c)}.$$

Suppose further that $A : \overline{P(\gamma, c)} \to K$, is completely continuous, $A|_{\partial P(\gamma,c)} : \partial P(\gamma, c) \to \overline{P(\gamma, c)}$ and there exist $h, d, a, b \ge 0$ with $0 < d < a$ such that each of the following is satisfied:

$(C_1)\{x \in P(\gamma, \theta, \alpha, a, b, c) | \alpha(x) > a\} \neq \emptyset$ and $\alpha(\Theta \circ Ax) > a$ for $x \in P(\gamma, \theta, \alpha, a, b, c)$,

$(C_2)\{x \in Q(\gamma, \beta, \psi, h, d, c) | \beta(x) < d\} \neq \emptyset$ and $\beta(\Theta \circ Ax) < d$ for $x \in Q(\gamma, \beta, \psi, h, d, c)$,

$(C_3)\alpha(\Theta \circ Ax) > a$ provided $x \in P(\gamma, \alpha, a, c)$ with $\theta(\Theta \circ Ax) > b$,

$(C_4)\beta(\Theta \circ Ax) < d$ provided $x \in Q(\gamma, \beta, d, c)$ with $\psi(\Theta \circ Ax) < h$,

where $\Theta : K \to \overline{P(\gamma, c)}$ is a contraction operator such that

$$\Theta u = u \ \text{ for } \ u \in \overline{P(\gamma, c)}; \qquad \Theta u \in \partial P(\gamma, c) \ \text{ for } \ u \notin \overline{P(\gamma, c)}.$$

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < d, a < \alpha(x_2), \ \text{ and } \ d < \beta(x_3) \ \text{ with } \ \alpha(x_3) < a.$$

Proof. By the definitions of A and Θ, $\Theta \circ A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous. By Theorem 2.1 and $(C_1) - (C_4)$, $\Theta \circ A$ has at least three fixed points, i.e., $(\Theta \circ A)x_i = x_i, i = 1, 2, 3$. We claim $Ax_i = x_i$.
In fact, if $Ax_i \in \overline{P(\gamma, c)}$, then $(\Theta \circ A)x_i = Ax_i = x_i$; if $Ax_i \notin \overline{P(\gamma, c)}$, then $x_i = (\Theta \circ A)x_i \in \partial P(\gamma, c)$. So $Ax_i \in \overline{P(\gamma, c)}$, a contradiction.
Thus $Ax_i \in \overline{P(\gamma, c)}$ and $Ax_i = x_i, i = 1, 2, 3$.

3. Main Result

In this paper we will use the following conditions

(H1) the nonlinear term $f(t, y)$ satisfies $f(t, y) \leq g(y) + h(y)$ for $t \in [0, 1]$ with f continuous on $(0, \infty), g > 0$ continuous, non-increasing on $(0, \infty)$ and h continuous on $[0, \infty)$,

(H2) there exists an $\varepsilon > 0$ such that $f(t, u)$ is non-increasing in $u \leq \varepsilon$ for all $t \in [0, 1]$,

(H3) $\int_0^1 e(s)g(ms)ds < \infty, \ \forall\ m > 0$,

(H4) for each constant $r > 0$, there exists a function φ_r continuous on $[0, 1]$ and positive $(0, 1)$ such that $f(t, u) \geq \varphi_r(t)$ on $[0, 1] \times (0, r]$.

The main result of this paper is the following

Theorem 3.1. *Suppose (H1)-(H4) hold. If there exist*

$$0 < t_1 < t_2 \leq 1/2, 0 < t_3 \leq 1/2, 0 < d < a < \frac{t_2}{t_1} a \leq c,$$

such that

$$f(t,u) \geq \max\left\{ \frac{\Phi_p(\frac{2a}{t_1})}{\int_{t_1}^{\frac{t_1+t_2}{2}} e(\theta)d\theta}, \frac{\Phi_p(\frac{2a}{1-t_2})}{\int_{\frac{t_1+t_2}{2}}^{t_2} e(\theta)d\theta} \right\},\ (t,u) \in [0,1] \times [t_1 a, \frac{t_2}{t_1} a + 1], \tag{2}$$

$$\int_0^1 e(\theta) \left(g(\alpha\eta(1-\eta)dt_3\theta) + \max_{u\in[0,d+1]} h(u) \right) d\theta \leq \Phi_p(d), \tag{3}$$

$$\int_0^1 e(\theta) \left(g(\alpha\eta(1-\eta)c\theta) + \max_{u\in[0,c+1]} h(u) \right) d\theta < \Phi_p(c). \tag{4}$$

Then problem (1.1) has at least three positive solutions $u_i \in C^1[0, 1], \Phi_p u_i' \in C^1(0, 1]$, *with* $u_i \in \overline{P(\gamma, c)}$, $i = 1, 2, 3$ *such that*

$$\max_{t\in[0,1]} u_1(t) \leq d,\ \frac{u_2(t_1) + u_2(t_2)}{2} \geq a,$$

and

$$\max_{t\in[0,1]} u_3(t) \geq d,\ with\ \frac{u_3(t_1) + u_3(t_2)}{2} \leq a.$$

4. Related Lemmas and Proof of Theorem 3.1

To prove the main result, we need some lemmas as follows:

Lemma 4.1. *Suppose $y : [0,1] \to (0,\infty)$ is continuous, $0 \le \alpha < 1$. Then boundary value problem*

$$\begin{cases} (\Phi_p u')' + y(t) = 0, \quad t \in [0,1], \\ u(0) = 0, u(1) = \alpha u(\eta) \end{cases} \tag{5}$$

has a unique solution $u \in C^1[0,1], \Phi_p u' \in C^1[0,1]$,

$$u(t) = \begin{cases} \displaystyle\int_0^t \Phi_q(\int_s^{\sigma_y} y(\theta)d\theta)ds, \\ 0 \le t \le \sigma_y \le 1, \eta \le \sigma_y; \\ \displaystyle\int_t^1 \Phi_q(\int_{\sigma_y}^s y(\theta)d\theta)ds + \alpha\int_0^\eta \Phi_q(\int_s^{\sigma_y} y(\theta)d\theta)ds, \\ \quad 0 \le \sigma_y \le t \le 1, \eta \le \sigma_y; \\ \displaystyle\int_t^1 \Phi_q(\int_{\sigma_y}^s y(\theta)d\theta)ds + \frac{\alpha}{1-\alpha}\int_\eta^1 \Phi_q(\int_{\sigma_y}^s y(\theta)d\theta)ds \\ \quad 0 \le \sigma_y \le t \le 1, \sigma_y \le \eta, \end{cases} \tag{6}$$

where

$$\int_0^{\sigma_y} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds = \int_{\sigma_y}^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right)ds + \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds$$

$$= \int_{\sigma_y}^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right)ds + \frac{\alpha}{1-\alpha}\int_\eta^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right)ds.$$

Proof. Firstly, integrating BVP(4.1) from 0 to s, then integrating again from 0 to t in s, the solution of BVP(4.1) may be represented as :

$$u(t) = \int_0^t \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds \tag{7}$$

where σ_y satisfies

$$\int_\eta^1 \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds + (1-\alpha)\int_0^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds = 0,$$

i.e.

$$\int_0^1 \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds = \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds \tag{8}$$

Set

$$\begin{aligned} F(\sigma) &= \int_0^1 \Phi_q\left(\int_s^{\sigma} y(\theta)d\theta\right)ds - \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma} y(\theta)d\theta\right)ds \\ &= (1-\alpha)\int_0^\eta \phi_q\left(\int_s^{\sigma} y(\theta)d\theta\right)ds + \int_\eta^1 \phi_q\left(\int_s^{\sigma} y(\theta)d\theta\right)ds. \end{aligned}$$

Clearly $F(\sigma)$ is continuous and increasing with respect to σ and

$$F(\sigma_1) < 0 \text{ for } \sigma_1 = 0; \qquad F(\sigma_2) > 0 \text{ for } \sigma_2 = 1. \tag{9}$$

So there exists a unique $\sigma_y \in [0,1]$ satisfying $F(\sigma_y) = 0$. Thus there is a unique solution u to BVP(4.1).

If $\eta \le \sigma_y$, by (4.3) (4.4) and $t \ge \sigma_y$ we have

$$\begin{aligned} u(t) &= -\int_t^1 \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds + \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds \\ &= \int_t^1 \Phi_q\left(\int_{\sigma_y}^{s} y(\theta)d\theta\right)ds + \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds. \end{aligned}$$

If $\eta \ge \sigma_y$, by (4.3) (4.4) and $t \ge \sigma_y$ we have

$$\begin{aligned} u(t) =& -\int_t^1 \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds + \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds \\ =& \int_t^1 \Phi_q\left(\int_{\sigma_y}^{s} y(\theta)d\theta\right)ds + \alpha\int_0^{\sigma_y} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds \\ &+ \alpha\int_{\sigma_y}^\eta \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right)ds. \end{aligned} \tag{10}$$

Then

$$\begin{aligned}
&\int_0^{\sigma_y} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds \\
= \;& u(\sigma_y) \\
= \;& \int_{\sigma_y}^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right) ds + \alpha \int_0^{\sigma_y} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds \\
& +\alpha \int_{\sigma_y}^{\eta} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds.
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_0^{\sigma_y} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds \\
&= \frac{1}{1-\alpha}\left(\int_{\sigma_y}^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right) ds + \alpha \int_{\sigma_y}^{\eta} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds\right).
\end{aligned}$$

So (4.6) becomes

$$\begin{aligned}
u(t) &= \int_t^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right) ds + \frac{\alpha}{1-\alpha}\left(\int_{\sigma_y}^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right) ds\right. \\
&\quad \left. +\alpha \int_{\sigma_y}^{\eta} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds + \alpha \int_{\sigma_y}^{\eta} \Phi_q\left(\int_s^{\sigma_y} y(\theta)d\theta\right) ds\right. \\
&= \int_t^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right) ds + \frac{\alpha}{1-\alpha}\int_{\eta}^1 \Phi_q\left(\int_{\sigma_y}^s y(\theta)d\theta\right) ds.
\end{aligned}$$

Thus (4.2) holds.

Let $E = C[0,1]$, $\|u\| = \sup_{t\in[0,1]} |u(t)|$ for $u \in E$,

$K = \{u \in E : u(t)$ is nonnegative, concave valued on $[0,1], u(0) = 0, u(1) = \alpha u(\eta)\}$.

Finally on K we define the nonnegative continuous concave functionals α, ψ and nonnegative continuous convex functionals β, θ, γ by

$$\gamma(u) = \max_{t\in[0,1]} u(t),\ \psi(u) = \min_{t\in[t_3,1-t_3]} u(t),\ \beta(u) = \max_{t\in[0,1]} u(t),$$

$$\alpha(u) = \frac{u(t_1) + u(t_2)}{2}, \ \theta(u) = \max_{t \in [t_1, t_2]} u(t),$$

for $0 < t_1 < t_2 \leq 1/2$, $0 < t_3 \leq 1/2$. It is obvious that for each $u \in P, \alpha(u) \leq \beta(u)$.

Lemma 4.2 Let $u \in K$. Then $u(t) \geq \alpha\eta(1-\eta)\|u\|t, \ \forall \ t \in [0, 1]$.

Proof. Without loss of generality, suppose $u(\xi) = u(1)$. By the concavity of u we have

$$u(t) \geq u(\xi)t = u(1)t, \ t \in [0, \xi].$$

In addition, $u(t) \geq u(1) \geq u(1)t$ for $t \in [\xi, 1]$.
By $u \in P$, we have $u(1) = \alpha u(\eta) \geq \alpha\eta(1-\eta)\|u\|$. So $u(t) \geq \alpha\eta(1-\eta)\|u\|t, t \in [0, 1]$.

Consider boundary value problem

$$\begin{cases} (\Phi_p x')' + e(t)f(t, [u]^* + \dfrac{1}{n}) = 0, \quad 0 < t < 1, \\ x(0) = 0, x(1) = \alpha x(\eta), \end{cases} \tag{11}$$

where $[u]^* = \begin{cases} u, & u \geq 0; \\ 0, & u \leq 0. \end{cases}$ Define the operator T by

$$(Tu)(t) = \begin{cases} \displaystyle\int_0^t \Phi_q \left(\int_s^{\sigma_u} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta \right) ds, \\ \quad 0 \leq t \leq \sigma_u \leq 1, \eta \leq \sigma_u; \\ \displaystyle\int_t^1 \Phi_q \left(\int_{\sigma_u}^s e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta \right) ds \\ \displaystyle + \alpha \int_0^\eta \Phi_q \left(\int_s^{\sigma_u} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta \right) ds, \\ \quad 0 \leq \sigma_u \leq t \leq 1, \eta \leq \sigma_u; \\ \displaystyle\int_t^1 \Phi_q \left(\int_{\sigma_u}^s e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta \right) ds \\ \displaystyle + \frac{\alpha}{1-\alpha} \int_\eta^1 \Phi_q \left(\int_{\sigma_u}^s e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta \right) ds, \\ \quad 0 \leq \sigma_u \leq t \leq 1, \sigma_u \leq \eta. \end{cases} \tag{12}$$

Lemma 4.3. *Suppose the conditions in Theorem 3.1 hold, then T has at least three fixed points, $u_{in} \in \overline{P(\gamma, c)}, i = 1, 2, 3$ satisfying*

$$\max_{t\in[0,1]} u_{1n}(t) < d, \quad \frac{u_{2n}(t_1) + u_{2n}(t_2)}{2} > a,$$

$$\text{and } \max_{t\in[0,1]} u_{3n}(t) > d, \quad \frac{u_{3n}(t_1) + u_{3n}(t_2)}{2} < a.$$

Proof. We will finish the proof applying Theorem 2.2.

First we show $T : \partial P(\gamma, c) \to \overline{P(\gamma, c)}$.

For any $u \in \partial P(\gamma, c)$, we have $c \geq u(t) \geq \alpha\eta(1-\eta)\|u\|t = \alpha\eta(1-\eta)ct, t \in [0, 1]$, by Lemma 4.2. So $[u]^* = u$.

By Lemma 4.1, (H1), (H3) and (3.3) we have

$$\begin{aligned}
\gamma(Tu) &= \max_{t\in[0,1]} Tu(t) = (Tu)(\sigma_u) = \int_0^{\sigma_u} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds \\
&\leq \Phi_q\left(\int_0^1 e(\theta)f(\theta, u(\theta) + \frac{1}{n})d\theta\right) \\
&\leq \Phi_q\left(\int_0^1 e(\theta)\left(g(\alpha\eta(1-\eta)c\theta) + \max_{u\in[0,c+1]} h(u)\right) d\theta\right) < c
\end{aligned}$$

So $T : \partial P(\gamma, c) \to \overline{P(\gamma, c)}$.

In addition, standard argument shows that $T : \overline{P(\gamma, c)} \to K$ is completely continuous.

Following we will show $(C_1) - (C_4)$ hold. Therefore T has at least three fixed points.

Let $b = \frac{t_2}{t_1}a, h = dt_3$,

$\{u \in P(\gamma, \theta, \alpha, a, \frac{t_2}{t_1}a, c) | \alpha(u) > a\} \neq \emptyset$

$\{u \in Q(\gamma, \beta, \psi, dt_3, d, c) | \beta(u) < d\} \neq \emptyset$.

In the following claims, we verify the remaining conditions of Theorem 2.2.

Let $(\Theta u)(t) = \min\{u(t), c\}$. So $\Theta : K \to \overline{P(\gamma, c)}$ is a contraction operator.

Claim 1. If $u \in P(\gamma, \theta, \alpha, a, \frac{t_2}{t_1}a, c)$, then $\alpha(\Theta \circ Tu) > a$.
By $u \in P(\gamma, \theta, \alpha, a, \frac{t_2}{t_1}a, c)$, we claim $u(t) \in [t_1 a, \frac{t_2}{t_1}a]$ holds for $t \in [t_1, t_2]$. In fact, by $u \in K$ we have

$$\min_{t\in[t_1,t_2]} u(t) = u(t_1) \geq t_1\|u\| \geq t_1\alpha(u) \geq t_1 a, \quad \text{for} \quad t_1 < t_2 \leq \sigma_u;$$

$$\min_{t\in[t_1,t_2]} u(t) = u(t_2) \geq (1-t_2)\|u\| \geq t_1\alpha(u) \geq t_1 a \quad \text{for} \quad \sigma_u \leq t_1 < t_2;$$

$$\min_{t\in[t_1,t_2]} u(t) = \min\{u(t_1), u(t_2)\} \geq t_1 a \quad \text{for} \quad t_1 \leq \sigma_u \leq t_2.$$

When $t_1 \leq t_2 \leq \sigma_u$, it follows from (3.1) that

$$\begin{aligned}
\alpha(\Theta\circ Tu) &= \frac{(\Theta\circ T)u(t_1) + (\Theta\circ T)u(t_2)}{2} > (\Theta\circ T)u(t_1)\\
&= \min\left\{\int_0^{t_1} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds, c\right\}\\
&\geq \min\left\{\int_0^{t_1} \Phi_q\left(\int_{t_1}^{t_2} e(\theta)f(\theta,u(\theta) + \frac{1}{n})d\theta\right)ds, c\right\}\\
&\geq \min\left\{t_1\Phi_q\left(\int_{t_1}^{t_2} e(\theta)\frac{\Phi_p(\frac{2a}{t_1})}{\int_{t_1}^{\frac{t_1+t_2}{2}} e(s)ds}d\theta\right), c\right\} > a.
\end{aligned}$$

When $\sigma_u \leq t_1 \leq t_2$, similar to the above process we obtain

$$\alpha(\Theta\circ Tu) = \frac{(\Theta\circ T)u(t_1) + (\Theta\circ T)u(t_2)}{2} > (\Theta\circ T)u(t_2) > a.$$

When $t_1 \leq \sigma_u \leq t_2$, it follows from (3.1) that

$$\begin{aligned}
&\alpha(\Theta\circ Tu) = \frac{(\Theta\circ T)u(t_1) + (\Theta\circ T)u(t_2)}{2}\\
> &\frac{1}{2}\min\left\{\int_0^{t_1} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right)ds, c\right\}\\
&+\min\left\{\int_{t_2}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right)ds, c\right\}\\
> &\frac{1}{2}\min\left\{t_1\Phi_q\left(\int_{t_1}^{\frac{t_1+t_2}{2}} e(\theta)f(\theta,u(\theta) + \frac{1}{n})d\theta\right), c\right\}\\
&or\frac{1}{2}\min\left\{(1-t_2)\Phi_q\left(\int_{\frac{t_1+t_2}{2}}^{t_2} e(\theta)f(\theta,u(\theta) + \frac{1}{n})d\theta\right), c\right\} > a.
\end{aligned}$$

Claim 2. If $u \in Q(\gamma,\beta,\psi,dt_3,d,c)$, then $\beta(\Theta\circ Tu) < a$.
For $u \in Q(\gamma,\beta,\psi,dt_3,d,c)$, we have $\beta(u) = \max_{t\in[0,1]} u(t) \leq d, \psi(u) = \min_{t\in[t_3,1-t_3]} u(t) \geq$

dt_3. It follows from Lemma 4.2 that $u(t) \geq \alpha\eta(1-\eta)\|u\|t \geq \alpha\eta(1-\eta)dt_3t, t \in [0,1]$. So $[u]^* = u$. By (H1) (H3) (3.2) we have

$$\beta(\Theta \circ Tu) = \max_{t\in[0,1]}(\Theta \circ T)u(t) = \min\{\max_{t\in[0,1]} Tu(t), c\} = \min\{Tu(\sigma_u), c\}$$
$$\leq \min\left\{\Phi_q\left(\int_0^1 e(\theta)\left(g(\alpha\eta(1-\eta)dt_3\theta) + \max_{u\in[0,d+1]} h(u)\right)d\theta\right), c\right\} < d.$$

Claim 3. If $u \in P(\gamma,\alpha,a,c)$ with $\theta(\Theta \circ Tu) > \frac{t_2}{t_1}a$, then $\alpha(\Theta \circ Tu) > a$.
For $\eta \leq \sigma_u, \sigma_u \leq t_1$ we have

$$\alpha(\Theta \circ Tu) = \frac{(\Theta \circ T)u(t_1) + (\Theta \circ T)u(t_2)}{2} > (\Theta \circ T)u(t_2)$$
$$= \min\left\{\int_{t_2}^1 \Phi_q\left(\int_{\sigma_u}^s e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds\right.$$
$$\left. +\alpha\int_0^\eta \Phi_q\left(\int_{\sigma_u}^s e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds, c\right\}$$
$$\geq \min\left\{\frac{1-t_2}{1-t_1}\int_{t_1}^1 \Phi_q\left(\int_{\sigma_u}^s e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds\right.$$
$$\left. +\alpha\int_0^\eta \Phi_q\left(\int_{\sigma_u}^s e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds, c\right\}$$
$$= \frac{1-t_2}{1-t_1}\max_{t\in[t_1,t_2]}\Theta \circ Tu(t) = \frac{1-t_2}{1-t_1}\theta(\Theta \circ Tu) > \frac{t_1}{t_2}\theta(\Theta \circ Tu) > a.$$

For $\eta \leq \sigma_u, t_2 \leq \sigma_u$ we have

$$\alpha(\Theta \circ Tu) = \frac{(\Theta \circ T)u(t_1) + (\Theta \circ T)u(t_2)}{2} > (\Theta \circ T)u(t_1)$$
$$= \min\left\{\int_0^{t_1} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds, c\right\}$$
$$\geq \min\left\{\frac{t_1}{t_2}\int_0^{t_2} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right)ds, c\right\}$$
$$= \frac{t_1}{t_2}\min_{t\in[t_1,t_2]}\Theta \circ Tu(t) = \frac{t_1}{t_2}\theta(\Theta \circ Tu) > a.$$

For $\eta \leq \sigma_u, t_1 \leq \sigma_u \leq t_2$, similar to the above process we have

$$\alpha(\Theta \circ Tu) = \frac{(\Theta \circ T)u(t_1) + (\Theta \circ T)u(t_2)}{2} > a.$$

For $\eta \geq \sigma_u, \sigma_u \leq t_1$ we have

$$\begin{aligned}
& \alpha(\Theta \circ Tu) = \frac{(\Theta \circ T)u(t_1) + \Theta \circ T)u(t_2)}{2} > (\Theta \circ T)u(t_2) \\
= \;& \min\left\{\int_{t_2}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds \right. \\
& \left. + \frac{\alpha}{1-\alpha}\int_{\eta}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
\geq \;& \min\left\{\frac{1-t_2}{1-t_1}\int_{t_1}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds \right. \\
& \left. + \frac{\alpha}{1-\alpha}\int_{\eta}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
\geq \;& \frac{1-t_2}{1-t_1}\max_{t\in[t_1,t_2]} \Theta \circ Tu(t) = \frac{1-t_2}{1-t_1}\theta(\Theta \circ Tu) > \frac{t_1}{t_2}\theta(\Theta \circ Tu) > a.
\end{aligned}$$

For the case $\eta \geq \sigma_u, t_2 \leq \sigma_u$ and $\eta \geq \sigma_u, t_1 \leq \sigma_u \leq t_2$. The proof is similar to $\eta \leq \sigma_u, t_2 \leq \sigma_u$ and $\eta \leq \sigma_u, t_1 \leq \sigma_u \leq t_2$, respectively. So we omit it here.

Claim 4. If $u \in Q(\gamma, \beta, d, c)$ with $\psi(\Theta \circ Tu) < dt_3$, then $\beta(\Theta \circ Tu) < d$.

For $u \in Q(\gamma, \beta, d, c), u(t) \in [0, d], t \in [0, 1]$. So $[u]^* = u$.

For $\eta \leq \sigma_u, 1 - t_3 \leq \sigma_u$ we have

$$\begin{aligned}
& \beta(\Theta \circ Tu) = (\Theta \circ Tu)(\sigma_u) \\
= \;& \min\left\{\int_{0}^{\sigma_u} \Phi_q\left(\int_{s}^{\sigma_u} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
\leq \;& \min\left\{\frac{\sigma_u}{t_3}\int_{0}^{t_3} \Phi_q\left(\int_{s}^{\sigma_u} e(\theta)f(\theta, [u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
\leq \;& \frac{1}{t_3}\min\{\min_{t\in[t_3,1-t_3]} Tu(t), c\} = \frac{1}{t_3}\Theta(\psi(Tu)) = \frac{1}{t_3}\psi(\Theta \circ Tu) < d.
\end{aligned}$$

For $\eta \le \sigma_u, \sigma_u \le t_3 \le 1-t_3$ we have

$$\begin{aligned}
\beta(\Theta \circ Tu) &= (\Theta \circ Tu)(\sigma_u) \\
&= \min\left\{\int_{\sigma_u}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds \right. \\
&\quad \left. +\alpha \int_0^{\eta} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
&\le \min\left\{\frac{1-\sigma_u}{t_3}\int_{1-t_3}^{1} \Phi_q\left(\int_{\sigma_u}^{s} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds \right. \\
&\quad \left. +\alpha \int_0^{\eta} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
&= \frac{1}{t_3}\min\{\min_{t\in[t_3,1-t_3]} Tu(t), c\} = \frac{1}{t_3}\psi(\Theta \circ Tu) < d.
\end{aligned}$$

For $\eta \le \sigma_u, t_3 \le \sigma_u \le 1-t_3$, we have the following two inequlities hold.

$$\begin{aligned}
&\int_0^{\sigma_u} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds \\
&\le \frac{\sigma_u}{t_3}\int_0^{t_3} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,u(\theta) + \frac{1}{n})d\theta\right) ds \le \frac{Tu(t_3)}{t_3}
\end{aligned}$$

and

$$\int_0^{\sigma_u} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds \le \frac{1}{t_3}Tu(1-t_3).$$

Then

$$\begin{aligned}
&\beta(\Theta \circ T_1u) = (\Theta \circ T_1u)(\sigma_u) \\
= &\min\left\{\int_0^{\sigma_u} \Phi_q\left(\int_s^{\sigma_u} e(\theta)f(\theta,[u(\theta)]^* + \frac{1}{n})d\theta\right) ds, c\right\} \\
\le &\min\left\{\frac{1}{t_3}Tu(t_3), \frac{1}{t_3}Tu(1-t_3), c\right\} \le \frac{1}{t_3}\min\{Tu(t_3), Tu(1-t_3), c\} \\
= &\frac{1}{t_3}\min\{\psi(Tu), c\} = \frac{1}{t_3}\psi(\Theta \circ Tu) < d.
\end{aligned}$$

For $\sigma_u \le \eta, 1-t_3 \le \sigma_u$, $\sigma_u \le \eta, \sigma_u \le t_3$ and $\sigma_u \le \eta, t_3 \le \sigma_u \le 1-t_3$, we can show $\beta(\Theta \circ Tu) < d$ similarly. Therefore the hypotheses of Theorem 2.2 are satisfied and there exists three fixed points $u_{1n}, u_{2n}, u_{3n} \in \overline{P(\gamma, c)}$ such that

$$\beta(u_{1n}) = \max_{t\in[0,1]} u_{1n}(t) < d, \qquad \alpha(u_{2n}) = \frac{u_{2n}(t_1) + u_{2n}(t_2)}{2} > a. \tag{13}$$

$$\beta(u_{3n}) = \max_{t\in[0,1]} u_{3n}(t) > d, \qquad \alpha(u_{3n}) = \frac{u_{3n}(t_1) + u_{3n}(t_2)}{2} < a. \tag{14}$$

Lemma 4.5. *Set*

$$S = \{u_n | u_n \textit{is a fixed point of operator } T\},$$

$$A = \{\sigma_{u_n} | \sigma_{u_n} \in (0,1), u_n^{'}(\sigma_{u_n}) = 0, u_n \textit{is a fixed point of operator } T\}.$$

Then $0 < infA \leq supA < 1$.

Proof. For any $u_n \in S$, by the conclusion of Lemma 4.1, for $\eta \leq \sigma_{u_n}, \sigma_{u_n}$ satisfies

$$\begin{aligned}
&\int_0^{\sigma_{u_n}} \Phi_q\left(\int_s^{\sigma_{u_n}} e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right) ds \\
&= \int_{\sigma_{u_n}}^1 \Phi_q\left(\int_{\sigma_{u_n}}^s e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right) ds + \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_{u_n}} e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right) ds.
\end{aligned} \tag{15}$$

If $\sigma_{u_n} \to 1$, taking limits on both sides of (4.12), we have

$$\begin{aligned}
&\int_0^1 \Phi_q\left(\int_s^1 e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right) ds \\
= \;& \alpha\int_0^\eta \Phi_q\left(\int_s^{\sigma_{u_n}} e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right) ds < \int_0^1 \Phi_q\left(\int_s^1 e(\theta)f(\theta,[u]^* + \frac{1}{n})d\theta\right) ds,
\end{aligned}$$

a contradiction. For $\sigma_{u_n} \leq \eta$, similarly we get a contradiction. The proof is completed.

Lemma 4.6. *Let u_n be a fixed point of operator T. Suppose (H4) holds, then there exists $m > 0$ (independent of n) such that $u_n(t) \geq mt$.*

Proof. Noticing $u_n(0) = 0, u^{''}(0) \leq 0, t \in [0,1]$, we have $u_n(t) \geq u_n(1)t$. So it is sufficient to prove there exists m (independent of m) such that $u_n(1) \geq m$.

$u_n(1)$ satisfied

$$\begin{aligned} u_n(1) &= \left\{ \begin{array}{l} \alpha \int_0^\eta \Phi_q \left(\int_s^{\sigma_{u_n}} e(\theta) f(\theta, [u(\theta)]^* + \frac{1}{n}) d\theta \right) ds, \quad \eta \le \sigma_{u_n}; \\ \frac{\alpha}{1-\alpha} \int_\eta^1 \Phi_q \left(\int_{\sigma_{u_n}}^s e(\theta) f(\theta, [u(\theta)]^* + \frac{1}{n}) d\theta \right) ds \quad \sigma_{u_n} \le \eta, \end{array} \right. \\ &\ge \left\{ \begin{array}{l} \alpha \int_0^\eta \Phi_q \left(\int_s^\eta e(\theta) \varphi_r(\theta) d\theta \right) ds, \quad \eta \le \sigma_{u_n}; \\ \frac{\alpha}{1-\alpha} \int_\eta^1 \Phi_q \left(\int_\eta^s e(\theta) \varphi_r(\theta) d\theta \right) ds \quad \sigma_{u_n} \le \eta. \end{array} \right. \end{aligned}$$

So there exists m (independent of n) such that $u_n(1) \ge m$. The proof is completed.

Proof of Theorem 3.1. The proof is achieved in three steps.
Step 1. Let the operator T be defined by (4.8) for every $n \in N$. It follows from Lemma 4.4 that T has at least three fixed points $u_{in}, i = 1, 2, 3$ such that $u_{in} \in \overline{P(\gamma, c)}$, i.e. $0 \le u_{in}(t) \le c$ on $[0, 1]$ and satisfied

$$\begin{cases} (\Phi_p u_{in}^{'})^{'} + e(t) f(t, u_{in} + \dfrac{1}{n}) = 0, \\ u_{in}(0) = 0, \quad u_{in}(1) = \alpha u_{in}(\eta). \end{cases} \tag{16}$$

Step 2. Let S, A be defined as Lemma 4.5, we will show there exists an infinite subset N^+ of S such that $\{u_{in}\}_{n \in N^+}$ is a relative compact set of $C[0, 1]$. From (4.10), (4.11), Lemma 4.2 and Lemma 4.6 we obtain

$$d > u_{1n}(t) \ge mt, \qquad c \ge u_{3n}(t) \ge \alpha\eta(1-\eta)\|u_{3n}\| t > \alpha\eta(1-\eta) dt,$$

$$c \ge u_{2n}(t) \ge \alpha\eta(1-\eta)\|u_{2n}\| t \ge \alpha\eta(1-\eta)\alpha(u_{2n}) t > \alpha\eta(1-\eta) at.$$

Thus $\{u_{in}\}_{N^+}, i = 1, 2, 3$, are uniformly bounded. Without loss of generality, we suppose $m' t \le u_{in}(t) \le c, t \in [0, 1], i = 1, 2, 3$.

Standard argument shows that $\{u_{in}\}_{n \in N^+}, i = 1, 2, 3$, are equi-continuous family on $[0, 1]$. The Arzelà-Ascoli theorem guarantees the existence of the subsequence N^+ of S and a function $u_i \in C[0, 1]$ with u_{in} converging uniformly on $[0, 1]$ to u_i as $n \to \infty$ through N^+. Also $u_i(0) = 0, u_i(1) = \alpha u_i(\eta), i = 1, 2, 3$. $u_i(t) > 0$ for $t \in (0, 1]$. In addition,

$$mt \le u_1(t) \le d, \ \alpha\eta(1-\eta) at \le u_2(t) \le c, \ \alpha\eta(1-\eta) dt \le u_3(t) \le c,$$

and

$$\frac{u_2(t_1) + u_2(t_2)}{2} \ge a, \ \max_{t \in [0,1]} u_3(t) \ge d, \ \frac{u_3(t_1) + u_3(t_2)}{2} \le a.$$

Step 3. For any $u_{in} \in \{u_{in}\}_{n\in N_1}$ we have

$$u_{in}(t) = u_{in}(\xi) + \int_\xi^t \Phi_q\left(\Phi_p u_{in}^{'}(\xi) - \int_\xi^s e(\theta)f(\theta, u_{in}(\theta) + \frac{1}{n})d\theta\right)ds \qquad (17)$$

for $t \in (0,1)\backslash\{\xi\}$ where $\xi = \frac{1}{2}$,

$$\int_\xi^t \Phi_q\left(\Phi_p u_{in}^{'}(\xi) - \int_\xi^s e(\theta)f(\theta, u_{in}(\theta) + \frac{1}{n})d\theta\right)ds = u_{in}(t) - u_{in}(\xi).$$

Fixed $t \in (0,1)\backslash\{\xi\}$. Without loss of generality, suppose $t = t_0$. Let $u_{in} \to u_i$ uniformly on $[\xi, t] \cup [t, \xi]$. Following we show $\lim_{n\to\infty} u_{in}^{'}(\xi) = u_i^{'}(\xi)$.

$$\begin{aligned} & u_{in}(t_0) - u_{in}(\xi) - u_i(t_0) + u_i(\xi) \\ = & \int_\xi^{t_0} \Phi_q\left(\Phi_p u_{in}^{'}(\xi) - \int_\xi^s e(\theta)f(\theta, u_{in}(\theta) + \frac{1}{n})d\theta\right) \\ & -\Phi_q\left(\Phi_p u_i^{'}(\xi) - \int_\xi^s e(\theta)f(\theta, u_i(\theta))d\theta\right)ds. \end{aligned}$$

By the mean value theorem for integrals then implies that there exists $\eta_{in} \in [0,1]$ with

$$\begin{aligned} & u_{in}(t_0) - u_{in}(\xi) - u_i(t_0) + u_i(\xi) \\ = & \Phi_q\left(\Phi_p u_{in}^{'}(\xi) - \int_\xi^{\eta_{in}} e(\theta)f(\theta, u_{in}(\theta) + \frac{1}{n})d\theta\right) \\ & -\Phi_q\left(\Phi_p u_i^{'}(\xi) - \int_\xi^{\eta_{in}} e(\theta)f(\theta, u_i(\theta))d\theta\right) \end{aligned}$$

and now let $u_{in} \to u_i$ uniformly on $[\xi, t_0] \cup [t_0, \xi]$, we have $\lim_{n\to\infty} u_{in}^{'}(\xi) = u_i^{'}(\xi)$. So we have

$$u_i(t) = u_i(\xi) + \int_\xi^t \Phi_q\left(\Phi_p u_i^{'}(\xi) - \int_\xi^s e(\theta)f(\theta, u_i(\theta))d\theta\right)ds, t \in (0,1)\backslash\{\xi\}.$$

By direct computation, we have for $t \in (0,1)\backslash\{\xi\}$,

$$(\Phi_p u_i^{'})^{'} + e(t)f(t, u_i(t)) = 0 \qquad (4.15)$$

If we take $\xi = 1/4$ in (4.16), in a similar way above we see that (4.15) also holds for $t = 1/2$. Obviously $u_i(0) = 0, u_i(1) = \alpha u_i(\eta)$. So $u_i(t), i = 1,2,3$ are three positive solutions of BVP (1.1).

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