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A Survey on Inequalities for Hermitian Forms*

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Abstract

Classical and recent inequalities for Hermitian forms on real or complex linear spaces are surveyed.

1. General Properties

1.1 Schwarz's Inequality

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. *A functional* $(\cdot,\cdot) : X \times X \to \mathbb{K}$ *is said to be a Hermitian form on* X *if*

(H1) $(ax + by, z) = a(x, z) + b(y, z)$ for $a, b \in \mathbb{K}$ and $x, y, z \in X$;

(H2) $(x, y) = \overline{(y, x)}$ for all $x, y \in X$.

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The functional $(\cdot,\cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y,y)\geq 0$ for every $y\in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y,y)=0$, $y\in Y$ implies $y=0$.

The functional $(\cdot,\cdot)$ is said to be *definite* on Y provided that either $(\cdot,\cdot)$ or $-(\cdot,\cdot)$ is positive semi-definite on Y.

When a Hermitian functional $(\cdot,\cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle\cdot,\cdot\rangle$.

We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X:

$$X_0 := \{x\in X|\,(x,x)=0\},$$
$$K := \{x\in X|\,(x,x)<0\}$$

and, for a given $z\in X$,

$$X^{(z)} := \{x\in X|\,(x,z)=0\} \quad \text{and} \quad L(z) := \{az|a\in\mathbb{K}\}.$$

The following fundamental facts concerning Hermitian forms hold [**?**]:

Theorem 1 (Kurepa, 1968). *Let X and $(\cdot,\cdot)$ be as above.*

1. *If $e\in X$ is such that $(e,e)\neq 0$, then we have the decomposition*
$$X = L(e)\bigoplus X^{(e)}, \tag{1.1}$$
where $\bigoplus$ denotes the direct sum of the linear subspaces $X^{(e)}$ and $L(e)$;

2. *If the functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e\in K$, then $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$ for each $f\in K$;*

3. *The functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ with $e\in K$ if and only if the inequality*
$$|(x,y)|^2 \geq (x,x)(y,y) \tag{1.2}$$
holds for all $x\in K$ and all $y\in X$;

4. *The functional* $(\cdot,\cdot)$ *is semi-definite on* X *if and only if the Schwarz's inequality*
$$|(x,y)|^2 \leq (x,x)\,(y,y) \tag{1.3}$$
holds for all $x,y \in X$;

5. *The case of equality holds in (**??**) for* $x,y \in X$ *and in (**??**), for* $x \in K$, $y \in X$, *respectively; if and only if there exists a scalar* $a \in \mathbb{K}$ *such that*
$$y - ax \in X_0^{(x)} := X_0 \cap X^{(x)}.$$

Proof. We follow the argument in [**?**].

If $(e,e) \neq 0$, then the element
$$x := y - \frac{(y,e)}{(e,e)}e$$
has the property that $(x,e) = 0$, i.e., $x \in X^{(e)}$. This proves that X is a sum of the subspaces $L(e)$ and $X^{(s)}$. The fact that the sum is direct is obvious.

Suppose that $(e,e) \neq 0$ and that $(\cdot,\cdot)$ is positive semi-definite on X. Then for each $y \in X$ we have $y = ae + z$ with $a \in \mathbb{K}$ and $z \in X^{(e)}$, from where we get
$$|(e,y)|^2 - (e,e)\,(y,y) = -\,(e,e)\,(z,z)\,. \tag{1.4}$$

From (**??**) we get the inequality (**??**), with $x = e$, in the case that $(e,e) > 0$ and (**??**) in the case that $(e,e) < 0$. In addition to this, from (**??**) we observe that the case of equality holds in (**??**) or in (**??**) if and only if $(z,z) = 0$, i.e., if and only if $y - ae \in X_0^{(e)}$.

Conversely, if (**??**) holds for all $x,y \in X$, then (x,x) has the same sign over the whole of X, i.e., $(\cdot,\cdot)$ is semi-definite on X. In the same manner, from (**??**), for $y \in X^{(e)}$, we get $(e,e)\cdot(y,y) \leq 0$, which implies $(y,y) \geq 0$, i.e., $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$.

Now, suppose that $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e \in K$. Let us prove that $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$ for each $f \in K$.

For a given $f \in K$, consider the vector
$$e' := e - \frac{(e,f)}{(f,f)}f. \tag{1.5}$$

Now,

$$(e',e') = (e',e) = \frac{(e,e)(f,f) - |(e,f)|^2}{(f,f)}, \quad (e',f) = 0$$

and together with

$$|(e,y)|^2 \geq (e,e)(y,y) \quad \text{for any} \quad y \in X$$

imply $(e',e') \geq 0$.

There are two cases to be considered: $(e',e') > 0$ and $(e',e') = 0$.

If $(e',e') > 0$, then for any $x \in X^{(f)}$, the vector

$$x' := x - ae' \quad \text{with} \quad a = \frac{(x,e')}{(e',e')}$$

satisfies the conditions

$$(x',e) = 0 \quad \text{and} \quad (x',f) = 0$$

which implies

$$x' \in X^{(e)} \quad \text{and} \quad (x,x) = |a|^2 (e',e') + (x',x') \geq 0.$$

Therefore $(\cdot,\cdot)$ is a positive semi-definite functional on $X^{(f)}$.

From the parallelogram identity:

$$(x+y,x+y) + (x-y,x-y) = 2[(x,x) + (y,y)], \quad x,y \in X \tag{1.6}$$

we conclude that the set $X_0^{(e)} = X_0 \cap X^{(e)}$ is a linear subspace of X.

Since

$$(x,y) = \frac{1}{4}[(x+y,x+y) + (x-y,x-y)], \quad x,y \in X \tag{1.7}$$

in the case of real spaces, and

$$(x,y) = \frac{1}{4}[(x+y,x+y) + (x-y,x-y)] + \frac{i}{4}[(x+iy,x+iy) - (x-iy,x-iy)], \quad x,y \in X \tag{1.8}$$

in the case of complex spaces, hence $(x,y) = 0$ provided that x and y belong to $X_0^{(e)}$.

If $(e', e') = 0$, then $(e', e) = (e', e') = 0$ and then we can conclude that $e' \in X_0^{(e)}$. Also, since $(e', e') = 0$ implies $(e, f) \neq 0$, hence we have

$$f = b\,(e - e') \quad \text{with} \quad b = \frac{(f, f)}{(e, f)}.$$

Now write

$$X^{(e)} = X_0^{(e)} \bigoplus X_+^{(e)},$$

where $X_+^{(e)}$ is any direct complement of $X_0^{(e)}$ in the space $X^{(e)}$. If $y \neq 0$, then $y \in X_+^{(e)}$ implies $(y, y) > 0$. For such a vector y, the vector

$$y' := e' - \frac{(e', y)}{(y, y)} \cdot y.$$

is in $X^{(e)}$ and therefore $(y', y') \geq 0$.

On the other hand

$$(y', y') = (e', y') = -\frac{|(e', y)|^2}{(y, y)}.$$

Hence $y \in X_+^{(e)}$ implies that $(e', y) = 0$, i.e.,

$$(e, y) = \frac{(e, f)}{(f, f)}\,(f, y)\,,$$

which together with $y \in X^{(e)}$ leads to $(f, y) = 0$. Thus $y \in X_+^{(e)}$ implies $y \in X^{(f)}$.

On the other hand $x \in X_0^{(e)}$ and $f = b\,(e - e')$ imply $(f, x) = -b\,(e', x) = 0$ due to the fact that $e', x \in X_0^{(e)}$.

Hence $x \in X_0^{(e)}$ implies $(x, f) = 0$, i.e., $x \in X^{(f)}$.

From $X_0^{(e)} \subseteq X^{(f)}$ and $X_+^{(e)} \subseteq X^{(f)}$ we get $X^{(e)} \subseteq X^{(f)}$. Since $e \notin X^{(f)}$ and $X = L\,(e) \bigoplus X^{(e)}$, we deduce $X^{(e)} = X^{(f)}$ and then $(\cdot, \cdot)$ is positive semi-definite on $X^{(f)}$.

The theorem is completely proved.

In the case of complex linear spaces we may state the following result as well [**?**]:

Theorem 2 (Kurepa, 1968). *Let X be a complex linear space and $(\cdot,\cdot)$ a hermitian functional on X.*

1. *The functional $(\cdot,\cdot)$ is semi-definite on X if and only if there exists at least one vector $e \in X$ with $(e,e) \neq 0$ such that*
$$[Re(e,y)]^2 \leq (e,e)(y,y), \tag{1.9}$$
for all $y \in X$;

2. *There is no nonzero Hermitian functional $(\cdot,\cdot)$ such that the inequality*
$$[Re(e,y)]^2 \geq (e,e)(y,y), \quad (e,e) \neq 0, \tag{1.10}$$
holds for all $y \in X$ and for an $e \in X$.

Proof. We follow the proof in [**?**].

Let σ and τ be real numbers and $x \in X^{(e)}$ a given vector. For $y := (\sigma + i\tau)e + x$ we get

$$[Re(e,y)]^2 - (e,e)(y,y) = -\tau^2 (e,e)^2 - (e,e)(x,x). \tag{1.11}$$

If $(\cdot,\cdot)$ is semi-definite on X, then (**??**) implies (**??**).

Conversely, if (**??**) holds for all $y \in X$ and for at least one $e \in X$, then $(\cdot,\cdot)$ is semi-definite on $X^{(e)}$. But (**??**) and (**??**) for $\tau = 0$ lead to $-(e,e)(x,x) \leq 0$ from which it follows that (e,e) and (x,x) are of the same sign so that $(\cdot,\cdot)$ is semi-definite on X.

Suppose that $(\cdot,\cdot) \neq 0$ and that (**??**) holds. We can assume that $(e,e) < 0$. Then (**??**) implies that $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$. On the other hand, if τ is such that
$$\tau^2 > -\frac{(x,x)}{(e,e)},$$
then (**??**) leads to $[Re(e,y)]^2 < (e,e)(y,y)$, contradicting (**??**).

Hence, if a Hermitian functional $(\cdot,\cdot)$ is not semi-definite and if $-(e,e) \neq 0$, then the function $y \longmapsto [Re(e,y)]^2 - (e,e)(y,y)$ takes both positive and negative values.

The theorem is completely proved.

1.2 Schwarz's Inequality for the Complexification of a Real Space

Let X be a real linear space. The *complexification* $X_{\mathbb{C}}$ of X is defined as a complex linear space $X \times X$ of all ordered pairs $\{x, y\}$ $(x, y \in X)$ endowed with the operations:

$$\begin{aligned} \{x, y\} + \{x', y'\} &:= \{x + x', y + y'\}, \\ (\sigma + i\tau) \cdot \{x, y\} &:= \{\sigma x - \tau y, \sigma x + \tau y\}, \end{aligned}$$

where $x, y, x', y' \in X$ and $\sigma, \tau \in \mathbb{R}$ (see for instance [**?**]).

If $z = \{x, y\}$, then we can define the conjugate vector $\bar{z}$ of z by $\bar{z} := \{x, -y\}$. Similarly, with the scalar case, we denote

$$Rez = \{x, 0\} \quad \text{and} \quad Imz := \{0, y\}.$$

Formally, we can write $z = x + iy = Rez + iImz$ and $\bar{z} = x - iy = Rez - iImz$.

Now, let $(\cdot, \cdot)$ be a Hermitian functional on X. We may define on the complexification $X_{\mathbb{C}}$ of X, the *complexification* of $(\cdot, \cdot)$, denoted by $(\cdot, \cdot)_{\mathbb{C}}$ and defined by:

$$(x + iy, x' + iy')_{\mathbb{C}} := (x, x') + (y, y') + i\left[(y, x') - (x, y')\right],$$

for $x, y, x', y' \in X$.

The following result may be stated [**?**]:

Theorem 3 (Kurepa, 1968). *Let X, $X_{\mathbb{C}}$, $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\mathbb{C}}$ be as above. An inequality of type (**??**) and (**??**) holds for the functional $(\cdot, \cdot)_{\mathbb{C}}$ in the space $X_{\mathbb{C}}$ if and only if the same type of inequality holds for the functional $(\cdot, \cdot)$ in the space X.*

Proof. We follow the proof in [**?**].

Firstly, observe that $(\cdot, \cdot)$ is semi-definite if and only if $(\cdot, \cdot)_{\mathbb{C}}$ is semi-definite.

Now, suppose that $e \in X$ is such that

$$|(e, y)|^2 \geq (e, e)(y, y), \quad (e, e) < 0$$

for all $y \in X$. Then for $x, y \in X$ we have

$$\begin{aligned} |(e, x + iy)_{\mathbb{C}}|^2 &= [(e, x)]^2 + [(e, y)]^2 \\ &\geq (e, e)\left[(x, x) + (y, y)\right] \\ &= (e, e)(x + iy, x + iy)_{\mathbb{C}}. \end{aligned}$$

Hence, if for the functional $(\cdot,\cdot)$ on X an inequality of type (**??**) holds, then the same type of inequality holds in $X_{\mathbb{C}}$ for the corresponding functional $(\cdot,\cdot)_{\mathbb{C}}$.

Conversely, suppose that $e,f\in X$ are such that

$$\left|(e+if,x+iy)_{\mathbb{C}}\right|^{2}\geq(e+if,e+if)_{\mathbb{C}}\,(x+iy,x+iy)_{\mathbb{C}} \tag{1.12}$$

holds for all $x,y\in X$ and that

$$(e+if,e+if)_{\mathbb{C}}=(e,e)+(f,f)<0. \tag{1.13}$$

If $e=af$ with a real number a, then (**??**) implies that $(f,f)<0$ and (**??**) for $y=0$ leads to

$$[(f,x)]^{2}\geq(f,f)\,(x,x)\,,$$

for all $x\in X$. Hence, in this case, we have an inequality of type (**??**) for the functional $(\cdot,\cdot)$ in X.

Suppose that e and g are linearly independent and by $Y=L\,(e,f)$ let us denote the subspace of X consisting of all linear combinations of e and f. On Y we define a hermitian functional D by setting $D\,(x,y)=(x,y)$ for $x,y\in Y$. Let $D_{\mathbb{C}}$ be the complexification of D. Then (**??**) implies:

$$\left|D_{\mathbb{C}}\,(e+if,x+iy)\right|^{2}\geq D_{\mathbb{C}}\,(e+if,e+if)\,D_{\mathbb{C}}\,(x+iy,x+iy)\,,\quad x,y\in X \tag{1.14}$$

and (**??**) implies

$$D\,(e,e)+D\,(f,f)<0. \tag{1.15}$$

Further, consider in Y a base consisting of the two vectors $\{u_1,u_2\}$ on which D is diagonal, i.e., D satisfies

$$D\,(x,y)=\lambda_1x_1y_1+\lambda_2x_2y_2,$$

where

$$x=x_1u_1+x_2u_2,\quad y=y_1u_1+y_2u_2,$$

and

$$\lambda_1=D\,(u_1,u_1)\,,\quad \lambda_2=D\,(u_2,u_2)\,.$$

Since for the functional D we have the relations (**??**) and (**??**), we conclude that D is not a semi-definite functional on Y. Hence $\lambda_1\cdot\lambda_2<0$, so we can take $\lambda_1<0$ and $\lambda_2>0$.

Set

$$X^{+}:=\{x|\,(x,e)=(x,f)=0,\ x\in X\}\,.$$

Obviously, $(x,e)=(x,f)=0$ if and only if $(x_1u_1)=(x_2u_2)=0$.

Now, if $y\in X$, then the vector

$$x:=y-\frac{(y,u_1)}{(u_1,u_1)}u_1-\frac{(y,u_2)}{(u_2,u_2)}u_2 \tag{1.16}$$

belongs to X^+. From this it follows that

$$X=L(e,f)\bigoplus X^+.$$

Now, replacing in (**??**) the vector $x+iy$ with $z\in X^+$, we get from (**??**) that

$$[(e,e)+(f,f)](z,z)\le 0,$$

which, together with (**??**) leads to $(z,z)\ge 0$. Therefore the functional $(\cdot,\cdot)$ is positive semi-definite on X^+.

Now, since any $y\in X$ is of the form (**??**), hence for $y\in X^{(u_1)}$ we get

$$(y,y)=(x,x)+\frac{[(y,u_2)]^2}{\lambda_2},$$

which is a nonnegative number. Thus, $(\cdot,\cdot)$ is positive semi-definite on the space $X^{(u_1)}$. Since $(u_1,u_1)<0$ we have $[(u_1,y)]^2\ge(u_1,u_1)(y,y)$ for any $y\in X$ and the theorem is completely proved.

2. Superadditivity and Monotonicity Properties

2.1 The Convex Case of Nonnegative Hermitian Forms

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X, i.e., the mapping $(\cdot,\cdot):X\times X\to\mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions

(i) $(x,x)\ge 0$ for all x in X;

(ii) $(\alpha x+\beta y,z)=\alpha(x,z)+\beta(y,z)$ for all $x,y\in X$ and $\alpha,\beta\in\mathbb{K}$

(iii) $(y,x)=\overline{(x,y)}$ for all $x,y\in X$.

If $(\cdot,\cdot)\in\mathcal{H}(X)$, then the functional $\|\cdot\|=(\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2\|y\|^2\geq|(x,y)|^2\quad\text{or}\quad\|x\|\|y\|\geq|(x,y)|\tag{2.1}$$

for any $x,y\in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1,(\cdot,\cdot)_2\in\mathcal{H}(X)$ implies that $(\cdot,\cdot)_1+(\cdot,\cdot)_2\in\mathcal{H}(X)$;

(ee) $\alpha\geq0$ and $(\cdot,\cdot)\in\mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot)\in\mathcal{H}(X)$.

We can introduce on $\mathcal{H}(X)$ the following binary relation [**?**]:

$$(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$$

if and only if $\|x\|_2\geq\|x\|_1$ for all $x\in X$.(2.2)We observe that the following properties hold:

(b) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ for all $(\cdot,\cdot)\in\mathcal{H}(X)$;

(bb) $(\cdot,\cdot)_3\geq(\cdot,\cdot)_2$ and $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_3\geq(\cdot,\cdot)_1$;

(bbb) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$ implies that $(\cdot,\cdot)_2=(\cdot,\cdot)_1$;

i.e., the binary relation defined by (**??**) is *an order relation* on $\mathcal{H}(X)$.

While (b) and (bb) are obvious from the definition, we should remark, for (bbb), that if $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$, then obviously $\|x\|_2=\|x\|_1$ for all $x\in X$, which implies, by the following well known identity:

$$(x,y)_k:=\frac{1}{4}\left[\|x+y\|_k^2-\|x-y\|_k^2+i\left(\|x+iy\|_k^2-\|x-iy\|_k^2\right)\right]\tag{2.3}$$

with $x,y\in X$ and $k\in\{1,2\}$, that $(x,y)_2=(x,y)_1$ for all $x,y\in X$.

2.2 The Superadditivity and Monotonicity of $\sigma-$Mapping

Let us consider the following mapping [?]:

$$\sigma : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \quad \sigma((\cdot,\cdot);x,y) := \|x\|\|y\| - |(x,y)|,$$

which is closely related to Schwarz's inequality (??).

The following simple properties of σ are obvious:

(s) $\sigma(\alpha(\cdot,\cdot);x,y) = \alpha\sigma((\cdot,\cdot);x,y)$;

(ss) $\sigma((\cdot,\cdot);y,x) = \sigma((\cdot,\cdot);x,y)$;

(sss) $\sigma((\cdot,\cdot);x,y) \geq 0$ (Schwarz's inequality);

for any $\alpha \geq 0$, $(\cdot,\cdot) \in \mathcal{H}(X)$ and $x,y \in X$.

The following result concerning the functional properties of σ as a function depending on the nonnegative hermitian form $(\cdot,\cdot)$ has been obtained in [?]:

Theorem 4 (Dragomir-Mond, 1994). *The mapping σ satisfies the following statements:*

(i) For every $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$ one has the inequality

$$\sigma((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) \geq \sigma((\cdot,\cdot)_1;x,y) + \sigma((\cdot,\cdot)_2;x,y) \qquad (\geq 0) \tag{2.4}$$

for all $x,y \in X$, i.e., the mapping $\sigma(\cdot;x,y)$ is superadditive on $\mathcal{H}(X)$;

(ii) For every $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$ with $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ one has

$$\sigma((\cdot,\cdot)_2;x,y) \geq \sigma((\cdot,\cdot)_1;x,y) \qquad (\geq 0) \tag{2.5}$$

for all $x,y \in X$, i.e., the mapping $\sigma(\cdot;x,y)$ is nondecreasing on $\mathcal{H}(X)$.

Proof. We follow the proof in [?].

(i) By the Cauchy-Bunyakovsky-Schwarz inequality for real numbers, we have

$$\left(a^2+b^2\right)^{\frac{1}{2}}\left(c^2+d^2\right)^{\frac{1}{2}} \geq ac+bd; \quad a,b,c,d \geq 0.$$

Therefore,

$$\begin{aligned}
&\sigma\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)\\
&=\left(\|x\|_1^2+\|x\|_2^2\right)^{\frac{1}{2}}\left(\|y\|_1^2+\|y\|_2^2\right)^{\frac{1}{2}}-\left|(x,y)_1+(x,y)_2\right|\\
&\geq\|x\|_1\|y\|_1+\|x\|_2\|y\|_2-\left|(x,y)_1\right|-\left|(x,y)_2\right|\\
&=\sigma\left((\cdot,\cdot)_1;x,y\right)+\sigma\left((\cdot,\cdot)_2;x,y\right),
\end{aligned}$$

for all $(\cdot,\cdot)_i\in\mathcal{H}(X)$ $(i=1,2)$ and $x,y\in X$, and the statement is proved.

(ii) Suppose that $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and define $(\cdot,\cdot)_{2,1}:=(\cdot,\cdot)_2-(\cdot,\cdot)_1$. It is obvious that $(\cdot,\cdot)_{2,1}$ is a nonnegative hermitian form and thus, by the above property one has,

$$\begin{aligned}
\sigma\left((\cdot,\cdot)_2;x,y\right)&\geq\sigma\left((\cdot,\cdot)_{2,1}+(\cdot,\cdot)_1;x,y\right)\\
&\geq\sigma\left((\cdot,\cdot)_{2,1};x,y\right)+\sigma\left((\cdot,\cdot)_1;x,y\right)
\end{aligned}$$

from where we get:

$$\sigma\left((\cdot,\cdot)_2;x,y\right)-\sigma\left((\cdot,\cdot)_1;x,y\right)\geq\sigma\left((\cdot,\cdot)_{2,1};x,y\right)\geq 0$$

and the proof of the theorem is completed.

Remark 1. *If we consider the related mapping [?]*

$$\sigma_r\left((\cdot,\cdot);x,y\right):=\|x\|\,\|y\|-Re\,(x,y),$$

*then we can show, as above, that $\sigma(\cdot;x,y)$ is **superadditive and nondecreasing** on $\mathcal{H}(X)$.*

Moreover, if we introduce another mapping, namely, [?]

$$\tau:\mathcal{H}(X)\times X^2\to\mathbb{R}_+,\quad \tau\left((\cdot,\cdot);x,y\right):=\left(\|x\|+\|y\|\right)^2-\|x+y\|^2,$$

*which is connected with the **triangle inequality***

$$\|x+y\|\leq\|x\|+\|y\|\quad \text{for any}\quad x,y\in X \tag{2.6}$$

then we observe that

$$\tau\left((\cdot,\cdot);x,y\right)=2\sigma_r\left((\cdot,\cdot);x,y\right) \tag{2.7}$$

*for all $(\cdot,\cdot)\in\mathcal{H}(X)$ and $x,y\in X$, therefore $\sigma(\cdot;x,y)$ is in its turn a **superadditive** and **nondecreasing** functional on $\mathcal{H}(X)$.*

2.3 The Superadditivity and Monotonicity of $\delta-$Mapping

Now consider another mapping naturally associated to Schwarz's inequality, namely [**?**]

$$\delta : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \quad \delta((\cdot,\cdot);x,y) := \|x\|^2 \|y\|^2 - |(x,y)|^2 .$$

It is obvious that the following properties are valid:

(i) $\delta((\cdot,\cdot);x,y) \geq 0$ (Schwarz's inequality);

(ii) $\delta((\cdot,\cdot);x,y) = \delta((\cdot,\cdot);y,x)$;

(iii) $\delta(\alpha(\cdot,\cdot);x,y) = \alpha^2 \delta((\cdot,\cdot);x,y)$

for all $x, y \in X$, $\alpha \geq 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$.

The following theorem incorporates some further properties of this functional [**?**]:

Theorem 5 (Dragomir-Mond, 1994). *With the above assumptions, we have:*

(i) If $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i = 1, 2)$, *then*

$$\begin{aligned} &\delta((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) - \delta((\cdot,\cdot)_1;x,y) - \delta((\cdot,\cdot)_2;x,y) \\ &\qquad \geq \left(\det \begin{bmatrix} \|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2 \end{bmatrix}\right)^2 \qquad (\geq 0); \end{aligned} \tag{2.8}$$

i.e., the mapping $\delta(\cdot;x,y)$ *is strong superadditive on* $\mathcal{H}(X)$.

(ii) If $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i = 1, 2)$, *with* $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$, *then*

$$\begin{aligned} &\delta((\cdot,\cdot)_2;x,y) - \delta((\cdot,\cdot)_1;x,y) \\ &\qquad \geq \left(\det \begin{bmatrix} \|x\|_1 & \|y\|_1 \\ \left(\|x\|_2^2 - \|x\|_1^2\right)^{\frac{1}{2}} & \left(\|y\|_2^2 - \|y\|_1^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \qquad (\geq 0); \end{aligned} \tag{2.9}$$

i.e., the mapping $\delta(\cdot;x,y)$ *is strong nondecreasing on* $\mathcal{H}(X)$.

Proof. (i) For all $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $(i=1,2)$ and $x,y \in X$ we have

$$\begin{aligned} &\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right) \\ &= \left(\|x\|_2^2-\|x\|_1^2\right)\left(\|y\|_2^2-\|y\|_1^2\right)-\left|(x,y)_2+(x,y)_1\right|^2 \\ &\geq \|x\|_2^2\|y\|_2^2+\|x\|_1^2\|y\|_1^2+\|x\|_1^2\|y\|_2^2+\|x\|_2^2\|y\|_1^2 \\ &\qquad -\left(\left|(x,y)_2\right|+\left|(x,y)_1\right|\right)^2 \\ &= \delta\left((\cdot,\cdot)_2;x,y\right)+\delta\left((\cdot,\cdot)_1;x,y\right) \\ &\qquad +\|x\|_1^2\|y\|_2^2+\|x\|_2^2\|y\|_1^2-2\left|(x,y)_2(x,y)_1\right|. \end{aligned} \tag{2.10}$$

By Schwarz's inequality we have

$$\left|(x,y)_2(x,y)_1\right| \leq \|x\|_1\|y\|_1\|x\|_2\|y\|_2, \tag{2.11}$$

therefore, by (**??**) and (**??**), we can state that

$$\begin{aligned} &\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right)-\delta\left((\cdot,\cdot)_2;x,y\right) \\ &\geq \|x\|_1^2\|y\|_2^2+\|x\|_2^2\|y\|_1^2-2\|x\|_1\|y\|_1\|x\|_2\|y\|_2 \\ &= \left(\|x\|_1\|y\|_2-\|x\|_2\|y\|_1\right)^2 \end{aligned}$$

and the inequality (**??**) is proved.

(ii) Suppose that $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ and, as in Theorem **??**, define $(\cdot,\cdot)_{2,1} := (\cdot,\cdot)_2-(\cdot,\cdot)_1$. Then $(\cdot,\cdot)_{2,1}$ is a nonnegative hermitian form and by (i) we have

$$\begin{aligned} \delta\left((\cdot,\cdot)_{2,1};x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right) &= \delta\left((\cdot,\cdot)_{2,1}+(\cdot,\cdot)_1;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right) \\ &\geq \delta\left((\cdot,\cdot)_{2,1};x,y\right)+\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \|x\|_{2,1} & \|y\|_{2,1}\end{bmatrix}\right)^2 \\ &\geq \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \|x\|_{2,1} & \|y\|_{2,1}\end{bmatrix}\right)^2. \end{aligned}$$

Since $\|z\|_{2,1} = \left(\|z\|_2^2-\|z\|_1^2\right)^{\frac{1}{2}}$ for $z \in X$, hence the inequality (**??**) is proved.

Remark 2. *If we consider the functional* $\delta_r\left((\cdot,\cdot);x,y\right) := \|x\|^2\|y\|^2-\left[Re\,(x,y)\right]^2$, *then we can state similar properties for it. We omit the details.*

2.4 Superadditivity and Monotonicity of $\beta-$Mapping

Consider the functional $\beta : \mathcal{H}(X) \times X^2 \to \mathbb{R}$ [?] defined by

$$\beta((\cdot,\cdot);x,y) = \left(\|x\|^2\|y\|^2 - |(x,y)|^2\right)^{\frac{1}{2}}. \tag{2.12}$$

It is obvious that $\beta((\cdot,\cdot);x,y) = [\delta((\cdot,\cdot);x,y)]^{\frac{1}{2}}$ and thus it is monotonic nondecreasing on $\mathcal{H}(X)$. Before we prove that $\beta(\cdot;x,y)$ is also superadditive, which apparently does not follow from the properties of δ pointed out in the subsection above, we need the following simple lemma:

Lemma 1. *If $(\cdot,\cdot)$ is a nonnegative Hermitian form on X, $x,y \in X$ and $\|y\| \neq 0$, then*

$$\inf_{\lambda\in\mathbb{K}} \|x-\lambda y\|^2 = \frac{\|x\|^2\|y\|^2 - |(x,y)|^2}{\|y\|^2}. \tag{2.13}$$

Proof. Observe that

$$\|x-\lambda y\|^2 = \|x\|^2 - 2Re[\lambda(x,y)] + |\lambda|^2\|y\|^2$$

and, for $\|y\| \neq 0$,

$$\frac{\|x\|^2\|y\|^2 - |(x,y)|^2 + \left|\mu\|y\|^2 - (x,y)\right|^2}{\|y\|^2} = \|x\|^2 - 2Re\left[\mu\overline{(x,y)}\right] + |\mu|^2\|y\|^2,$$

and since $Re\left[\bar{\lambda}(x,y)\right] = Re\overline{\left[\bar{\lambda}(x,y)\right]} = Re\left[\lambda\overline{(x,y)}\right]$, we deduce the equality

$$\|x-\lambda y\|^2 = \frac{\|x\|^2\|y\|^2 - |(x,y)|^2 + \left|\mu\|y\|^2 - (x,y)\right|^2}{\|y\|^2}, \tag{2.14}$$

for any $x,y \in X$ with $\|y\| \neq 0$.

Taking the infimum over $\lambda \in \mathbb{K}$ in (**??**), we deduce the desired result (**??**).

For the subclass $\mathcal{JP}(X)$, of all inner products defined on X, of $\mathcal{H}(X)$ and $y \neq 0$, we may define

$$\gamma((\cdot,\cdot);x,y) = \frac{\|x\|^2\|y\|^2 - |(x,y)|^2}{\|y\|^2} = \frac{\delta((\cdot,\cdot);x,y)}{\|y\|^2}.$$

The following result may be stated (see also [?]):

Theorem 6 (Dragomir-Mond, 1996). *The functional* $\gamma(\cdot; x, y)$ *is superadditive and monotonic nondecreasing on* $\mathcal{JP}(X)$ *for any* $x, y \in X$ *with* $y \neq 0$.

Proof. Let $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{JP}(X)$. Then

$$\begin{aligned}
&\gamma\left((\cdot,\cdot)_1 + (\cdot,\cdot)_2 ; x, y\right) \\
&= \frac{\left(\|x\|_1^2 + \|x\|_2^2\right)\left(\|y\|_1^2 + \|y\|_2^2\right) - \left|(x,y)_1 + (x,y)_2\right|^2}{\|y\|_1^2 \|y\|_2^2} \\
&= \inf_{\lambda \in \mathbb{K}} \left[\|x - \lambda y\|_1^2 + \|x - \lambda y\|_2^2\right],
\end{aligned} \tag{2.15}$$

and for the last equality we have used Lemma **??**.

Also,

$$\begin{aligned}
\gamma\left((\cdot,\cdot)_i ; x, y\right) &= \frac{\|x\|_i^2 \|y\|_i^2 - \left|(x,y)_i\right|^2}{\|y\|_i^2} \\
&= \inf_{\lambda \in \mathbb{K}} \|x - \lambda y\|_i^2, \qquad i = 1, 2.
\end{aligned} \tag{2.16}$$

Utilising the infimum property that

$$\inf_{\lambda \in \mathbb{K}} \left(f(\lambda) + g(\lambda)\right) \geq \inf_{\lambda \in \mathbb{K}} f(\lambda) + \inf_{\lambda \in \mathbb{K}} g(\lambda),$$

we can write that

$$\inf_{\lambda \in \mathbb{K}} \left[\|x - \lambda y\|_1^2 + \|x - \lambda y\|_2^2\right] \geq \inf_{\lambda \in \mathbb{K}} \|x - \lambda y\|_1^2 + \inf_{\lambda \in \mathbb{K}} \|x - \lambda y\|_2^2,$$

which proves the superadditivity of $\gamma(\cdot; x, y)$.

The monotonicity follows by the superadditivity property and the theorem is completely proved.

Corollary 1. *If* $(\cdot,\cdot)_i \in \mathcal{JP}(X)$ *with* $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ *and* $x, y \in X$ *are such that* $x, y \neq 0$, *then:*

$$\begin{aligned}
\delta\left((\cdot,\cdot)_2 ; x, y\right) &\geq \max\left\{\frac{\|y\|_2^2}{\|y\|_1^2}, \frac{\|x\|_2^2}{\|x\|_1^2}\right\} \delta\left((\cdot,\cdot)_1 ; x, y\right) \\
&\quad \left(\geq \delta\left((\cdot,\cdot)_1 ; x, y\right)\right)
\end{aligned} \tag{2.17}$$

or equivalently, [?]

$$\delta\left((\cdot,\cdot)_2;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right) \geq \max\left\{\frac{\|y\|_2^2-\|y\|_1^2}{\|y\|_1^2},\frac{\|x\|_2^2-\|x\|_1^2}{\|x\|_1^2}\right\}\delta\left((\cdot,\cdot)_1;x,y\right). \quad (2.18)$$

The following strong superadditivity property of $\delta(\cdot;x,y)$ that is different from the one in Subsection **??** holds [**?**]:

Corollary 2 (Dragomir-Mond, 1996). *If* $(\cdot,\cdot)_i \in \mathcal{JP}(X)$ *and* $x,y\in X$ *with* $x,y\neq 0$, *then*

$$\begin{aligned}&\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right)-\delta\left((\cdot,\cdot)_2;x,y\right)\\ &\geq\max\left\{\left(\frac{\|y\|_2}{\|y\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|y\|_1}{\|y\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right);\right.\\ &\left.\left(\frac{\|x\|_2}{\|x\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|x\|_1}{\|x\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right)\right\}\quad(\geq 0). \quad (2.19)\end{aligned}$$

Proof. Utilising the identities (**??**) and (**??**) and taking into account that $\gamma(\cdot;x,y)$ is superadditive, we can state that

$$\begin{aligned}&\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right) \quad (2.20)\\ &\geq\frac{\|y\|_1^2+\|y\|_2^2}{\|y\|_1^2}\delta\left((\cdot,\cdot)_1;x,y\right)+\frac{\|y\|_1^2+\|y\|_2^2}{\|y\|_2^2}\delta\left((\cdot,\cdot)_2;x,y\right)\\ &=\delta\left((\cdot,\cdot)_1;x,y\right)+\delta\left((\cdot,\cdot)_2;x,y\right)\\ &\quad+\left(\frac{\|y\|_2}{\|y\|_1}\right)^2\delta\left((\cdot,\cdot)_1;x,y\right)+\left(\frac{\|y\|_1}{\|y\|_2}\right)^2\delta\left((\cdot,\cdot)_2;x,y\right)\end{aligned}$$

and a similar inequality with x instead of y. These show that the desired inequality (**??**) holds true.

Remark 3. *Obviously, all the inequalities above remain true if* $(\cdot,\cdot)_i$, $i=1,2$ *are nonnegative Hermitian forms for which we have* $\|x\|_i$, $\|y\|_i\neq 0$.

Finally, we may state and prove the superadditivity result for the mapping β (see [**?**]):

Theorem 7 (Dragomir-Mond, 1996). *The mapping β defined in (**??**) is superadditive on $\mathcal{H}(X)$.*

Proof. Without loss of generality, if $(\cdot,\cdot)_i \in \mathcal{H}(X)$ and $x, y \in X$, we may assume, for instance, that $\|y\|_i \neq 0$, $i = 1, 2$.

If so, then

$$\left(\frac{\|y\|_2}{\|y\|_1}\right)^2 \delta\left((\cdot,\cdot)_1; x, y\right) + \left(\frac{\|y\|_1}{\|y\|_2}\right)^2 \delta\left((\cdot,\cdot)_2; x, y\right) \geq 2\left[\delta\left((\cdot,\cdot)_1; x, y\right)\delta\left((\cdot,\cdot)_2; x, y\right)\right]^{\frac{1}{2}},$$

and by making use of (**??**) we get:

$$\delta\left((\cdot,\cdot)_1 + (\cdot,\cdot)_2; x, y\right) \geq \left\{\left[\delta\left((\cdot,\cdot)_1; x, y\right)\right]^{\frac{1}{2}} + \left[\delta\left((\cdot,\cdot)_2; x, y\right)\right]^{\frac{1}{2}}\right\}^2,$$

which is exactly the superadditivity property for β.

3. Applications for Inner Product Spaces

3.1 Inequalities for Orthonormal Families

Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$). The family of vectors $E := \{e_i\}_{i\in I}$ (I is a finite or infinite) is an *orthonormal family* of vectors if $\langle e_i, e_j\rangle = \delta_{ij}$ for $i, j \in I$, where δ_{ij} is Kronecker's delta.

The following inequality is well known in the literature as Bessel's inequality:

$$\sum_{i\in F} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \tag{3.1}$$

for any F a finite part of I and x a vector in H.

If by $\mathcal{F}(I)$ we denote the family of all finite parts of I (including the empty set $\varnothing$), then for any $F \in \mathcal{F}(I) \setminus \{\varnothing\}$ the functional $(\cdot,\cdot)_F : H \times H \to \mathbb{K}$ given by

$$(x, y)_F := \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \tag{3.2}$$

is a Hermitian form on H.

It is obvious that if $F_1, F_2 \in \mathcal{F}(I) \setminus \{\varnothing\}$ and $F_1 \cap F_2 = \varnothing$, then $(\cdot,\cdot)_{F_1 \cup F_2} = (\cdot,\cdot)_{F_1} + (\cdot,\cdot)_{F_2}$.

We can define the functional $\sigma : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ by

$$\sigma(F; x, y) := \|x\|_F \|y\|_F - |(x,y)_F|, \tag{3.3}$$

where

$$\|x\|_F := \left(\sum_{i \in F} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} = [(x,x)_F]^{\frac{1}{2}}, \qquad x \in H.$$

The following proposition may be stated (see also [?]):

Proposition 1 (Dragomir-Mond, 1995). *The mapping σ satisfies the following*

(i) If $F_1, F_2 \in \mathcal{F}(I) \setminus \{\varnothing\}$ with $F_1 \cap F_2 = \varnothing$, then

$$\sigma(F_1 \cup F_2; x, y) \geq \sigma(F_1; x, y) + \sigma(F_2; x, y) \qquad (\geq 0)$$

for any $x, y \in H$, i.e., the mapping $\sigma(\cdot; x, y)$ is an index set superadditive mapping on $\mathcal{F}(I)$;

(ii) If $\varnothing \neq F_1 \subseteq F_2$, $F_1, F_2 \in \mathcal{F}(I)$, then

$$\sigma(F_2; x, y) \geq \sigma(F_1; x, y) \qquad (\geq 0),$$

i.e., the mapping $\sigma(\cdot; x, y)$ is an index set monotonic mapping on $\mathcal{F}(I)$.

The proof is obvious by Theorem **??** and we omit the details.

We can also define the mapping $\sigma_r(\cdot;\cdot,\cdot) : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ by

$$\sigma_r(F; x, y) := \|x\|_F \|y\|_F - \mathrm{Re}\,(x,y)_F,$$

which also has the properties (i) and (ii) of Proposition **??**.

Since, by Bessel's inequality the hermitian form $(\cdot,\cdot)_F \leq \langle \cdot,\cdot \rangle$ in the sense of Definition (**??**) then by Theorem **??** we may state the following *refinements* of Schwarz's inequality [?]:

Proposition 2 (Dragomir-Mond, 1994). *For any $F \in \mathcal{F}(I) \setminus \{0\}$, we have the inequalities*

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \left(\sum_{i\in F} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \tag{3.4}$$

and

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \left(\|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left(\|y\|^2 - \sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} - \left|\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \tag{3.5}$$

and the corresponding versions on replacing $|\cdot|$ by $\mathrm{Re}(\cdot)$, where x, y are vectors in H.

Remark 4. *Note that the inequality (**??**) and its version for $\mathrm{Re}(\cdot)$ has been established for the first time and utilising a different argument by Dragomir and Sándor in 1994 (see [**?**, Theorem 5 and Remark 2]).*

If we now define the mapping $\delta : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ by

$$\delta(F; x, y) := \|x\|_F^2 \|y\|_F^2 - |(x, y)_F|^2$$

and making use of Theorem **??**, we may state the following result [**?**].

Proposition 3 (Dragomir-Mond, 1995). *The mapping δ satisfies the following properties:*

(i) If $F_1, F_2 \in \mathcal{F}(I)$ with $F_1 \cap F_2 = \varnothing$, then

$$\delta(F_1 \cup F_2; x, y) - \delta(F_1; x, y) - \delta(F_2; x, y) \geq \left(\det \begin{bmatrix} \|x\|_{F_1} & \|y\|_{F_1} \\ \|x\|_{F_2} & \|y\|_{F_2} \end{bmatrix}\right)^2 \quad (\geq 0), \tag{3.6}$$

i.e., the mapping $\delta(\cdot; x, y)$ is strong superadditive as an index set mapping;

(ii) If $\varnothing \neq F_1 \subseteq F_2,\ F_1, F_2 \in \mathcal{F}(I)$, *then*

$$\delta(F_2; x, y) - \delta(F_1; x, y) \geq \left(\det \begin{bmatrix} \|x\|_{F_1} & \|y\|_{F_1} \\ \left(\|x\|_{F_2}^2 - \|x\|_{F_1}^2\right)^{\frac{1}{2}} & \left(\|y\|_{F_2}^2 - \|y\|_{F_1}^2\right)^{\frac{1}{2}} \end{bmatrix} \right)^2 \quad (\geq 0), \quad (3.7)$$

i.e., the mapping $\delta(\cdot; x, y)$ *is strong nondecreasing as an index set mapping.*

On applying the same general result in Theorem **??**, (ii) for the hermitian functionals $(\cdot,\cdot)_F$ $(F \in \mathcal{F}(I) \setminus \{\varnothing\})$ and $\langle\cdot,\cdot\rangle$ for which, by Bessel's inequality we know that $(\cdot,\cdot)_F \leq \langle\cdot,\cdot\rangle$, we may state the following result as well, which provides refinements for the Schwarz inequality.

Proposition 4 (Dragomir-Mond, 1994). *For any* $F \in \mathcal{F}(I) \setminus \{\varnothing\}$, *we have the inequalities:*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \sum_{i \in F} |\langle x, e_i\rangle|^2 \sum_{i \in F} |\langle y, e_i\rangle|^2 - \left| \sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle \right|^2 \quad (\geq 0) \quad (3.8)$$

and

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \left(\|x\|^2 - \sum_{i \in F} |\langle x, e_i\rangle|^2 \right) \left(\|y\|^2 - \sum_{i \in F} |\langle y, e_i\rangle|^2 \right) - \left| \langle x, y\rangle - \sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle \right|^2 \quad (\geq 0), \quad (3.9)$$

for any $x, y \in H$.

On utilising Corollary **??** we may state the following different superadditivity property for the mapping $\delta(\cdot; x, y)$.

Proposition 5. *If $F_1, F_2 \in \mathcal{F}(I) \setminus \{\varnothing\}$ with $F_1 \cap F_2 = \varnothing$, then*

$$\begin{aligned}\delta(F_1 \cup F_2; x, y) - \delta(F_1; x, y) - \delta(F_2; x, y) \\ \geq \max\left\{ \left(\frac{\|y\|_{F_2}}{\|y\|_{F_1}}\right)^2 \delta(F_1; x, y) + \left(\frac{\|y\|_{F_1}}{\|y\|_{F_2}}\right)^2 \delta(F_2; x, y); \right. \\ \left. \left(\frac{\|x\|_{F_2}}{\|x\|_{F_1}}\right)^2 \delta(F_1; x, y) + \left(\frac{\|x\|_{F_1}}{\|x\|_{F_2}}\right)^2 \delta(F_2; x, y) \right\} \quad (\geq 0) \quad (3.10)\end{aligned}$$

for any $x, y \in H \setminus \{0\}$.

Further, for $y \notin M^{\perp}$ where $M = Sp\{e_i\}_{i \in I}$ is the linear space generated by $E = \{e_i\}_{i \in I}$, we can also consider the functional $\gamma : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ defined by

$$\gamma(F; x, y) := \frac{\delta(F; x, y)}{\|y\|_F^2} = \frac{\|x\|_F^2 \|y\|_F^2 - |(x, y)_F|^2}{\|y\|_F^2},$$

where $x \in H$ and $F \neq \varnothing$.

Utilising Theorem **??**, we may state the following result concerning the properties of the functional $\gamma(\cdot; x, y)$ with x and y as above.

Proposition 6. *For any $x \in H$ and $y \in H \backslash M^{\perp}$, the functional $\gamma(\cdot; x, y)$ is superadditive and monotonic nondecreasing as an index set mapping on $\mathcal{F}(I)$.*

Since $\langle \cdot, \cdot \rangle \geq (\cdot, \cdot)_F$ for any $F \in \mathcal{F}(I)$, on making use of Corollary **??**, we may state the following refinement of Schwarz's inequality:

Proposition 7. *Let $x \in H$ and $y \in H \backslash M_F^{\perp}$, where $M_F := Sp\{e_i\}_{i \in I}$ and $F \in \mathcal{F}(I) \setminus \{\varnothing\}$ is given. Then*

$$\begin{aligned}\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \max\left\{ \frac{\|y\|^2}{\sum_{i \in F} |\langle y, e_i\rangle|^2}, \frac{\|x\|^2}{\sum_{i \in F} |\langle x, e_i\rangle|^2} \right\} \\ \times \left(\sum_{i \in F} |\langle x, e_i\rangle|^2 \sum_{i \in F} |\langle y, e_i\rangle|^2 - \left| \sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle \right|^2 \right) \\ \left(\geq \sum_{i \in F} |\langle x, e_i\rangle|^2 \sum_{i \in F} |\langle y, e_i\rangle|^2 - \left| \sum_{i \in F} \langle x, e_i\rangle \langle e_i, y\rangle \right|^2 \right), \quad (3.11)\end{aligned}$$

*which is a refinement of (**??**) in the case that $y \in H \backslash M_F^{\perp}$.*

Finally, consider the functional $\beta : \mathcal{F}(I) \times H^2 \to \mathbb{R}_+$ given by

$$\beta(F; x, y) := [\delta(F; x, y)]^{\frac{1}{2}} = \left(\|x\|_F^2 \|y\|_F^2 - |(x, y)_F|^2\right)^{\frac{1}{2}}.$$

Utilising Theorem **??**, we may state the following.

Proposition 8. *The functional $\beta(\cdot; x, y)$ is superadditive as an index set mapping on $\mathcal{F}(I)$ for each $x, y \in H$.*

As a dual approach, one may also consider the following form $(\cdot, \cdot)_{C,F}$: $H \times H \to \mathbb{R}$ given by:

$$(x, y)_{C,F} := \langle x, y\rangle - (x, y)_F = \langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle. \tag{3.12}$$

By Bessel's inequality, we observe that $(\cdot, \cdot)_{C,F}$ is a nonnegative hermitian form and, obviously

$$(\cdot, \cdot)_I + (\cdot, \cdot)_{C,F} = \langle \cdot, \cdot\rangle.$$

Utilising the superadditivity properties from Section **??**, one may state the following refinement of the Schwarz inequality:

$$\begin{aligned} &\|x\| \|y\| - |\langle x, y\rangle| \\ &\qquad \geq \left(\sum_{i\in F} |\langle x, e_i\rangle|^2 \sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \\ &\qquad + \left(\|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left(\|y\|^2 - \sum_{i\in F} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} \\ &\qquad\qquad - \left|\langle x, y\rangle - \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle\right| \qquad (\geq 0), \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \\
&\qquad\geq \sum_{i\in F}|\langle x,e_i\rangle|^2\sum_{i\in F}|\langle y,e_i\rangle|^2-\left|\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \\
&\qquad+\left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right) \\
&\qquad\qquad-\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \qquad (\geq 0) \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right)^{\frac{1}{2}} \\
&\qquad\geq \left[\sum_{i\in F}|\langle x,e_i\rangle|^2\sum_{i\in F}|\langle y,e_i\rangle|^2-\left|\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2\right]^{\frac{1}{2}} \\
&\qquad+\left[\left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right)\right. \\
&\qquad\qquad\left.-\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2\right]^{\frac{1}{2}} \qquad (\geq 0), \qquad (3.15)
\end{aligned}
$$

for any $x,y\in H$ and $F\in\mathcal{F}(I)\setminus\{\varnothing\}$.

3.2 Inequalities for Gram Determinants

Let $\{x_1, \ldots, x_n\}$ be vectors in the inner product space $(H, \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Consider the *gram matrix* associated to the above vectors:

$$G(x_1, \ldots, x_n) := \begin{bmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_n\rangle \\ \langle x_2, x_1\rangle & & \cdots & \langle x_2, x_n\rangle \\ \cdots & & \cdots & \cdots \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & \cdots & \langle x_n, x_n\rangle \end{bmatrix}.$$

The determinant

$$\Gamma(x_1, \ldots, x_n) := \det G(x_1, \ldots, x_n)$$

is called the Gram determinant associated to the system $\{x_1, \ldots, x_n\}$.

If $\{x_1, \ldots, x_n\}$ does not contain the null vector 0, then [?]

$$0 \leq \Gamma(x_1, \ldots, x_n) \leq \|x_1\|^2 \|x_2\|^2 \cdots \|x_n\|^2. \tag{3.16}$$

The equality holds on the left (respectively right) side of (??) if and only if $\{x_1, \ldots, x_n\}$ is linearly dependent (respectively orthogonal). The first inequality in (??) is known in the literature as *Gram's inequality* while the second one is known as *Hadamard's inequality.*

The following result obtained in [?] may be regarded as a refinement of Gram's inequality:

Theorem 8 (Dragomir-Sándor, 1994). *Let $\{x_1, \ldots, x_n\}$ be a system of nonzero vectors in H. then for any $x, y \in H$ one has:*

$$\Gamma(x, x_1, \ldots, x_n)\, \Gamma(y, x_1, \ldots, x_n) \geq |\Gamma(x_1, \ldots, x_n)(x, y)|^2, \tag{3.17}$$

where $\Gamma(x_1, \ldots, x_n)(x, y)$ is defined by:

$$\Gamma(x_1, \ldots, x_n)(x, y) := \det \begin{bmatrix} \langle x, y\rangle & \langle x, x_1\rangle & \cdots & \langle x, x_n\rangle \\ \langle x_1, y\rangle & & & \\ \cdots & & G(x_1, \ldots, x_n) & \\ \langle x_n, y\rangle & & & \end{bmatrix}.$$

Proof. We will follow the proof from [?].

Let us consider the mapping $p : H \times H \to \mathbb{K}$ given by

$$p(x, y) = \Gamma(x_1, \ldots, x_n)(x, y).$$

Utilising the properties of determinants, we notice that

$$\begin{aligned} p(x,y) &= \Gamma(x,x_1,\dots,x_n) \geq 0, \\ p(x+y,z) &= \Gamma(x_1,\dots,x_n)(x+y,z) \\ &= \Gamma(x_1,\dots,x_n)(x,z) + \Gamma(x_1,\dots,x_n)(y,z) \\ &= p(x,z) + p(y,z), \\ p(\alpha x,y) &= \alpha p(x,y), \\ p(y,x) &= \overline{p(x,y)}, \end{aligned}$$

for any $x,y,z \in H$ and $\alpha \in \mathbb{K}$, showing that $p(\cdot,\cdot)$ is a nonnegative hermitian from on X. Writing Schwarz's inequality for $p(\cdot,\cdot)$ we deduce the desired result (**??**).

In a similar manner, if we define $q : H \times H \to \mathbb{K}$ by

$$\begin{aligned} q(x,y) &:= (x,y) \prod_{i=1}^{n} \|x_i\|^2 - p(x,y) \\ &= (x,y) \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1,\dots,x_n)(x,y), \end{aligned}$$

then, using Hadamard's inequality, we conclude that $q(\cdot,\cdot)$ is also a nonnegative hermitian form. Therefore, by Schwarz's inequality applied for $q(\cdot,\cdot)$, we can state the following result as well [**?**].

Theorem 9 (Dragomir-Sándor, 1994). *With the assumptions of Theorem* **??***, we have:*

$$\left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x,x_1,\dots,x_n)\right]\left[\|y\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(y,x_1,\dots,x_n)\right] \geq \left|\langle x,y\rangle \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1,\dots,x_n)(x,y)\right|^2, \quad (3.18)$$

for each $x,y \in H$.

Observing that, for a given set of nonzero vectors $\{x_1,\dots,x_n\}$,

$$p(x,y) + q(x,y) = (x,y) \prod_{i=1}^{n} \|x_i\|^2,$$

for any $x, y \in H$, then, on making use of the superadditivity properties of the various functionals defined in Section **??**, we can state the following refinements of the Schwarz inequality in inner product spaces:

$$\begin{aligned}
&[\|x\|\,\|y\| - |\langle x, y\rangle|] \prod_{i=1}^{n} \|x_i\|^2 \\
&\quad \geq [\Gamma(x, x_1, \dots, x_n)\, \Gamma(y, x_1, \dots, x_n)]^{\frac{1}{2}} - |\Gamma(x_1, \dots, x_n)(x, y)| \\
&\qquad + \left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x, x_1, \dots, x_n)\right]^{\frac{1}{2}} \\
&\qquad \times \left[\|y\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(y, x_1, \dots, x_n)\right]^{\frac{1}{2}} \\
&\qquad - \left|\langle x, y\rangle \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1, \dots, x_n)(x, y)\right| \qquad (\geq 0), \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
&\left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \prod_{i=1}^{n} \|x_i\|^4 \\
&\quad \Gamma(x, x_1, \dots, x_n)\, \Gamma(y, x_1, \dots, x_n) - |\Gamma(x_1, \dots, x_n)(x, y)|^2 \\
&\qquad + \left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x, x_1, \dots, x_n)\right] \\
&\qquad \times \left[\|y\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(y, x_1, \dots, x_n)\right] \\
&\qquad - \left|\langle x, y\rangle \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1, \dots, x_n)(x, y)\right|^2 \qquad (\geq 0), \qquad (3.20)
\end{aligned}$$

and

$$
\begin{aligned}
& [\|x\| \|y\| - |\langle x, y\rangle|]^{\frac{1}{2}} \prod_{i=1}^{n} \|x_i\|^2 \\
& \geq \left[\Gamma(x, x_1, \ldots, x_n) \Gamma(y, x_1, \ldots, x_n) - |\Gamma(x_1, \ldots, x_n)(x, y)|^2\right]^{\frac{1}{2}} \\
& \quad + \left\{\left[\|x\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x, x_1, \ldots, x_n)\right]\right. \\
& \quad \times \left[\|y\|^2 \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(y, x_1, \ldots, x_n)\right] \\
& \quad \left. - \left|\langle x, y\rangle \prod_{i=1}^{n} \|x_i\|^2 - \Gamma(x_1, \ldots, x_n)(x, y)\right|^2\right\}^{\frac{1}{2}} \quad (\geq 0). \quad (3.21)
\end{aligned}
$$

3.3 Inequalities for Linear Operators

Let $A : H \to H$ be a linear bounded operator and

$$\|A\| := \sup\{\|Ax\|, \|x\| < 1\}$$

its norm.

If we consider the hermitian forms $(\cdot, \cdot)_2, (\cdot, \cdot)_1 : H \to H$ defined by

$$(x, y)_1 := \langle Ax, Ay\rangle, \qquad (x, y)_2 := \|A\|^2 \langle x, y\rangle$$

then obviously $(\cdot, \cdot)_2 \geq (\cdot, \cdot)_1$ in the sense of Definition (**??**) and utilising the monotonicity properties of the functional considered in Section **??**, we may state the following inequalities:

$$\|A\|^2 [\|x\| \|y\| - |\langle x, y\rangle|] \geq \|Ax\| \|Ay\| - |\langle Ax, Ay\rangle| \qquad (\geq 0), \qquad (3.22)$$

$$\|A\|^4 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \geq \|Ax\|^2 \|Ay\|^2 - |\langle Ax, Ay\rangle|^2 \qquad (\geq 0) \quad (3.23)$$

for any $x, y \in H$, and the corresponding versions on replacing $|\cdot|$ by $Re(\cdot)$.

The results (**??**) and (**??**) have been obtained by Dragomir and Mond in [**?**].

On using Corollary **??**, we may deduce the following inequality as well:

$$\|A\|^2 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \geq \max\left\{\frac{\|x\|^2}{\|Ax\|^2}, \frac{\|y\|^2}{\|Ay\|^2}\right\} \left[\|Ax\|^2 \|Ay\|^2 - |\langle Ax, Ay\rangle|^2\right] \qquad (\geq 0) \quad (3.24)$$

for any $x, y \in H$ with $Ax, Ay \neq 0$; which improves (**??**) for x, y specified before.

Similarly, if $B : H \to H$ is a linear operator satisfying the condition

$$\|Bx\| \geq m \|x\| \quad \text{for any} \quad x \in H, \quad (3.25)$$

where $m > 0$ is given, then the hermitian forms $[x, y]_2 := \langle Bx, By\rangle$, $[x, y]_1 := m^2 \langle x, y\rangle$, have the property that $[\cdot, \cdot]_2 \geq [\cdot, \cdot]_1$. Therefore, from the monotonicity results established in Section **??**, we can state that

$$\|Bx\| \|By\| - |\langle Bx, By\rangle| \geq m^2 [\|x\| \|y\| - |\langle x, y\rangle|] \qquad (\geq 0), \quad (3.26)$$

$$\|Bx\|^2 \|By\|^2 - |\langle Bx, By\rangle|^2 \geq m^4 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \qquad (\geq 0) \quad (3.27)$$

for any $x, y \in H$, and the corresponding results on replacing $|\cdot|$ by $Re(\cdot)$.

The same Corollary **??**, would give the inequality

$$\|Bx\|^2 \|By\|^2 - |\langle Bx, By\rangle|^2 \geq m^2 \max\left\{\frac{\|Bx\|^2}{\|x\|^2}, \frac{\|By\|^2}{\|y\|^2}\right\} \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \quad (3.28)$$

for $x, y \neq 0$, which is an improvement of (**??**).

We recall that a linear self-adjoint operator $P : H \to H$ is *nonnegative* if $\langle Px, x\rangle \geq 0$ for any $x \in H$. P is called *positive* if $\langle Px, x\rangle = 0$ and *positive definite with the constant* $\gamma > 0$ if $\langle Px, x\rangle \geq \gamma \|x\|^2$ for any $x \in H$.

If $A, B : H \to H$ are two linear self-adjoint operators such that $A \geq B$ (this means that $A - B$ is nonnegative), then the corresponding hermitian forms $(x, y)_A := \langle Ax, y\rangle$ and $(x, y)_B := \langle Bx, y\rangle$ satisfies the property that $(\cdot, \cdot)_A \geq (\cdot, \cdot)_B$.

If by $\mathcal{P}(H)$ we denote the *cone* of all linear self-adjoint and nonnegative operators defined in the Hilbert space H, then, on utilising the results of

Section **??**, we may state that the functionals $\sigma_0, \delta_0, \beta_0 : \mathcal{P}(H) \times H^2 \to [0, \infty]$ given by

$$\begin{aligned}\sigma_0(P; x, y) &:= \langle Ax, x\rangle^{\frac{1}{2}} \langle Py, y\rangle^{\frac{1}{2}} - |\langle Px, y\rangle|, \\ \delta_0(P; x, y) &:= \langle Px, x\rangle \langle Py, y\rangle - |\langle Px, y\rangle|^2, \\ \beta_0(P; x, y) &:= \left[\langle Px, x\rangle \langle Py, y\rangle - |\langle Px, y\rangle|^2\right]^{\frac{1}{2}},\end{aligned}$$

are *superadditive* and *monotonic decreasing* on $\mathcal{P}(H)$, i.e.,

$$\gamma_0(P + Q; x, y) \geq \gamma_0(P; x, y) + \gamma_0(Q; x, y) \qquad (\geq 0)$$

for any $P, Q \in \mathcal{P}(H)$ and $x, y \in H$, and

$$\gamma_0(P; x, y) \geq \gamma_0(Q; x, y) \qquad (\geq 0)$$

for any P, Q with $P \geq Q \geq 0$ and $x, y \in H$, where $\gamma \in \{\sigma, \delta, \beta\}$.

The superadditivity and monotonicity properties of σ_0 and δ_0 have been noted by Dragomir and Mond in [**?**].

If $u \in \mathcal{P}(H)$ is such that $I \geq U \geq 0$, where I is the identity operator, then on using the superadditivity property of the functionals σ_0, δ_0 and β_0 one may state the following refinements for the Schwarz inequality:

$$\begin{aligned}\|x\| \|y\| - |\langle x, y\rangle| &\geq \langle Ux, x\rangle^{\frac{1}{2}} \langle Uy, y\rangle^{\frac{1}{2}} - |\langle Ux, y\rangle| \\ &\quad + \langle (I - U)x, x\rangle^{\frac{1}{2}} \langle (I - U)y, y\rangle^{\frac{1}{2}} - |\langle (I - U)x, y\rangle| \qquad (\geq 0),\end{aligned} \tag{3.29}$$

$$\begin{aligned}\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &\geq \langle Ux, x\rangle \langle Uy, y\rangle - |\langle Ux, y\rangle|^2 \\ &\quad + \langle (I - U)x, x\rangle \langle (I - U)y, y\rangle - |\langle (I - U)x, y\rangle|^2 \qquad (\geq 0),\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\left(\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right)^{\frac{1}{2}} &\geq \left(\langle Ux, x\rangle \langle Uy, y\rangle - |\langle Ux, y\rangle|^2\right)^{\frac{1}{2}} \\ &\quad + \left(\langle (I - U)x, x\rangle \langle (I - U)y, y\rangle - |\langle (I - U)x, y\rangle|^2\right)^{\frac{1}{2}} \qquad (\geq 0)\end{aligned} \tag{3.31}$$

for any $x, y \in H$.

Note that (**??**) is a better result than (**??**).

Finally, if we assume that $D \in \mathcal{P}(H)$ with $D \geq \gamma I$, where $\gamma > 0$, i.e., D is positive definite on H, then we may state the following inequalities

$$\langle Dx, x\rangle^{\frac{1}{2}} \langle Dy, y\rangle^{\frac{1}{2}} - |\langle Dx, y\rangle| \geq \gamma \left[\|x\| \|y\| - |\langle x, y\rangle|\right] \qquad (\geq 0), \qquad (3.32)$$

$$\langle Dx, x\rangle \langle Dy, y\rangle - |\langle Dx, y\rangle|^2 \geq \gamma^2 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \qquad (\geq 0), \qquad (3.33)$$

for any $x, y \in H$ and

$$\langle Dx, x\rangle \langle Dy, y\rangle - |\langle Dx, y\rangle|^2 \geq \gamma \max\left\{\frac{\langle Dx, x\rangle}{\|x\|^2}, \frac{\langle Dy, y\rangle}{\|y\|^2}\right\} \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \qquad (\geq 0) \quad (3.34)$$

for any $x, y \in H \setminus \{0\}$.

The results (**??**) and (**??**) have been obtained by Dragomir and Mond in [**?**].

Note that (**??**) is a better result than (**??**).

The above results (**??**) – (**??**) also hold for $Re(\cdot)$ instead of $|\cdot|$.

4. Applications for Sequences of Vectors in Inner Product Spaces

4.1 The Case of Mapping σ

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the natural number set $\mathbb{N}$, $\mathcal{S}_+(\mathbb{R})$ the cone of nonnegative real sequences and for a given inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$, $\mathcal{S}(H)$ the linear space of all sequences of vectors from H, i.e.,

$$\mathcal{S}(H) := \left\{\mathbf{x} | \mathbf{x} = (x_i)_{i\in\mathbb{N}},\ x_i \in H,\ i \in \mathbb{N}\right\}.$$

We may define the mapping σ by

$$\sigma(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) := \left(\sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i \langle x_i, y_i\rangle\right|, \qquad (4.1)$$

where $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

We observe that, for a fixed $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $I \in \mathcal{P}_f(\mathbb{N})$, the functional $\langle\cdot,\cdot\rangle_{\mathbf{p},I} \geq \langle\cdot,\cdot\rangle_{\mathbf{q},I}$.

Using Theorem **??**, we may state the following result.

Proposition 9. *Let $I \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$. Then the functional $\sigma(\cdot, I, \mathbf{x}, \mathbf{y})$ is superadditive and monotonic nondecreasing on $\mathcal{S}_+(\mathbb{R})$.*

If $I, J \in \mathcal{P}_f(\mathbb{N})$, with $I \cap J = \varnothing$ and if we consider, for a given $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, we observe that

$$\langle \cdot, \cdot \rangle_{\mathbf{p}, I \cup J} = \langle \cdot, \cdot \rangle_{\mathbf{p}, I} + \langle \cdot, \cdot \rangle_{\mathbf{p}, J}. \tag{4.2}$$

Taking into account this property and on making use of Theorem **??**, we may state the following result.

Proposition 10. *Let $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.*

(i) For any $I, J \in \mathcal{P}_f(\mathbb{N})$, with $I \cap J = \varnothing$, we have

$$\sigma(\mathbf{p}, I \cup J, \mathbf{x}, \mathbf{y}) \geq \sigma(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) + \sigma(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \qquad (\geq 0), \tag{4.3}$$

i.e., $\sigma(\mathbf{p}, \cdot, \mathbf{x}, \mathbf{y})$ is superadditive as an index set mapping on $\mathcal{P}_f(\mathbb{N})$.

(ii) If $\varnothing \neq J \subseteq I$, $I, J \in \mathcal{P}_f(\mathbb{N})$, then

$$\sigma(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \geq \sigma(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \qquad (\geq 0), \tag{4.4}$$

i.e., $\sigma(\mathbf{p}, \cdot, \mathbf{x}, \mathbf{y})$ is monotonic nondecreasing as an index set mapping on $\mathcal{S}_+(\mathbb{R})$.

It is well known that the following Cauchy-Bunyakovsky-Schwarz (CBS) type inequality for sequences of vectors in an inner product space holds true:

$$\sum_{i \in I} p_i \|x_i\|^2 \sum_{i \in I} p_i \|y_i\|^2 \geq \left| \sum_{i \in I} p_i \langle x_i, y_i \rangle \right|^2 \tag{4.5}$$

for $I \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

If $p_i > 0$ for all $i \in I$, then equality holds in (**??**) if and only if there exists a scalar $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$, $i \in I$.

Utilising the above results for the functional, we may state the following inequalities related to the (CBS)-inequality (**??**).

(1) Let $\alpha_i \in \mathbb{R}$, $x_i, y_i \in H$, $i \in \{1, \dots, n\}$. Then one has the inequality:

$$\begin{aligned} &\sum_{i=1}^{n} \|x_i\|^2 \sum_{i=1}^{n} \|y_i\|^2 - \left| \sum_{i=1}^{n} \langle x_i, y_i \rangle \right| \\ &\quad \geq \left(\sum_{i=1}^{n} \|x_i\|^2 \sin^2 \alpha_i \sum_{i=1}^{n} \|y_i\|^2 \sin^2 \alpha_i \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{n} \langle x_i, y_i \rangle \sin^2 \alpha_i \right| \\ &\quad + \left(\sum_{i=1}^{n} \|x_i\|^2 \cos^2 \alpha_i \sum_{i=1}^{n} \|y_i\|^2 \cos^2 \alpha_i \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{n} \langle x_i, y_i \rangle \cos^2 \alpha_i \right| \geq 0. \end{aligned} \tag{4.6}$$

(2) Denote $S_n(\mathbf{1}) := \{\mathbf{p} \in \mathcal{S}_+(\mathbb{R}) \,|\, p_i \leq 1 \text{ for all } i \in \{1, \dots, n\}\}$. Then for all $x_i, y_i \in H$, $i \in \{1, \dots, n\}$, we have the bound:

$$\begin{aligned} &\left(\sum_{i=1}^{n} \|x_i\|^2 \sum_{i=1}^{n} \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{n} \langle x_i, y_i \rangle \right| \\ &= \sup_{\mathbf{p} \in S_n(\mathbf{1})} \left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle \right| \right] \geq 0. \end{aligned} \tag{4.7}$$

(3) Let $p_i \geq 0$, $x_i, y_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality:

$$\begin{aligned} &\left(\sum_{i=1}^{2n} p_i \|x_i\|^2 \sum_{i=1}^{2n} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{2n} p_i \langle x_i, y_i \rangle \right| \\ &\qquad \geq \left(\sum_{k=1}^{n} p_{2k} \|x_{2k}\|^2 \sum_{k=1}^{n} p_{2k} \|y_{2k}\|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^{n} p_{2k} \langle x_{2k}, y_{2k} \rangle \right| \end{aligned} \tag{4.8}$$

$$+ \left(\sum_{k=1}^{n} p_{2k-1} \|x_{2k-1}\|^2 \sum_{k=1}^{n} p_{2k-1} \|y_{2k-1}\|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^{n} p_{2k-1} \langle x_{2k-1}, y_{2k-1} \rangle \right| \qquad (\geq 0).$$

(4) We have the bound:

$$\left[\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right]^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right| = \sup_{\varnothing \neq I \subseteq \{1,\dots,n\}} \left(\left[\sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2\right]^{\frac{1}{2}} - \left|\sum_{i\in I} p_i \langle x_i, y_i\rangle\right|\right) \geq 0. \tag{4.9}$$

(5) The sequence S_n given by

$$S_n := \left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|$$

is nondecreasing, i.e.,

$$S_{k+1} \geq S_k, \quad k \geq 2 \tag{4.10}$$

and we have the bound

$$S_n \geq \max_{1\leq i<j\leq n} \Big\{ \left(p_i \|x_i\|^2 + p_j \|x_j\|^2\right)^{\frac{1}{2}} \left(p_i \|y_i\|^2 + p_j \|y_j\|^2\right)^{\frac{1}{2}} - \left|p_i \langle x_i, y_i\rangle + p_j \langle x_j, y_j\rangle\right| \Big\} \geq 0, \tag{4.11}$$

for $n \geq 2$ and $x_i, y_i \in H$, $i \in \{1, \dots, n\}$.

Remark 5. *The results in this subsection have been obtained by Dragomir and Mond in [**?**] for the particular case of scalar sequences* $\mathbf{x}$ *and* $\mathbf{y}$.

4.2 The Case of Mapping δ

Under the assumptions of the above subsection, we can define the following functional

$$\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) := \sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2 - \left|\sum_{i\in I} p_i \langle x_i, y_i\rangle\right|^2,$$

where $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

Utilising Theorem **??**, we may state the following results.

Proposition 11. *We have*

(i) *For any* $\mathbf{p},\mathbf{q}\in\mathcal{S}_{+}(\mathbb{R})$, $I\in\mathcal{P}_f(\mathbb{N})$ *and* $\mathbf{x},\mathbf{y}\in\mathcal{S}(H)$ *we have*

$$\delta(\mathbf{p}+\mathbf{q},I,\mathbf{x},\mathbf{y})-\delta(\mathbf{p},I,\mathbf{x},\mathbf{y})-\delta(\mathbf{q},I,\mathbf{x},\mathbf{y}) \geq \left(\det\begin{bmatrix}\left(\sum_{i\in I}p_i\|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I}p_i\|y_i\|^2\right)^{\frac{1}{2}}\\ \left(\sum_{i\in I}q_i\|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I}q_i\|y_i\|^2\right)^{\frac{1}{2}}\end{bmatrix}\right)^2\geq 0. \quad (4.12)$$

(ii) *If* $\mathbf{p}\geq\mathbf{q}\geq\mathbf{0}$, *then*

$$\delta(\mathbf{p},I,\mathbf{x},\mathbf{y})-\delta(\mathbf{q},I,\mathbf{x},\mathbf{y}) \geq \left(\det\begin{bmatrix}\left(\sum_{i\in I}p_i\|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I}p_i\|y_i\|^2\right)^{\frac{1}{2}}\\ \left(\sum_{i\in I}(p_i-q_i)\|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I}(p_i-q_i)\|y_i\|^2\right)^{\frac{1}{2}}\end{bmatrix}\right)^2\geq 0. \quad (4.13)$$

Proposition 12. *We have*

(i) *For any* $I,J\in\mathcal{P}_f(\mathbb{N})$, *with* $I\cap J=\varnothing$ *and* $\mathbf{p}\in\mathcal{S}_{+}(\mathbb{R})$, $\mathbf{x},\mathbf{y}\in\mathcal{S}(H)$, *we have*

$$\delta(\mathbf{p},I\cup J,\mathbf{x},\mathbf{y})-\delta(\mathbf{p},I,\mathbf{x},\mathbf{y})-\delta(\mathbf{p},J,\mathbf{x},\mathbf{y}) \geq \left(\det\begin{bmatrix}\left(\sum_{i\in I}p_i\|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I}p_i\|y_i\|^2\right)^{\frac{1}{2}}\\ \left(\sum_{i\in J}p_i\|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in J}p_i\|y_i\|^2\right)^{\frac{1}{2}}\end{bmatrix}\right)^2\geq 0. \quad (4.14)$$

(ii) If $\varnothing \neq J \subseteq I,\ I \neq J,\ I, J \in \mathcal{P}_f(\mathbb{N})$, *then we have*

$$\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \geq \left(\det \begin{bmatrix} \left(\sum_{i\in I} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I} p_i \|y_i\|^2\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I\setminus J} p_i \|x_i\|^2\right)^{\frac{1}{2}} & \left(\sum_{i\in I\setminus J} p_i \|y_i\|^2\right)^{\frac{1}{2}} \end{bmatrix} \right)^2 \geq 0. \quad (4.15)$$

The following particular instances that provide refinements for the (CBS)-inequality may be stated as well:

$$\begin{aligned} &\sum_{i\in I} \|x_i\|^2 \sum_{i\in I} \|y_i\|^2 - \left|\sum_{i\in I} \langle x_i, y_i\rangle\right|^2 \qquad (4.16) \\ &\geq \sum_{i\in I} \|x_i\|^2 \sin^2\alpha_i \sum_{i\in I} \|y_i\|^2 \sin^2\alpha_i - \left|\sum_{i\in I} \langle x_i, y_i\rangle \sin^2\alpha_i\right|^2 \\ &\qquad + \sum_{i\in I} \|x_i\|^2 \cos^2\alpha_i \sum_{i\in I} \|y_i\|^2 \cos^2\alpha_i - \left|\sum_{i\in I} \langle x_i, y_i\rangle \cos^2\alpha_i\right|^2 \\ &\geq \left(\det \begin{bmatrix} \left(\sum_{i\in I} \|x_i\|^2 \sin^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I} \|y_i\|^2 \sin^2\alpha_i\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I} \|x_i\|^2 \cos^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I} \|y_i\|^2 \cos^2\alpha_i\right)^{\frac{1}{2}} \end{bmatrix} \right)^2 \geq 0, \end{aligned}$$

where $x_i, y_i \in H$, $\alpha_i \in \mathbb{R}$, $i \in I$ and $I \in \mathcal{P}_f(\mathbb{N})$.

Suppose that $p_i \geq 0,\ x_i, y_i \in H,\ i \in \{1, \ldots, 2n\}$. Then

$$\begin{aligned} &\sum_{i=1}^{2n} p_i \|x_i\|^2 \sum_{i=1}^{2n} p_i \|y_i\|^2 - \left|\sum_{i=1}^{2n} p_i \langle x_i, y_i\rangle\right|^2 \\ &\geq \sum_{k=1}^{n} p_{2k} \|x_{2k}\|^2 \sum_{k=1}^{n} p_{2k} \|y_{2k}\|^2 - \left|\sum_{k=1}^{n} p_{2k} \langle x_{2k}, y_{2k}\rangle\right|^2 \\ &+ \sum_{k=1}^{n} p_{2k-1} \|x_{2k-1}\|^2 \sum_{k=1}^{n} p_{2k-1} \|y_{2k-1}\|^2 - \left|\sum_{k=1}^{n} p_{2k-1} \langle x_{2k-1}, y_{2k-1}\rangle\right|^2 \\ &\geq \left(\det \begin{bmatrix} \left(\sum_{k=1}^{n} p_{2k} \|x_{2k}\|^2\right)^{\frac{1}{2}} & \left(\sum_{k=1}^{n} p_{2k} \|y_{2k}\|^2\right)^{\frac{1}{2}} \\ \left(\sum_{k=1}^{n} p_{2k-1} \|x_{2k-1}\|^2\right)^{\frac{1}{2}} & \left(\sum_{k=1}^{n} p_{2k-1} \|y_{2k-1}\|^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \geq 0. \end{aligned} \tag{4.17}$$

Remark 6. *The above results (**??**) – (**??**) have been obtained for the case where* $\mathbf{x}$ *and* $\mathbf{y}$ *are real or complex numbers by Dragomir and Mond [**?**].*

Further, if we use Corollaries **??** and **??**, then we can state the following propositions as well.

Proposition 13. *We have*

(i) For any $\mathbf{p}, \mathbf{q} \in \mathcal{S}_+(\mathbb{R}),\ I \in \mathcal{P}_f(\mathbb{N})$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$ *we have*

$$\begin{aligned} &\delta(\mathbf{p}+\mathbf{q}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) \\ &\geq \max\left\{\frac{\sum_{i\in I} p_i \|x_i\|^2}{\sum_{i\in I} q_i \|x_i\|^2}\delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) + \frac{\sum_{i\in I} q_i \|x_i\|^2}{\sum_{i\in I} p_i \|x_i\|^2}\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}),\right. \\ &\left.\frac{\sum_{i\in I} p_i \|y_i\|^2}{\sum_{i\in I} q_i \|y_i\|^2}\delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) + \frac{\sum_{i\in I} q_i \|y_i\|^2}{\sum_{i\in I} p_i \|y_i\|^2}\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y})\right\} \geq 0. \end{aligned} \tag{4.18}$$

(ii) If $\mathbf{p} \geq \mathbf{q} \geq \mathbf{0}$ *and* $I \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$, *then:*

$$\begin{aligned} &\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{q}, I, \mathbf{x}, \mathbf{y}) \\ &\geq \max\left\{\frac{\sum_{i\in I} (p_i - q_i) \|x_i\|^2}{\sum_{i\in I} p_i \|x_i\|^2}, \frac{\sum_{i\in I} (p_i - q_i) \|y_i\|^2}{\sum_{i\in I} p_i \|y_i\|^2}\right\}\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \geq 0. \end{aligned} \tag{4.19}$$

Proposition 14. *We have*

(i) For any $I, J \in \mathcal{P}_f(\mathbb{N})$, *with* $I \cap J = \varnothing$ *and* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$, *we have*

$$\begin{aligned}
&\delta(\mathbf{p}, I \cup J, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \\
&\geq \max \left\{ \frac{\sum_{i\in I} p_i \|x_i\|^2}{\sum_{j\in J} p_j \|x_j\|^2} \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) + \frac{\sum_{j\in J} p_j \|x_j\|^2}{\sum_{i\in I} p_i \|x_i\|^2} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}), \right. \\
&\left. \frac{\sum_{i\in I} p_i \|y_i\|^2}{\sum_{j\in J} p_j \|y_j\|^2} \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) + \frac{\sum_{j\in J} p_j \|y_j\|^2}{\sum_{i\in I} p_i \|y_i\|^2} \delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) \right\} \geq 0. \quad (4.20)
\end{aligned}$$

(ii) If $\varnothing \neq J \subseteq I$, $I \neq J$, $I, J \in \mathcal{P}_f(\mathbb{N})$ *and* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R}) \setminus \{0\}$, $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H) \setminus \{0\}$, *then*

$$\begin{aligned}
&\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \\
&\geq \max \left\{ \frac{\sum_{k\in I\setminus J} p_k \|x_k\|^2}{\sum_{i\in I} p_i \|x_i\|^2}, \frac{\sum_{k\in I\setminus J} p_k \|y_k\|^2}{\sum_{i\in I} p_i \|y_i\|^2} \right\} \delta(\mathbf{p}, J, \mathbf{x}, \mathbf{y}) \geq 0. \quad (4.21)
\end{aligned}$$

Remark 7. *The results in Proposition* **??** *have been obtained by Dragomir and Mond in [*?*] for the case of scalar sequences* $\mathbf{x}$ *and* $\mathbf{y}$.

4.3 The Case of Mapping β

With the assumptions in the first subsections, we can define the following functional

$$\begin{aligned}
\beta(\mathbf{p}, I, \mathbf{x}, \mathbf{y}) &:= [\delta(\mathbf{p}, I, \mathbf{x}, \mathbf{y})]^{\frac{1}{2}} \\
&= \left[\sum_{i\in I} p_i \|x_i\|^2 \sum_{i\in I} p_i \|y_i\|^2 - \left| \sum_{i\in I} p_i \langle x_i, y_i \rangle \right|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

where $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

Utilising Theorem **??**, we can state the following results:

Proposition 15. *We have*

(i) The functional $\beta(\cdot, I, \mathbf{x}, \mathbf{y})$ *is superadditive on* $\mathcal{S}_+(\mathbb{R})$ *for any* $I \in \mathcal{P}_f(\mathbb{N})$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

(ii) The functional $\beta(\mathbf{p}, \cdot, \mathbf{x}, \mathbf{y})$ *is superadditive as an index set mapping on* $\mathcal{P}_f(\mathbb{N})$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$.

As simple consequences of the above proposition, we may state the following refinements of the (CBS)-inequality.

(a) If $\mathbf{x}, \mathbf{y} \in \mathcal{S}(H)$ and $\alpha_i \in \mathbb{RA}$, $i \in I$ with $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{0\}$, then

$$\left(\sum_{i\in I}\|x_i\|^2\sum_{i\in I}\|y_i\|^2-\left|\sum_{i\in I}\langle x_i,y_i\rangle\right|^2\right)^{\frac{1}{2}}$$
$$\geq\left(\sum_{i\in I}\|x_i\|^2\sin^2\alpha_i\sum_{i\in I}\|y_i\|^2\sin^2\alpha_i-\left|\sum_{i\in I}\langle x_i,y_i\rangle\sin^2\alpha_i\right|^2\right)^{\frac{1}{2}}$$
$$+\left(\sum_{i\in I}\|x_i\|^2\cos^2\alpha_i\sum_{i\in I}\|y_i\|^2\cos^2\alpha_i-\left|\sum_{i\in I}\langle x_i,y_i\rangle\cos^2\alpha_i\right|^2\right)^{\frac{1}{2}}\geq 0. \tag{4.22}$$

(b) If $x_i, y_i \in H$, $p_i > 0$, $i \in \{1, \ldots, 2n\}$, then

$$\left(\sum_{i=1}^{2n}p_i\|x_i\|^2\sum_{i=1}^{2n}p_i\|y_i\|^2-\left|\sum_{i=1}^{2n}p_i\langle x_i,y_i\rangle\right|^2\right)^{\frac{1}{2}}$$
$$\geq\left(\sum_{k=1}^{n}p_{2k}\|x_{2k}\|^2\sum_{k=1}^{n}p_{2k}\|y_{2k}\|^2-\left|\sum_{k=1}^{n}p_{2k}\langle x_{2k},y_{2k}\rangle\right|^2\right)^{\frac{1}{2}}$$
$$+\left(\sum_{k=1}^{n}p_{2k-1}\|x_{2k-1}\|^2\sum_{k=1}^{n}p_{2k-1}\|y_{2k-1}\|^2\right.$$
$$\left.-\left|\sum_{k=1}^{n}p_{2k-1}\langle x_{2k-1},y_{2k-1}\rangle\right|^2\right)^{\frac{1}{2}}\quad(\geq 0). \tag{4.23}$$

Remark 8. *Part (i) of Proposition* **??** *and the inequality (***??***) have been obtained by Dragomir and Mond in [***?***] for the case of scalar sequences* $\mathbf{x}$ *and* $\mathbf{y}$.

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