Tamsui Oxford Journal of Mathematical Sciences **23(1)** (2007) 41-47
Aletheia University

A Simple Bjective Proof of Generalized Schur's Theorem *

Padmavathamma[†], R. Raghavendra[‡], and B. M. Chandrashekara [§]

Department of Studies in Mathematics, University of Mysore,
Manasagangotri, Mysore 570 006, Karnataka, India

Received April 22, 2005, Accepted July 26, 2006.

Abstract

The object of this paper is to give a simple bijective proof of the generalized version of Schur's theorem stated and proved by D.M. Bressoud in the year 1980.

Keywords and Phrases: *Schur's theorem, Generalized version, Bijective Proof.*

1. Introduction

In 1980, D. M. Bressoud [4] gave a combinatorial proof of Schur's 1926 theorem by establishing a one-to-one correspondence between the two types of partitions counted in the theorem. In fact he proved the following generalized version of Schur's theorem:

---

*2000 *Mathematics Subject Classification.* Primary 11P81, 11P82, 11P83; Secondary 05A15, 05A17, 05A19

[†]E-mail: vathamma@yahoo.com
[‡]E-mail: maths@yahoo.co.in
[§]E-mail: alur@yahoo.com

Theorem 1 (Generalized Schur's theorem). *Given positive integers r and m such that $r < m/2$, let $C_{r,m}(n)$ denote the number of partitions of n into distinct parts $\equiv \pm\, r\ (m)$ and let $D_{r,m}(n)$ denote the number of partitions of n of the form $b_1 + \cdots + b_s$ such that $b_i \equiv 0, \pm r(m)$, $b_i - b_{i+1} \geq m$, and $b_i - b_{i+1} \geq 2m$ when $b_i \equiv b_{i+1} \equiv 0(m)$. Then $C_{r,m}(n) = D_{r,m}(n)$ for all n.*

In the year 2003, Padmavathamma and M. Ruby Salestina [5] gave a different combinatorial proof of the above theorem for the case when $m = 4$ and $r = 1$. The object of this paper is to give a simple bijective proof of Theorem 1.

2. Proof

We construct a mapping from the partitions enumerated by $C_{r,m}(n)$ to those enumerated by $D_{r,m}(n)$. Let $\pi = b_1 + b_2 + \cdots + b_s$ denote a partition enumerated by $C_{r,m}(n)$. If every pair of b_i and b_{i+1} satisfies $b_i - b_{i+1} \geq m$, then π is a partition enumerated by $D_{r,m}$ also. We adopt the following procedure to map the rest of partition of $C_{r,m}(n)$ into $D_{r,m}(n)$.

Step CD_1 : List the parts of π in a column in decreasing order. Let π^1 denote this partition.

Step CD_2 : From the **top** look for the first i say α for which $b_\alpha - b_{\alpha+1} < m$. The only two possibilities are:
(i) $b_\alpha = m(k+1) - r$ and $b_{\alpha+1} = mk + r$ or
(ii) $b_\alpha = mk + r$ and $b_{\alpha+1} = mk - r$

In both cases we replace the two consecutive parts b_α and $b_{\alpha+1}$ with just one part $(b_{i_1} + b_{i_1+1})$. We note that the sum will always be $\equiv 0(m)$. In the first case $(b_\alpha + b_{\alpha+1}) = m(2k+1)$ while in the second case $(b_\alpha + b_{\alpha+1}) = m(2k)$.

Eg: Let $m = 5$ and $r = 1$

(i) $\begin{matrix} 4 \\ 1 \end{matrix} \longrightarrow 5$ (ii) $\begin{matrix} 6 \\ 4 \end{matrix} \longrightarrow 10$

Let π^2 denote the resulting partition. We now get two possibilities.

<u>Case 1:</u> $b_{\alpha-1} - (b_\alpha + b_{\alpha+1}) < m.$
<u>Case 2:</u> $b_{\alpha-1} - (b_\alpha + b_{\alpha+1}) > m$

We note that the possibility that $b_{\alpha-1} - (b_\alpha + b_{\alpha+1}) = m$ will not arise since $b_{\alpha-1} \not\equiv 0(m)$ and $(b_\alpha + b_{\alpha+1}) \equiv 0(m)$.

In case 1, we replace the pair

$$\begin{pmatrix} b_{\alpha-1} \\ b_\alpha + b_{\alpha+1} \end{pmatrix} \text{ by } \begin{pmatrix} b_\alpha + b_{\alpha+1} + m \\ b_{\alpha-1} - m \end{pmatrix}$$

Eg: Let $m = 8$ and $r = 3$

$$\text{i)} \begin{matrix} 27 \\ 11 \\ 5 \end{matrix} \longrightarrow \begin{matrix} 27 \\ 16 \end{matrix} \qquad \text{ii)} \begin{matrix} 13 \\ 5 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 13 \\ 8 \end{matrix} \longrightarrow \begin{matrix} 16 \\ 5 \end{matrix}$$

Once again we get two possibilities for case 1.

$$b_{\alpha-2} - (b_\alpha + b_{\alpha+1} + m) < m \quad \text{and} \quad b_{\alpha-2} - (b_\alpha + b_{\alpha+1} + m) > m.$$

As before, in the first case we apply the procedure explained in case 1. This procedure is continued until we meet the second case or we find $(b_\alpha + b_{\alpha+1} + tm)$ on the top.

In case 2 from the **top** we look for the next i say β for which $b_\beta - b_{\beta+1} < m$ and we follow the same procedure explained in Step CD_2 until we meet the second case for β.

The requirement of the minimal difference between multiples of m is clearly satisfied in our mapping for the following reason.

Let $\pi = (\cdots, a, b, \cdots, c, d, \cdots)$ where (a, b) and (c, d) are two consecutive pairs who need to be treated by Step CD_2. Clearly, $a + b \geq c + d + (t + 2)m$ where t counts the number of parts between b and c. The procedure still needs to lift parts (a, b) and (c, d) up if necessary. For every lifting, each part is increased by m; but there is no way to lift the part caused by (c, d) above the one caused by (a, b). Therefore, the final two parts caused by (a, b) and (c, d) must have minimal difference $2m$.

Following the procedure explained in Step CD_2 (in a finite number of steps) we arrive at a stage where difference condition is satisfied for all the parts of π. We associate this resulting partition π^4 which is enumerated by $D_{r,m}(n)$ to π.

We illustrate our procedure by an example by taking $m = 5$ and $r = 2$.

Let $\quad \pi = 47 + 42 + 38 + 37 + 28 + 27 + 23 + 18 + 13 + 12 + 3 + 2$

be a partition enumerated by $C_{2,5}(290)$.

$$C_{r,m}(n) \to D_{r,m}(n)$$

$$\begin{array}{c} 47 \\ \left.\begin{array}{c} 42 \\ 38 \end{array}\right\} \\ 37 \\ 28 \\ 27 \\ 23 \\ 18 \\ 13 \\ 12 \\ 3 \\ 2 \end{array} \to \begin{array}{c} \left.\begin{array}{c} 47 \\ 80 \end{array}\right\} \\ 37 \\ 28 \\ 27 \\ 23 \\ 18 \\ 13 \\ 12 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 42 \\ 37 \\ \left.\begin{array}{c} 28 \\ 27 \end{array}\right\} \\ 23 \\ 18 \\ 13 \\ 12 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 42 \\ \left.\begin{array}{c} 37 \\ 55 \end{array}\right\} \\ 23 \\ 18 \\ 13 \\ 12 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ \left.\begin{array}{c} 42 \\ 60 \end{array}\right\} \\ 32 \\ 23 \\ 18 \\ 13 \\ 12 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 65 \\ 37 \\ 32 \\ 23 \\ 18 \\ \left.\begin{array}{c} 13 \\ 12 \end{array}\right\} \\ 3 \\ 2 \end{array}$$

$$\to \begin{array}{c} 85 \\ 65 \\ 37 \\ 32 \\ 23 \\ \left.\begin{array}{c} 18 \\ 25 \end{array}\right\} \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 65 \\ 37 \\ 32 \\ \left.\begin{array}{c} 23 \\ 30 \end{array}\right\} \\ 13 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 65 \\ 37 \\ \left.\begin{array}{c} 32 \\ 35 \end{array}\right\} \\ 18 \\ 13 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 65 \\ \left.\begin{array}{c} 37 \\ 40 \end{array}\right\} \\ 27 \\ 18 \\ 13 \\ 3 \\ 2 \end{array} \to \begin{array}{c} 85 \\ 65 \\ 45 \\ 32 \\ 27 \\ 18 \\ 13 \\ \left.\begin{array}{c} 3 \\ 2 \end{array}\right\} \end{array} \to \begin{array}{c} 85 \\ 65 \\ 45 \\ 32 \\ 27 \\ 18 \\ 13 \\ 5 \end{array}$$

The last partition is the associated partition of π enumerated by $D_{2,5}(290)$.

We now give the reverse mapping from $D_{r,m}(n)$ to $C_{r,m}(n)$. Let ψ be a partition enumerated by $D_{r,m}(n)$. If no part is a multiple of m, then it is a partition enumerated by $C_{r,m}(n)$ also. We adopt the following procedure to map the rest of partition of $D_{r,m}(n)$ into $C_{r,m}(n)$.

Step DC_1 **:** Let the parts of ψ be arranged in a column in decreasing order. Let ψ^1 denote this partition.

Step DC_2 **:** From the **bottom** look for the first multiple of m say x. We split x into (α, β) tentatively as below:

<u>*TABLE*</u>

$$x = m * (2k) \rightarrow (mk + r, mk - r).$$
$$x = m * (2k + 1) \rightarrow (m(k + 1) - r, mk + r).$$

Suppose y lies below x. If $y < \beta$ then the tentative splitting is just what we want; otherwise, we replace

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} y + m \\ x - m \end{pmatrix}$$

Now split $x - m$ into (α, β) tentatively as before, and then apply the same procedure on $x - m$ and the part below it. This process is continued till the end. Let the resulting partition be ψ^2.

Eg1: Let $m = 8$ and $r = 3$

$$\begin{matrix} 16 \\ 5 \end{matrix} \longrightarrow \begin{matrix} 13 \\ 8 \end{matrix} \longrightarrow \begin{matrix} 13 \\ 5 \\ 3 \end{matrix}$$

Eg2: Let $m = 8$ and $r = 1$

$$\begin{matrix} 56 \\ 32 \\ 17 \\ 7 \end{matrix} \longrightarrow \begin{matrix} 56 \\ 25 \\ 24 \\ 7 \end{matrix} \longrightarrow \begin{matrix} 56 \\ 25 \\ 15 \\ 9 \\ 7 \end{matrix}$$

Step DC_2 will not create multiples of m. This is obvious if $y < \beta$. When $y \geq \beta$, the step involves only addition or subtraction of m which does not change the congurecy of x or y (mod m).

From the **bottom** look for the next multiple of m say x^1 and follow the same procedure explained in Step DC_2 to split x^1.

We apply Step DC_2 until all the multiples of ψ are split into parts $\equiv \pm r(m)$. The resulting partition will be a partition enumerated by $C_{r,m}(n)$.

We also claim: Let $\pi = (\cdots, x, \cdots, y, \cdots)$ where x and y are two consecutive multiples of m. Clearly, $x \geq y + (t + 2)m$ where t counts the number of parts between x and y. During the procedure x and y would be moved downward with m subtracted each time. However, the splitting caused by x will never go under nor between the ones caused by y. This is obvious because if the resulting parts obtained are x' and y' then x' will be $\geq y' + 2m$ always. And β part of x' will be $> \alpha$ part of y'.

We now illustrate the reverse map by taking the same partition,

$$\psi = 85 + 65 + 45 + 32 + 27 + 18 + 13 + 5 \quad \text{where } m = 5 \text{ and } r = 2$$

obtained from

$$\pi = 47 + 42 + 38 + 37 + 28 + 27 + 23 + 18 + 13 + 12 + 3 + 2$$

$$D_{r,m}(n) \rightarrow C_{r,m}(n)$$

$$\begin{array}{c}85\\65\\45\\32\\27\\18\\13\\ \left.5\right\}\end{array} \rightarrow \begin{array}{c}85\\65\\ \left.\begin{array}{c}45\\32\end{array}\right\}\\27\\18\\13\\3\\2\end{array} \rightarrow \begin{array}{c}85\\65\\37\\ \left.\begin{array}{c}40\\27\end{array}\right\}\\18\\13\\3\\2\end{array} \rightarrow \begin{array}{c}85\\65\\37\\32\\ \left.\begin{array}{c}35\\18\end{array}\right\}\\13\\3\\2\end{array} \rightarrow \begin{array}{c}85\\65\\37\\32\\23\\ \left.\begin{array}{c}30\\13\end{array}\right\}\\3\\2\end{array} \rightarrow \begin{array}{c}85\\65\\37\\32\\23\\18\\ \left.\begin{array}{c}25\\3\end{array}\right\}\\2\end{array}$$

$$\rightarrow \begin{array}{c}85\\ \left.\begin{array}{c}65\\37\end{array}\right\}\\32\\23\\18\\13\\12\\3\\2\end{array} \rightarrow \begin{array}{c}85\\42\\ \left.\begin{array}{c}60\\32\end{array}\right\}\\23\\18\\13\\12\\3\\2\end{array} \rightarrow \begin{array}{c}85\\42\\37\\ \left.\begin{array}{c}55\\23\end{array}\right\}\\18\\13\\12\\3\\2\end{array} \rightarrow \begin{array}{c}\left.\begin{array}{c}85\\42\end{array}\right\}\\37\\28\\27\\23\\18\\13\\12\\3\\2\end{array} \rightarrow \begin{array}{c}47\\ \left.\begin{array}{c}80\\37\end{array}\right\}\\28\\27\\23\\18\\13\\12\\3\\2\end{array} \rightarrow \begin{array}{c}47\\42\\38\\37\\28\\27\\23\\18\\13\\12\\3\\2\end{array}$$

The above two mappings $C_{r,m}(n) \rightarrow D_{r,m}(n)$ and $D_{r,m}(n) \rightarrow C_{r,m}(n)$ are inverse to each other follows from the reasons mentioned below.

i) $\begin{pmatrix} mk+r \\ mk-r \end{pmatrix} \leftrightarrow \ m(2k)$ and $\begin{pmatrix} m(k+1)-r \\ mk+r \end{pmatrix} \leftrightarrow \ m(2k+1)$.

ii) $\begin{pmatrix} x \\ mk+r \\ mk-r \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ m(2k) \end{pmatrix} \leftrightarrow \begin{pmatrix} m(2k+1) \\ x-m \end{pmatrix}$ where $x - m(2k) < m$,

since $x \geq mk + r + m \Leftrightarrow x - m \geq mk + r$ which is β part of $m(2k+1)$.

ii) $\begin{pmatrix} x \\ m(k+1)-r \\ mk+r \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ m(2k+1) \end{pmatrix} \leftrightarrow \begin{pmatrix} m(2k+2) \\ x-m \end{pmatrix}$ where $x - m(2k+1) < m$

since $x \geq m(k+1) - r + m \Leftrightarrow x - m \geq m(k+1) - r$ which is β part of $m(2k+2)$.

References

[1] A. J. Yee, A combinatorial proof of Andrews' partition functions related to Schur's partition theorem, (2001, Preprint).

[2] K. Alladi and B. Gordon, Generalizations of Schur's partition theorem, *Manus. Math.* **79** (1993), 113-126.

[3] C. Bessenrodtoud, A combinatorial proof of a refinemetn of the Andrews-Olsson partition identity, *European J. Combin.* **12** (1991), 271-276.

[4] D. M. Bressoud, A combinatorial proof of Schur's 1926 partition theorem, *Proc. Amer. Math. Soc.*, Vol.79, **2** (1980), 338-340.

[5] Padmavathamma and M. Ruby Salestina, A simple proof of Schur's 1926 theorem on partitions, *Applied Science Periodical*, SIWAN, Bihar, Vol V, **3** (August 2003), 135-138.