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A Note on Multivariate Ostrowski Type Inequalities*

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Abstract

In this note, we establish some multivariate Ostrowski type inequalities involving several functions.

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1. Introduction

Throughout, $\mathbf{R}$ and $\mathbf{R}^n$ denote the set of real numbers and the $n-$dimensional Euclidean space, respectively. Let $D = \{(x_1, \cdots, x_n) \in \mathbf{R}^n : a_i < x_i < b_i \ (i = 1, \cdots, n)\}$ and $\overline{D}$ be the closure of D. For a function $u(x) : \mathbf{R}^n \to \mathbf{R}$, we denote the first order partial derivative by $\frac{\partial u(x)}{\partial x_i}$ $(i = 1, \cdots, n)$ and $\int_D u(x)\, dx$ the $n-$fold integral $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$.

In 1938, Ostrowski [3, $p.468$] established the following integral inequality:

Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If the derivative f' is bounded on (a, b), that is,

$$\|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty,$$

then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \tag{1.1}$$

for all $x \in [a, b]$.

The inequality (1.1) is known in the literature as the *Ostrowski inequality.*

For some recent results which generalize, improve and extend the inequality (1.1), see $[1 - 8]$.

In [8] Pachpatte established the following two theorems about new multivariate Ostrowski type inequality:

Theorem A. *Let* $f, g : \mathbf{R}^n \to \mathbf{R}$ *be functions continuous on* $\overline{D}$, *differentiable on* D *and whose derivatives* $\frac{\partial f(x)}{\partial x_i}$ *and* $\frac{\partial g(x)}{\partial x_i}$ $(i = 1, \cdots, n)$ *are buonded, i.e.,*

$$\left\| \frac{\partial f}{\partial x_i} \right\|_\infty = \sup_{x \in D} \left| \frac{\partial f(x)}{\partial x_i} \right| < \infty$$

and

$$\left\| \frac{\partial g}{\partial x_i} \right\|_\infty = \sup_{x \in D} \left| \frac{\partial g(x)}{\partial x_i} \right| < \infty$$

for all $i = 1, \cdots, n$. *Let the function* $w(x)$ *be nonnegative, integrable for every*

$x \in D$ and $\int_D w(y)\,dy > 0$. Then for every $x \in \overline{D}$,

$$\begin{aligned} &\left| f(x)g(x) - \frac{1}{2M} f(x) \int_D g(y)\,dy - \frac{1}{2M} g(x) \int_D f(y)\,dy \right| \\ \leq\ & \frac{1}{2M} \sum_{i=1}^{n} \left[|f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty + |g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right] E_i(x) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &\left| f(x)g(x) - \left[\frac{f(x) \int_D w(y) g(y)\,dy + g(x) \int_D w(y) f(y)\,dy}{2 \int_D w(y)\,dy} \right] \right| \\ \leq\ & \frac{\int_D w(y) \sum_{i=1}^{n} \left[|f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty + |g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right] |x_i - y_i|\,dy}{2 \int_D w(y)\,dy}, \end{aligned} \tag{1.3}$$

where $M = mesD = \prod_{i=1}^{n} (b_i - a_i)$, $dy = dy_1 \cdots dy_n$ and $E_i(x) = \int_D |x_i - y_i|\,dy$.

Theorem B. *Let $f, g, \frac{\partial f(x)}{\partial x_i}$ and $\frac{\partial g(x)}{\partial x_i}$ $(i = 1, \cdots, n)$ be defined as in Theorem A. Then for every $x \in \overline{D}$,*

$$\begin{aligned} &\left| f(x)g(x) - \frac{f(x) \int_D g(y)\,dy}{M} \right. \\ &\left. - \frac{g(x) \int_D f(y)\,dy}{M} + \frac{\int_D f(y) g(y)\,dy}{M} \right| \\ \leq\ & \frac{\int_D \left[\left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty |x_i - y_i| \right) \left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty |x_i - y_i| \right) \right] dy}{M} \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\left| f(x)g(x) - \frac{f(x) \int_D g(y)\,dy}{M} \right. \\ &\left. - \frac{g(x) \int_D f(y)\,dy}{M} + \frac{\int_D f(y)\,dy \int_D g(y)\,dy}{M^2} \right| \\ \leq\ & \frac{1}{M^2} \left[\left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty E_i(x) \right) \left(\sum_{i=1}^{n} \left\| \frac{\partial g}{\partial x_i} \right\|_\infty E_i(x) \right) \right], \end{aligned} \tag{1.5}$$

where M, dy and $E_i(x)$ are defined as in Theorem A.

The main purpose of the present paper is to establish some generalizations of Theorems A and B.

2. Main Results

Theorem 1. *Let $f, g, w, M, dy, \frac{\partial f(x)}{\partial x_i}, \frac{\partial g(x)}{\partial x_i}$ and $E_i(x)$ $(i = 1, \cdots, n)$ be defined as in Theorem A and $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta = 1$. Then for every $x \in \overline{D}$, we have*

$$\left| f(x)g(x) - \frac{\alpha f(x) \int_D g(y)\, dy + \beta g(x) \int_D f(y)\, dy}{M} \right| \le \frac{\sum_{i=1}^{n} \left[|\alpha f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty + |\beta g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right] E_i(x)}{M} \tag{2.1}$$

and

$$\left| f(x)g(x) - \left[\frac{\alpha f(x) \int_D g(y) w(y)\, dy + \beta g(x) \int_D f(y) w(y)\, dy}{\int_D w(y)\, dy} \right] \right| \le \frac{\int_D w(y) \sum_{i=1}^{n} \left[|\alpha f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty + |\beta g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right] |x_i - y_i|\, dy}{\int_D w(y)\, dy}. \tag{2.2}$$

Proof. Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ $\left(x \in \overline{D},\ y \in D\right)$. By the assumptions on f, g and the $n-$dimensional version of the mean value theorem, we have (see [4, p. 121] or [9, p. 174])

$$f(x) - f(y) = \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i} (x_i - y_i) \tag{2.3}$$

and

$$g(x) - g(y) = \sum_{i=1}^{n} \frac{\partial g(d)}{\partial x_i} (x_i - y_i), \tag{2.4}$$

where

$$c = (y_1 + \gamma(x_1 - y_1), \cdots, y_n + \gamma(x_n - y_n))$$

and

$$d = \left(y_1 + \rho\left(x_1 - y_1\right), \cdots, y_n + \rho\left(x_n - y_n\right)\right)$$

for some $\gamma, \rho \in (0,1)$.

Multiplying both sides of (2.3) and (2.4) by $\beta g(x)$ and $\alpha f(x)$ respectively and adding the resulting identities, we get

$$\begin{aligned} & f(x)g(x) - \alpha f(x)\, g(y) - \beta g(x)\, f(y) \\ = \; & \alpha f(x) \sum_{i=1}^{n} \frac{\partial g(d)}{\partial x_i}\left(x_i - y_i\right) + \beta g(x) \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}\left(x_i - y_i\right). \end{aligned} \tag{2.5}$$

Integrating both sides of (2.5) with respect to y over D and using $M = mesD = \prod_{i=1}^{n}\left(b_i - a_i\right)$, we obtain

$$\begin{aligned} & f(x)g(x) - \frac{\alpha f(x) \int_D g(y)\, dy + \beta g(x) \int_D f(y)\, dy}{M} \\ = \; & \frac{\alpha f(x) \int_D \sum_{i=1}^{n} \frac{\partial g(d)}{\partial x_i}\left(x_i - y_i\right) dy + \beta g(x) \int_D \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}\left(x_i - y_i\right) dy}{M}. \end{aligned} \tag{2.6}$$

By (2.6) and the properties of modulus, we have

$$\begin{aligned} & \left| f(x)g(x) - \frac{\alpha f(x) \int_D g(y)\, dy + \beta g(x) \int_D f(y)\, dy}{M} \right| \\ \leq \; & \frac{\sum_{i=1}^{n} \left[\left|\alpha f(x)\right| \int_D \left| \frac{\partial g(d)}{\partial x_i} \right| \left|x_i - y_i\right| dy + \left|\beta g(x)\right| \int_D \left| \frac{\partial f(c)}{\partial x_i} \right| \left|x_i - y_i\right| dy \right]}{M} \\ \leq \; & \frac{\sum_{i=1}^{n} \left[\left|\alpha f(x)\right| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty + \left|\beta g(x)\right| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right] \int_D \left|x_i - y_i\right| dy}{M} \\ = \; & \frac{\sum_{i=1}^{n} \left[\left|\alpha f(x)\right| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty + \left|\beta g(x)\right| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right] E_i(x)}{M} \end{aligned}$$

which is the inequality (2.1).

Multiplying both sides of (2.5) by $w(y)$ and integrating the resulting identity with respect to y on D and following the proof of inequality (2.1), we obtain the inequality (2.2).

This completes the proof.

Remark 1. If we choose $n = 1$. $\alpha = 0$, $\beta = 1$ and $g(x) \equiv 1$ in Theorem 1, then (2.1) reduces to (1.1).

Remark 2. If we choose $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in Theorem 1, then Theorem 1 reduces to Theorem A.

Theorem 2. *Let* $f_1, f_{2,\cdots}, f_m : \mathbf{R}^n \to \mathbf{R}$ *be functions continuous on* $\overline{D}$*, differentiable on* D *and whose derivatives* $\frac{\partial f_k(x)}{\partial x_i}$ $(i = 1, \cdots, n,\ k = 1, \cdots, m)$ *are bounded. Further,let* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *in* $\mathbf{R}$ *with* $\sum\limits_{k=1}^{m} \alpha_k = 1$. *Then for every* $x \in \overline{D}$*, we have*

$$\left| \prod_{k=1}^{m} f_k(x) - \frac{\sum\limits_{k=1}^{m} \alpha_k \prod\limits_{l=1, l\neq k}^{m} f_l(x) \int_D f_k(y)\, dy}{M} \right| \leq \frac{\sum\limits_{k=1}^{m} \left\{ \sum\limits_{i=1}^{n} \left[\left| \alpha_k \prod\limits_{l=1, l\neq k}^{m} f_l(x) \right| \left\| \frac{\partial f_k}{\partial x_i} \right\|_\infty \right] E_i(x) \right\}}{M} \tag{2.7}$$

and

$$\left| \prod_{k=1}^{m} f_k(x) - \frac{\sum\limits_{k=1}^{m} \alpha_k \prod\limits_{l=1, l\neq k}^{m} f_l(x) \int_D f_k(y)\, w(y)\, dy}{\int_D w(y)\, dy} \right| \leq \frac{\sum\limits_{k=1}^{m} \int_D w(y) \left\{ \sum\limits_{i=1}^{n} \left[\left| \alpha_k \prod\limits_{l=1, l\neq k}^{m} f_l(x) \right| \left\| \frac{\partial f_k}{\partial x_i} \right\|_\infty \right] |x_i - y_i| \right\} dy}{\int_D w(y)\, dy}. \tag{2.8}$$

Proof. Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ $\left(x \in \overline{D},\ y \in D\right)$. By the assumptions on f_k $(k = 1, \cdots, m)$ and the $n-$dimensional version of the mean value theorem, we have

$$f_1(x) - f_1(y) = \sum_{i=1}^{n} \frac{\partial f_1(c_1)}{\partial x_i}(x_i - y_i), \tag{2.9}$$

$$f_2(x) - f_2(y) = \sum_{i=1}^{n} \frac{\partial f_2(c_2)}{\partial x_i}(x_i - y_i), \tag{2.10}$$

$$\vdots$$

$$f_m(x) - f_m(y) = \sum_{i=1}^{n} \frac{\partial f_m(c_m)}{\partial x_i}(x_i - y_i), \tag{2.11}$$

where $c_k = (y_1 + \gamma_k(x_1 - y_1), \cdots, y_n + \gamma_k(x_n - y_n))$ $(k = 1, \cdots, m)$ for some $0 < \gamma_k < 1$ $(k = 1, \cdots, m)$.

Multiplying both sides of (2.9.k) by $\alpha_k \prod_{l=1, l\neq k}^{m} f_l(x)$ $(k = 1, \cdots, m)$ and adding the resulting identities, we get

$$\begin{aligned} &\prod_{k=1}^{m} f_k(x) - \sum_{k=1}^{m} \left[\alpha_k \prod_{l=1, l\neq k}^{m} f_l(x)\right] f_k(y) \\ = &\sum_{k=1}^{m} \left\{\left[\alpha_k \prod_{l=1, l\neq k}^{m} f_l(x)\right]\left[\sum_{i=1}^{n} \frac{\partial f_k(c_k)}{\partial x_i}(x_i - y_i)\right]\right\}. \end{aligned} \tag{2.12}$$

Integrating both sides of (2.10) with respect to y over D and using $M = mesD = \prod_{i=1}^{n}(b_i - a_i)$, we obtain

$$\begin{aligned} &\prod_{k=1}^{m} f_k(x) - \frac{\sum_{k=1}^{m} \alpha_k \prod_{l=1, l\neq k}^{m} f_l(x) \int_D f_k(y)\,dy}{M} \\ = &\frac{\sum_{k=1}^{m} \left\{\left[\alpha_k \prod_{l=1, l\neq k}^{m} f_l(x)\right]\left[\int_D \sum_{i=1}^{n} \frac{\partial f_k(c_k)}{\partial x_i}(x_i - y_i)\,dy\right]\right\}}{M}. \end{aligned} \tag{2.13}$$

By (2.11) and the properties of modulus, we have

$$\begin{aligned}
&\left| \prod_{k=1}^{m} f_k(x) - \frac{\sum_{k=1}^{m} \alpha_k \prod_{l=1,l\neq k}^{m} f_l(x) \int_D f_k(y)\, dy}{M} \right| \\
&\leq \frac{\sum_{k=1}^{m} \left\{ \left| \alpha_k \prod_{l=1,l\neq k}^{m} f_l(x) \right| \int_D \sum_{i=1}^{n} \left| \frac{\partial f_k(c_k)}{\partial x_i} \right| |x_i - y_i|\, dy \right\}}{M} \\
&\leq \frac{\sum_{k=1}^{m} \left\{ \left| \alpha_k \prod_{l=1,l\neq k}^{m} f_l(x) \right| \sum_{i=1}^{n} \left\| \frac{\partial f_k}{\partial x_i} \right\|_\infty \int_D |x_i - y_i|\, dy \right\}}{M} \\
&\leq \frac{\sum_{k=1}^{m} \left\{ \sum_{i=1}^{n} \left[\left| \alpha_k \prod_{l=1,l\neq k}^{m} f_l(x) \right| \left\| \frac{\partial f_k}{\partial x_i} \right\|_\infty \right] E_i(x) \right\}}{M} \qquad (2.14)
\end{aligned}$$

which is the inequality (2.7).

Multiplying both sides of (2.11) by $w(y)$ and integrating the resulting identity with respect to y on D and following the proof of inequality (2.7), we obtain the inequality (2.8).

This completes the proof.

Remark 3. If we choose $n = 2$ in Theorem 2, then Theorem 2 reduces to Theorem 1.

Remark 4. If we choose $n = 2$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$ in Theorem 2, then Theorem 2 reduces to Theorem A.

Theorem 3. *Let* $f, g, w, dy, \frac{\partial f(x)}{\partial x_i}$ *and* $\frac{\partial g(x)}{\partial x_i}$ $(i = 1, \cdots, n)$ *be defined as in*

Theorem A. Then for every $x \in \overline{D}$, we have

$$\begin{aligned}
&\left| f(x)g(x) - \frac{f(x)\int_D g(y)\,w(y)\,dy}{\int_D w(y)\,dy} \right. \\
&\left. - \frac{g(x)\int_D f(y)\,w(y)\,dy}{\int_D w(y)\,dy} + \frac{\int_D f(y)\,g(y)\,w(y)\,dy}{\int_D w(y)\,dy} \right| \\
\leq\ & \frac{\int_D \left[\left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty |x_i - y_i| \right) \left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty |x_i - y_i| \right) w(y) \right] dy}{\int_D w(y)\,dy} \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
&\left| f(x)g(x) - \frac{f(x)\int_D g(y)\,w(y)\,dy}{\int_D w(y)\,dy} \right. \\
&\left. - \frac{g(x)\int_D f(y)\,w(y)\,dy}{\int_D w(y)\,dy} + \frac{\int_D f(y)\,w(y)\,dy \int_D g(y)\,w(y)\,dy}{\left(\int_D w(y)\,dy\right)^2} \right| \\
\leq\ & \frac{1}{\left(\int_D w(y)\,dy\right)^2} \left(\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \int_D w(y)\,|x_i - y_i|\,dy \right) \times \\
& \left(\sum_{i=1}^{n} \left\| \frac{\partial g}{\partial x_i} \right\|_\infty \int_D w(y)\,|x_i - y_i|\,dy \right). \qquad (2.16)
\end{aligned}$$

Proof. From the hypotheses, as in the proof of Theorem 1, the identities (2.3) and (2.4) hold. Multiplying the left and right sides of (2.3) and (2.4), we have

$$\begin{aligned}
& f(x)g(x) - f(x)g(y) - g(x)f(y) + f(y)g(y) \\
=\ & \left[\sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i} (x_i - y_i) \right] \left[\sum_{i=1}^{n} \frac{\partial g(d)}{\partial x_i} (x_i - y_i) \right]. \qquad (2.17)
\end{aligned}$$

Multiplying both sides of (2.15) by $w(y)$ and integrating the resulting identity with respect to y on D, we get

$$\begin{aligned}
&\left|f(x)g(x) - \frac{f(x)\int_D g(y)w(y)\,dy}{\int_D w(y)\,dy}\right. \\
&\left. - \frac{g(x)\int_D f(y)w(y)\,dy}{\int_D w(y)\,dy} + \frac{\int_D f(y)g(y)w(y)\,dy}{\int_D w(y)\,dy}\right| \\
\leq\ & \left|\left[\sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i - y_i)\right]\left[\sum_{i=1}^{n}\frac{\partial g(d)}{\partial x_i}(x_i - y_i)\right]\right| \\
\leq\ & \frac{\int_D\left[\left(\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty |x_i - y_i|\right)\left(\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty |x_i - y_i|\right)w(y)\right]dy}{\int_D w(y)\,dy}
\end{aligned}$$

which is the required inequality in (2.13).

Multiplying both sides of (2.3) and (2.4) by $w(y)$ and integrating the resulting identities with respect to y on D, we get

$$\begin{aligned}
& f(x) - \frac{\int_D f(y)w(y)\,dy}{\int_D w(y)\,dy} \\
=\ & \frac{1}{\int_D w(y)\,dy}\int_D \sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i - y_i)w(y)\,dy \qquad (2.18)
\end{aligned}$$

and

$$\begin{aligned}
& g(x) - \frac{\int_D g(y)w(y)\,dy}{\int_D w(y)\,dy} \\
=\ & \frac{1}{\int_D w(y)\,dy}\int_D \sum_{i=1}^{n}\frac{\partial g(d)}{\partial x_i}(x_i - y_i)w(y)\,dy, \qquad (2.19)
\end{aligned}$$

respectively. Multiplying the left and right sides of (2.16) and (2.17), we have

$$\begin{aligned}
& f(x)g(x)-\frac{f\left(x\right)\int_{D}g\left(y\right)w\left(y\right)dy}{\int_{D}w\left(y\right)dy} \\
& \qquad -\frac{g\left(x\right)\int_{D}f\left(y\right)w\left(y\right)dy}{\int_{D}w\left(y\right)dy}+\frac{\int_{D}f\left(y\right)w\left(y\right)dy\int_{D}g\left(y\right)w\left(y\right)dy}{\left(\int_{D}w\left(y\right)dy\right)^{2}} \\
= & \frac{1}{\left(\int_{D}w\left(y\right)dy\right)^{2}}\left[\int_{D}\sum_{i=1}^{n}\frac{\partial f\left(c\right)}{\partial x_{i}}\left(x_{i}-y_{i}\right)w\left(y\right)dy\right]\times \\
& \qquad \left[\int_{D}\sum_{i=1}^{n}\frac{\partial g\left(d\right)}{\partial x_{i}}\left(x_{i}-y_{i}\right)w\left(y\right)dy\right]. \qquad (2.20)
\end{aligned}$$

By (2.18) and the properties of modulus, we have

$$\begin{aligned}
& \left| f(x)g(x)-\frac{f\left(x\right)\int_{D}g\left(y\right)w\left(y\right)dy}{\int_{D}w\left(y\right)dy}\right. \\
& \left. -\frac{g\left(x\right)\int_{D}f\left(y\right)w\left(y\right)dy}{\int_{D}w\left(y\right)dy}+\frac{\int_{D}f\left(y\right)w\left(y\right)dy\int_{D}g\left(y\right)w\left(y\right)dy}{\left(\int_{D}w\left(y\right)dy\right)^{2}}\right| \\
\leq & \frac{1}{\left(\int_{D}w\left(y\right)dy\right)^{2}}\left[\int_{D}\sum_{i=1}^{n}\left|\frac{\partial f\left(c\right)}{\partial x_{i}}\right|\left(x_{i}-y_{i}\right)w\left(y\right)dy\right]\times \\
& \qquad \left[\int_{D}\sum_{i=1}^{n}\left|\frac{\partial g\left(d\right)}{\partial x_{i}}\right|\left(x_{i}-y_{i}\right)w\left(y\right)dy\right] \\
\leq & \frac{1}{\left(\int_{D}w\left(y\right)dy\right)^{2}}\left(\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}\int_{D}w\left(y\right)\left|x_{i}-y_{i}\right|dy\right)\times \\
& \qquad \left(\sum_{i=1}^{n}\left\|\frac{\partial g}{\partial x_{i}}\right\|_{\infty}\int_{D}w\left(y\right)\left|x_{i}-y_{i}\right|dy\right)
\end{aligned}$$

which is the inequality (2.14).

This completes the proof.

Remark 5. For $w\left(y\right)\equiv 1$, Theorem 3 reduces to Theorem B.

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