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Sturm-Liouville Problem in Quantum Calculus[*]

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Abstract

This paper aims to study the q-analogue of the Sturm Liouville problem and to give an asymptotic behaviour at infinity for its solution φ. Additionally, we establish an asymptotic expansion of the q-Bessel function j_α for $\alpha > -\frac{1}{2}$. We are not in situation to claim that our results are new but they have the advantage to show that the method used by Agranovich and Marchenko remain true.

Keywords and Phrases: *Quantum calculus, q-Analysis, q-Integral transform.*

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1. Introduction

In the classical spectral analysis (see [2] , [14],...), we denote by L a linear differential operator of the second-order given by on $[0,\infty[$ of the form

$$Lu = \frac{d^2u}{dx^2} - p(x)u, \tag{1.1}$$

where $p(x)$ is a real function, continuous which is integrable on $[0,\infty[$.

We note by $\varphi(x,\lambda^2)$ and $\theta(x,\lambda^2)$ the solutions of

$$Lu = \lambda^2 u \tag{1.2}$$

with initial conditions

$$\begin{array}{ll} \varphi(0,\lambda^2) = \sin\alpha, & \varphi'(0,\lambda^2) = -\cos\alpha; \\ \theta(0,\lambda^2) = \cos\alpha, & \theta'(0,\lambda^2) = \sin\alpha, \end{array}$$

where λ is an arbitrary positif real number and α is an arbitrary real number.

It is known that solving (1.2) is equivalent to solving the following volterra integral equation

$$\begin{aligned} u(x,\lambda^2) &= \sin\alpha\cos(\lambda x) - \frac{\sin(\lambda x)}{\lambda}\cos\alpha \\ &\quad + \int_0^x \frac{\sin(\lambda(x-y))}{\lambda} p(y)u(y,\lambda^2)dy \end{aligned} \tag{1.3}$$

where $x \in [0,+\infty[$, $\lambda \in \mathbb{R}_+^*$ and $p(x)$ is an continuous integrable function on $[0,+\infty[$.

Hence, for all $\lambda \geq \rho > 0$, $\varphi(x,\lambda^2)$ is a bounded function and have the asymptotic formulas

$$\varphi(x,\lambda^2) = \mu(\lambda^2)\cos(\lambda x) + \nu(\lambda^2)\sin(\lambda x) + O(1) \tag{1.4}$$

where

$$\mu(\lambda^2) = \sin\alpha - \int_0^\infty \frac{\sin(\lambda y)}{\lambda} p(y)\varphi(y,\lambda^2)dy, \tag{1.5}$$

$$\nu(\lambda^2) = -\frac{\cos\alpha}{\lambda} + \int_0^\infty \frac{\cos(\lambda y)}{\lambda} p(y)\varphi(y,\lambda^2)dy. \tag{1.6}$$

Similarly, we have

$$\theta(x,\lambda^2) = \mu_1(\lambda^2)\cos(\lambda x) + \nu_1(\lambda^2)\sin(\lambda x) + O(1) \tag{1.7}$$

where

$$\mu_1(\lambda^2) = \cos\alpha - \int_0^\infty \frac{\sin(\lambda y)}{\lambda} p(y)\theta(y,\lambda^2)dy, \tag{1.8}$$

$$\nu_1(\lambda^2) = \frac{\sin\alpha}{\lambda} + \int_0^\infty \frac{\cos(\lambda y)}{\lambda} p(y)\theta(y,\lambda^2)dy. \tag{1.9}$$

In the present paper we are concerned to give its q-analogue and study its asymptotic behaviour at infinity.
This paper is organized as follows: in section 2, we present some preliminaries results and notations that will be useful in the sequel. Further it is natural to consider in section 3, the asymptotic behaviour of $\varphi(x,\lambda^2;q^2)$ and $\theta(x,\lambda^2)$ for $\lambda \longrightarrow \infty$. the fundamental result is given in the following theorem

Theorem 1.1. *For λ in $\mathbb{R}_{q,+}$, we have:*

$$\mu(\lambda^2;q^2)\nu_1(\lambda^2;q^2) - \nu(\lambda^2;q^2)\mu_1(\lambda^2;q^2) = \frac{1}{q^{\frac{1}{2}}\lambda}. \tag{1.10}$$

Section 4, is devoted to finding precise asymptotic formulas of j_α: the q-Bessel function for large λ.

2. Notations and Preliminaries

We recall some usual notions and notations used in the q-theory. Let a and q be real numbers such that $0 < q < 1$. In all the sequel we suppose that and $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$.
The q-shifted factorials are defined by

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad ; n \in \mathbb{N}\backslash\{0\}, \tag{2.1}$$

$$(a;q)_0 = 1, \tag{2.2}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{2.3}$$

and more generally:

$$(a_1, \cdots, a_r; q)_n = \prod_{k=1}^{r} (a_k; q)_n. \tag{2.4}$$

The basic hypergeometric series or q-hypergeometric series is given for r , s integers by

$${}_r\phi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n (q, q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} x^n$$

here

$$r, s \in \mathbb{N}; a_1, \cdots, a_r \in \mathbb{C}; b_1, \cdots, b_s \in \mathbb{C}\backslash\{q^{-k}, k \in \mathbb{N}\}.$$

The q-derivative $D_{q,x}f$ of a function f on an open interval is given by:

$$D_{q,x}f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0 \tag{2.5}$$

and $(D_{q,x}f)(0) = f^{'}(0)$ provided $f^{'}(0)$ exist. The q-shift operators are

$$(\Lambda_{q,x}f)(x) = f(qx) \tag{2.6}$$

$$(\Lambda_{q,x}^{-1}f)(x) = f(q^{-1}x). \tag{2.7}$$

We consider the q-operator

$$\Delta_{q,x} = \Lambda_{q,x}^{-1} D_{q,x}^2. \tag{2.8}$$

The q-Jackson integral from 0 to a and from 0 to ∞ are respectively defined by

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.9}$$

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{-\infty}^{+\infty} f(q^n) q^n, \tag{2.10}$$

and from a to ∞,

$$\int_a^{\infty} f(x) d_q x = (1-q) a \sum_{n=1}^{+\infty} f(aq^{-n}) q^{-n}. \tag{2.11}$$

Some q-functional spaces will be used to establish our result. We begin by putting

$$\mathbb{R}_q = \{\pm q^k, k \in \mathbb{Z}\} \cup \{0\}, \tag{2.12}$$
$$\mathbb{R}_{q,+} = \{+q^k, k \in \mathbb{Z}\}. \tag{2.13}$$

Let $L_q^p(\mathbb{R}_{q,+})$, $p \in [1, +\infty[$ be the space of functions f such that

$$\| f \|_{q,p} = (\int_0^\infty | f(x) |^p d_q x)^{\frac{1}{p}} < +\infty, \tag{2.14}$$

and for $p = \infty$

$$\| f \|_{q,\infty} = ess \sup_{x \in \mathbb{R}_{q,+}} | f(x) | . \tag{2.15}$$

Note that for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_q$, we have

$$\int_0^\infty f(q^n x) d_q x = \frac{1}{q^n} \int_0^\infty f(x) d_q x. \tag{2.16}$$

$$\int_0^a f(q^n x) d_q x = \frac{1}{q^n} \int_0^{aq^n} f(x) d_q x. \tag{2.17}$$

The q-integration by parts is given for suitable function f and g by:

$$\int_0^\infty f(x) D_{q,x} g(x) d_q x = \Big[f(x) g(x) \Big]_0^\infty - \int_0^\infty D_{q,x}(f(q^{-1}x)) g(x) d_q x. \tag{2.18}$$

Jackson in [8] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}. \tag{2.19}$$

We take the definition of q-trigonometric given by T. H. Koornwinder and R. F. Swarttouw (see [11]) with simple changes and we write q-cosine and q-sinus as a series of functions

$$\cos(x;q^2) = {}_1\varphi_1(0, q, q^2; (1-q)^2 x^2) = \sum_{n=0}^\infty (-1)^n b_n(x;q^2), \tag{2.20}$$
$$\begin{aligned} \sin(x;q^2) &= (1-q) x {}_1\varphi_1(0, q^3, q^2; (1-q)^2 x^2) \\ &= \sum_{n=0}^\infty (-1)^n c_n(x;q^2), \end{aligned} \tag{2.21}$$

where we have put

$$b_n(x;q^2) = b_n(1;q^2)x^{2n} = q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}x^{2n}, \tag{2.22}$$

$$c_n(x;q^2) = c_n(1;q^2)x^{2n+1} = q^{n(n-1)}\frac{(1-q)^{2n+1}}{(q;q)_{2n+1}}x^{2n+1}. \tag{2.23}$$

The reader will notice that the previous definition (2.20) derived from those given in [11] with minor change. These functions satisfy

$$D_{q,x}\cos(x;q^2) = -q^{-1}\sin(qx;q^2), \tag{2.24}$$

$$D_{q,x}\sin(x;q^2) = \cos(x;q^2). \tag{2.25}$$

and we have the following estimations:

$$|\cos(x;q^2)| \leq \frac{1}{(q;q^2)^2_\infty}, \tag{2.26}$$

$$|\sin(x;q^2)| \leq \frac{1}{(q;q^2)^2_\infty}. \tag{2.27}$$

We recall the tow q-analogue of the exponential functions [10], defined by:

$$E(x;q) = (-(1-q)x;q)_\infty = \sum_{n=0}^{\infty}\frac{(1-q)^n}{(q;q)_n}q^{n(n-1)/2}x^n, \quad x \in \mathbb{R}, \tag{2.28}$$

$$e(x;q) = \frac{1}{((1-q)x;q)_\infty} = \sum_{n=0}^{\infty}\frac{(1-q)^n}{(q;q)_n}x^n, \quad |x| < \frac{1}{1-q}. \tag{2.29}$$

The function $E(x;q)$ is analytic and $e(z;q)$ is a meromorphic function on $\mathbb{C}$ having simple poles at $z = \frac{q^{-m}}{1-q}$, $m \in \mathbb{N}$. They satisfy

$$e(x;q)E(-x;q) = 1. \tag{2.30}$$

Proposition 2.1. *(1)- If F is any q-derivative of the function f, namely $D_{q,x}F(x) = f(x)$, continuous at $x = 0$, then*

$$\int_0^x f(t)d_qt = (F(x) - F(0)). \tag{2.31}$$

(2)- For any function f we have

$$D_{q,x}\left[\int_0^x f(t)d_qt\right] = f(x). \tag{2.32}$$

(3)- If G is any q-derivative of the function g, integrable over (x,∞); $x \geq 0$ then:

$$\int_x^{+\infty} g(t)d_qt = -\lim_{b\longrightarrow+\infty}(G(x)-G(b)) \quad = \quad G(\infty)-G(x) \tag{2.33}$$

(4)- For any function f integrable over $(x,+\infty)$, we have

$$D_{q,x}\left[\int_x^{+\infty} g(t)d_qt\right] = -g(x). \tag{2.34}$$

2.1 The q-Wronskian:

Let the following q-difference equation:

$$D_{q,x}^2u(x) + a(x)D_{q,x}u(x) + b(x)u(qx) = 0 \tag{2.35}$$

Proposition 2.2. *We define the q-Wronskian $W(x;q)$ For the two solutions u_1 and u_2 of the q-difference equation (2.35) by:*

$$W(x;q)(x) = u_1(qx)D_{q,x}u_2(x) - u_2(qx)D_{q,x}u_1(x) \tag{2.36}$$

$$= \frac{u_1(qx)u_2(x) - u_1(x)u_2(qx)}{(1-q)x}. \tag{2.37}$$

It satisfies the following q-difference equation:

$$D_{q,x}W(x;q) + a(x)W(x;q) = 0. \tag{2.38}$$

Furthermore:

$$W(x;q) = \frac{W(0;q)}{\displaystyle\prod_{k=0}^{\infty}[1+(1-q)q^kxa(q^kx)]}. \tag{2.39}$$

Proof. It easy to see that, if u_1 and u_2 are tow solutions of (2.35), we have the relation (2.38).
Using the q-derivative definition (2.5), the equation (2.38) can be rewritten as:

$$W(x;q) - W(qx;q) = -(1-q)xa(x)W(x;q) \tag{2.40}$$

then

$$W(qx,q) = \frac{1}{1+(1-q)qxa(qx)}W(q^2x;q). \tag{2.41}$$

So, by induction we have for $n \in \mathbb{N}$:

$$W(q^n x;q) = \frac{1}{1+(1-q)q^n xa(q^n x)}W(q^{n+1}x;q). \tag{2.42}$$

We deduce that

$$W(x;q) = \frac{W(0;q)}{\prod_{k=0}^{\infty}[1+(1-q)q^k xa(q^k x)]}. \tag{2.43}$$

Proposition 2.3. *The solution of the q-difference equation*

$$(E) = \begin{cases} D^2_{q,x}u(x) + u(qx) = 0 \\ u(0;q^2) = a \\ D_q u(0;q^2) = b \end{cases} \tag{2.44}$$

is given by:

$$u(x;q^2) = a\cos(x;q^2) + bq^{-\frac{1}{2}}sin(q^{\frac{1}{2}}x;q^2). \tag{2.45}$$

Proof. Let $u(x) = \sum_{n\geq 0} a_n x^n$. Then

$$D^2_{q,x}u(x) = \sum_{n\geq 0} a_{n+2}\frac{1-q^{n+2}}{1-q}\frac{1-q^{n+1}}{1-q}x^n. \tag{2.46}$$

If we replace in (2.44) we have the following recurrence relation:

$$\frac{1-q^{n+2}}{1-q}\frac{1-q^{n+1}}{1-q}a_{n+2} = q^n a_n. \tag{2.47}$$

then

If $n = 2p$; $p \in \mathbb{N}$, we have

$$q^{2p}a_{2p} = \frac{1-q^{2p+2}}{1-q}\frac{1-q^{2p+1}}{1-q}a_{2p+2}. \tag{2.48}$$

So, by induction on p, we obtain:

$$a_{2p} = (-1)^p \frac{(1-q)^{2p}}{(q;q)_{2p}} q^{p(p-1)} a_0. \tag{2.49}$$

Similarly,

if $n = 2p+1$, we obtain

$$a_{2p+1} = (-1)^p \frac{(1-q)^{2p+1}}{(q;q)_{2p+1}} q^{p(p-1)} a_1. \tag{2.50}$$

Using the definition (2.20), the solution of (2.44) is given.

Corollary 2.4. *For $x \in \mathbb{R}_q$ we have*

$$\cos(qx;q^2)\cos(q^{\frac{1}{2}}x;q^2) + q^{-\frac{3}{2}}sin(q^{\frac{3}{2}}x;q^2)sin(qx;q^2) = 1 \tag{2.51}$$

which tends to the classical trigonometric relation

$$cos^2(x) + sin^2(x) = 1 \tag{2.52}$$

when $q \to 1^-$.

Proof. Using the relation (2.38) and the condition $a(x) = 0$, we have

$$D_{q,x}W(x;q) = 0. \tag{2.53}$$

So,

$$W(x;q) = \lim_{n\to+\infty} W(q^n x;q) = W(0;q).$$

$$\begin{aligned}W(0;q) &= \lim_{x\to 0} W(x;q) = \lim_{x\to 0}\left[\cos(qx;q^2)\cos(q^{\frac{1}{2}}x;q^2) + q^{-\frac{3}{2}}\sin(q^{\frac{3}{2}}x;q^2)\sin(qx;q^2)\right]\\ &= \lim_{x\to 0}\frac{1}{(1-q)x}\left[q^{-\frac{1}{2}}\cos(x;q^2)\sin(q^{\frac{3}{2}}x;q^2) - q^{-\frac{1}{2}}\cos(qx;q^2)\sin(q^{\frac{1}{2}}x;q^2)\right]\\ &= 1.\end{aligned}$$

The result follows.

3. Asymptotic Expansion of Solutions at Infinity

In this section, we try to study for $\lambda \longrightarrow \infty$ the asymptotic expansion of solution $u(x,\lambda^2;q^2)$ of L_q the q-difference operator defined by:

$$L_q u(x) = D^2_{q,x}u(x) - p(x)u(x); \quad p(x) \in L^\infty(\mathbb{R}_{q,+}) \cap L^1(\mathbb{R}_{q,+}). \tag{3.1}$$

In the next we try to resolve the following q-difference problem

$$L_q u(x) = -\lambda^2 u(qx), \quad x \in \mathbb{R}_{q,+}, \lambda \in \mathbb{R}_{q,+}. \tag{3.2}$$

Proposition 3.1. (The q-Gronwall lemma:)
Let f and g be two positive functions, continuous at 0 and q-integrable over all finite interval of $[0,+\infty[$.
We suppose that

$$f(x) \le C_q + \int_0^x f(t)g((t)d_qt \tag{3.3}$$

where $C_q \in \mathbb{R}_{q,+}$.
Then

$$f(x) \le \frac{C_q}{\prod_{k=0}^{\infty}[1-(1-q)q^k x g(q^k x)]}. \tag{3.4}$$

Proof. Let the following q-Jackson integral

$$y(x) = \int_0^x f(t)g(t)d_qt,$$

we have,

$$D_{q,x}y(x) = f(x)g(x) \leq [C_q + y(x)]g(x)$$

then,

$$C_q + y(x) \leq \frac{1}{1-(1-q)xg(x)}(C_q + y(qx)$$

and by induction on n, we deduce that

$$C_q + y(q^n x) \leq \frac{1}{1-(1-q)q^n xg(q^n x)}(C_q + y(q^{n+1}x))$$

then

$$C_q + y(x) \leq \frac{1}{\prod_{k=0}^{\infty}[1-(1-q)q^k xg(q^k x)]}(C_q + y(0)).$$

The fact that $y(0) = 0$ leads to the result.

Corollary 3.2. *Let f be a positive function, continuous at 0 and q-integrable over all finite interval of $[0, +\infty[$. We suppose that there exist two constants C_q and M_q in $\mathbb{R}_{q,+}$, such that*

$$f(x) \leq C_q + M_q \int_0^x f(t)d_q t.$$

Then we have,

$$f(x) \leq C_q e(M_q(1-q)x; q^2) \tag{3.5}$$

where $e(x; q^2)$ is given by (2.29).

Definition 3.3. Let $U(x; q^2)$ and $V(x; q^2)$ be twice q-differentiable functions. We define $[U, V]_q$ by:

$$[U, V]_q = U(qx; q^2)D_{q,x}V(x; q^2) - V(qx; q^2)D_{q,x}U(x; q^2). \tag{3.6}$$

Proposition 3.4. *(q-Green formula)*
For $U(x; q^2)$ and $V(x; q^2)$ twice q-differentiable functions, we have

$$D_{q,x}[U, V]_q = V(qx; q^2)L_q U(x; q^2) - U(qx; q^2)L_q V(x; q^2). \tag{3.7}$$

3.1 q-Asymptotic behaviour of $\varphi(x,\lambda^2;q^2)$ when $\lambda \longrightarrow \infty$

For $\lambda \in \mathbb{R}_{q,+}$ let $\varphi(x,\lambda^2;q^2)$ the solution of the following q-difference problem (E_1)

$$(E_1) = \begin{cases} L_q U(x,\lambda^2;q^2) = -\lambda^2 U(qx,\lambda^2;q^2), \\ \\ U(0,\lambda^2;q^2) = q^{-1}\sin(q\alpha;q^2), \\ \\ D_q U(0,\lambda^2;q^2) = \cos(q\alpha;q^2) \quad ,\alpha \in \mathbb{R}. \end{cases} \tag{3.8}$$

where L_q is given by (3.2).

Theorem 3.5. *Let $p(x)$ in $L_q^\infty(\mathbb{R}_{q,+})$, then the solution $\varphi(x,\lambda^2;q^2)$ of (E_1) verifies the following q-integral equation:*

$$\begin{aligned} \varphi(x,\lambda^2;q^2) &= q^{-1}\sin(q\alpha;q^2)\cos(\lambda x;q^2) \\ &+ q^{-\frac{1}{2}}\frac{\cos(q\alpha;q^2)}{\lambda}\sin(q^{\frac{1}{2}}\lambda x;q^2) \\ &+ \frac{1}{\lambda}\int_0^x G(x,y,\lambda^2;q^2)p(y)\varphi(y,\lambda^2;q^2)d_q y, \end{aligned} \tag{3.9}$$

where $G(x,y,\lambda^2;q^2)$ is the Green kernel defined by

$$G(x,y,\lambda^2;q^2) = \cos(q\lambda y;q^2) sin(q^{\frac{1}{2}}\lambda x;q^2) - sin(q^{\frac{3}{2}}\lambda y;q^2)\cos(\lambda x;q^2). \tag{3.10}$$

Proof. We begin by resolving the following q-homogenous equation $(E_{1,h})$

$$(E_{1,h}): \qquad D^2_{q,x} U(x,\lambda^2;q^2) + \lambda^2 U(qx,\lambda^2;q^2) = 0.$$

For this way, if we use the same steps, given in proposition 2.3, we obtain

$$\varphi_{1,h}(x,\lambda^2;q^2) = a\cos(\lambda x;q^2) + b\frac{q^{-\frac{1}{2}}}{\lambda}\sin(q^{\frac{1}{2}}\lambda x;q^2) \quad ; a,b \in \mathbb{R}.$$

Now we are able to give a particular solution $\varphi_p(x,\lambda^2;q^2)$ of (E_1) . For deeps, we use the q-Method of variation of constant. Hence we write $\varphi_p(x,\lambda^2;q^2)$ in the following form

$$\varphi_p(x,\lambda^2;q^2) = a(x,\lambda^2;q^2)\cos(\lambda x;q^2) + b(x,\lambda^2;q^2)\frac{q^{-\frac{1}{2}}}{\lambda}\sin(q^{\frac{1}{2}}\lambda x;q^2)$$

Therefore, if we replace $D_{q,x}\varphi(x,\lambda^2;q^2)$ and $D^2_{q,x}\varphi(x,\lambda^2;q^2)$ in (E_1), it can be rewritten in the form

$$D_{q,x}\left[I_1\right] + I_2 = p(x)\varphi(x,\lambda^2;q^2), \tag{3.11}$$

where

$$I_1 = D_{q,x}a(x,\lambda^2;q^2)\cos(q\lambda x;q^2) + D_{q,x}b(x,\lambda^2;q^2)\sin(q^{\frac{3}{2}}\lambda x;q^2). \tag{3.12}$$

and

$$I_2 = D_{q,x}a(x,\lambda^2;q^2)D_{q,x}[\cos(\lambda x;q^2)] + D_{q,x}b(x,\lambda^2;q^2)D_{q,x}[\sin(q^{\frac{1}{2}}\lambda x;q^2)]. \tag{3.13}$$

On the other hand, if we use (2.24) and (2.25), we obtain the following system

$$\begin{cases} D_{q,x}a(x,\lambda^2;q^2)\cos(q\lambda x;q^2) + D_{q,x}b(x,\lambda^2;q^2)sin(q^{\frac{3}{2}}\lambda x;q^2) \\ = 0 \\ D_{q,x}a(x,\lambda^2;q^2)D_{q,x}[\cos(\lambda x;q^2)] + D_{q,x}b(x,\lambda^2;q^2)D_{q,x}[\sin(q^{\frac{1}{2}}\lambda x;q^2)] \\ = p(x)\varphi(x,\lambda^2;q^2) \end{cases} \tag{3.14}$$

Hence ,

$$\begin{aligned} D_{q,x}a(x,\lambda^2;q^2) &= \frac{1}{W(x,\lambda^2;q^2)}\begin{vmatrix} 0 & sin(q^{\frac{3}{2}}\lambda x;q^2) \\ p(x)\varphi(x,\lambda^2;q^2) & q^{\frac{1}{2}}\lambda\cos(q^{\frac{1}{2}}\lambda x;q^2) \end{vmatrix} \\ &= -\frac{p(x)\varphi(x,\lambda^2;q^2)\sin(q^{\frac{3}{2}}\lambda x;q^2)}{W(x,\lambda^2;q^2)}, \end{aligned}$$

where $W(x,\lambda^2;q^2)$ is given by (2.36). So by proposition 2.1 and proposition 2.3 we obtain respectively that

$$W(x,\lambda^2;q^2) = \lambda \tag{3.15}$$

and

$$a(x,\lambda^2;q^2) = -\frac{1}{\lambda}\int_0^x p(y)\varphi(y,\lambda^2;q^2)\sin(q^{\frac{3}{2}}\lambda y;q^2)d_qy. \tag{3.16}$$

In a similar way, we can show that

$$b(x,\lambda^2;q^2) = \frac{1}{\lambda}\int_0^x p(y)\varphi(y,\lambda^2;q^2)\cos(q\lambda y;q^2)d_qy. \tag{3.17}$$

Then the particular solution $\varphi_p(x,\lambda^2;q^2)$ of (E_1) is given by

$$\varphi_p(x,\lambda^2;q^2)=\frac{1}{\lambda}\int_0^x p(y)\varphi(y,\lambda^2;q^2)G(x,y,\lambda^2;q^2)d_qy, \tag{3.18}$$

where

$$G(x,y,\lambda^2;q^2)=cos(q\lambda y;q^2)sin(q^{\frac{1}{2}}\lambda x;q^2)-sin(q^{\frac{3}{2}}\lambda y;q^2)\cos(q\lambda x;q^2). \tag{3.19}$$

Furthermore, by the fact that

$$\varphi(0,\lambda^2)=a=q^{-1}sin(q\alpha;q^2)$$

and

$$D_q\varphi(0;q^2)=b=cos(q\alpha;q^2)$$

we can deduce the result.

Proposition 3.6. *Let $p(x)$ a boundary function on $\mathbb{R}_{q,+}$. Then $\varphi(x,\lambda^2;q^2)$ verifies:*

1. *For $\lambda\in\mathbb{R}_{q,+}$,*

$$\varphi(x,\lambda^2;q^2)=\mathcal{O}\left(e(C_\lambda(1-q)x;q^2)\right) \tag{3.20}$$

where

$$C_\lambda=\frac{2\parallel p\parallel_{q,\infty}}{\mid\lambda\mid(q;q^2)^2_\infty} \tag{3.21}$$

and $e(x;q^2)$ is given by (2.29).

2. *Additionally, if $\lambda\longrightarrow\infty$, we have*

$$\varphi(x,\lambda^2;q^2)=\mathcal{O}(1;q^2). \tag{3.22}$$

Proof. To prove the first result, it suffices to use theorem 3.5, (2.26) and (2.27). Therefore,

$$\begin{aligned}\mid\varphi(x,\lambda^2;q^2)\mid &\le \frac{\mid a\mid}{(q;q^2)^2_\infty}+\frac{\mid b\mid}{q\mid\lambda\mid(q;q^2)^2_\infty}\\ &\quad+\frac{2}{\mid\lambda\mid(q;q^2)^2_\infty}\int_0^x\mid\varphi(y,\lambda^2;q^2)\mid\mid p(y)\mid d_qy,\\ &\le \frac{\mid a\mid}{(q;q^2)^2_\infty}+\frac{\mid b\mid}{q|\lambda|(q;q^2)^2_\infty}\\ &\quad+\frac{2}{\mid\lambda\mid(q;q^2)^2_\infty}\parallel p\parallel_{q,\infty}\int_0^x\mid\varphi(y,\lambda^2;q^2)\mid d_qy,\\ \mid\varphi(x,\lambda^2;q^2)\mid &\le A_\lambda+C_\lambda\int_0^x\mid\varphi(y,\lambda^2;q^2)\mid d_qy,\end{aligned}$$

where

$$A_\lambda = \frac{| a |}{(q;q^2)^2_\infty} + \frac{| b |}{q | \lambda | (q;q^2)^2_\infty}$$

and

$$C_\lambda = \frac{\| p \|_{q,\infty}}{| \lambda | (q;q^2)^2_\infty}$$

By proposition 3.2, we obtain that

$$| \varphi(x,\lambda^2;q^2) | \leq A_\lambda e(C_\lambda(1-q)x;q^2). \tag{3.23}$$

and the result follows immediately.

Theorem 3.7. *For $p(x)$ in $L^\infty_q(\mathbb{R}_{q,+}) \cap L^1_q(\mathbb{R}_{q,+})$, we have*

1. *For $\lambda \geq \xi > 0$, $\varphi(x,\lambda^2;q^2)$ is an uniformly bounded function.*

2. *For a large λ , we have the following estimation*

$$\varphi(x,\lambda^2;q^2) = \mu(\lambda^2;q^2)\cos(\lambda x;q^2) + \nu(\lambda^2;q^2)q^{-\frac{1}{2}}\sin(q^{\frac{1}{2}}\lambda x;q^2) + \mathcal{O}(1;q^2) \tag{3.24}$$

where

$$\mu(\lambda^2;q^2) = q^{-1}\sin(q\alpha;q^2) - \frac{1}{\lambda}\int_0^\infty sin(q^{\frac{3}{2}}\lambda y;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy \tag{3.25}$$

and

$$\nu(\lambda^2;q^2) = \frac{\cos(q\alpha;q^2)}{\lambda} + \frac{q^{\frac{1}{2}}}{\lambda}\int_0^\infty \cos(q\lambda y;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy. \tag{3.26}$$

Proof. The first result follows immediately by proposition 3.6.
Proving the second relation, by theorem 3.5 we have

$$\begin{aligned}\varphi(x,\lambda^2;q^2) &= q^{-1}\sin(q\alpha;q^2)\cos(\lambda x;q^2) - q^{-\frac{1}{2}}\frac{\cos(q\alpha;q^2)}{\lambda}\sin(q^{\frac{1}{2}}\lambda x;q^2)\\ &+\frac{1}{\lambda}\int_0^\infty G(x,y,\lambda^2;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy\\ &-\frac{1}{\lambda}\int_x^\infty G(x,y,\lambda^2;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy\end{aligned}$$

Taking account of the fact that p in $L^1_q(\mathbb{R}_{q,+})$ and by (2.26) , (2.27)

$$\begin{aligned} \mid \int_x^\infty G(x,y,\lambda^2;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy \mid &\leq C_q \int_x^\infty \mid p(y)\varphi(y,\lambda^2;q^2) \mid d_qy \\ &\leq C_q \parallel \varphi \parallel_{q,\infty} \int_0^\infty \mid p(y) \mid d_qy \\ &< +\infty. \end{aligned}$$

leads to the result.

On the same way, if we note by $\theta(x,\lambda^2;q^2)$ the solution of the q-difference problem

$$(E_2) = \begin{cases} L_qU = -\lambda^2 U; \\ U(0,\lambda^2;q^2) = q^{\frac{1}{2}} \cos(q^{\frac{1}{2}}\alpha;q^2); \\ D_qU(0,\lambda^2;q^2) = -\sin(q^{\frac{3}{2}}\alpha;q^2) \quad ,\alpha \in \mathbb{C}. \end{cases} \tag{3.27}$$

We show that

$$\theta(x,\lambda^2;q^2) = \mu_1(\lambda^2;q^2)\cos(\lambda x;q^2) + \nu_1(\lambda^2;q^2)q^{-\frac{1}{2}}\sin(q^{\frac{1}{2}}\lambda x;q^2) + \mathcal{O}(1;q^2), \tag{3.28}$$

where

$$\mu_1(\lambda^2;q^2) = q^{\frac{1}{2}}\cos(q^{\frac{1}{2}}\alpha;q^2) - \frac{1}{\lambda}\int_0^\infty sin(q^{\frac{3}{2}}\lambda y;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy \tag{3.29}$$

and

$$\nu_1(\lambda^2;q^2) = -q^{\frac{1}{2}}\frac{\cos(q\alpha;q^2)}{\lambda} - q^{\frac{1}{2}}\frac{\sin(q^{\frac{3}{2}}\alpha;q^2)}{\lambda}\int_0^\infty \cos(q\lambda y;q^2)p(y)\varphi(y,\lambda^2;q^2)d_qy. \tag{3.30}$$

Theorem 3.8. *For λ in $\mathbb{R}_{q,+}$, we have:*

$$\mu(\lambda^2;q^2)\nu_1(\lambda^2;q^2) - \nu(\lambda^2;q^2)\mu_1(\lambda^2;q^2) = \frac{1}{q^{\frac{1}{2}}\lambda}. \tag{3.31}$$

Proof. Using theorem 3.7, we obtain that

$$\varphi(x,\lambda^2;q^2) = \mu(\lambda^2;q^2)\cos(\lambda x;q^2) + \nu(\lambda^2;q^2)q^{-\frac{1}{2}}\sin(q^{\frac{1}{2}}\lambda x;q^2) + \mathcal{O}(1;q^2)$$

and

$$\theta(x,\lambda^2;q^2) = \mu_1(\lambda^2;q^2)\cos(\lambda x;q^2) + \nu_1(\lambda^2;q^2)q^{-\frac{1}{2}}\sin(q^{\frac{1}{2}}\lambda x;q^2) + \mathcal{O}(1;q^2).$$

We can deduce that

$$D_{q,x}\varphi(x,\lambda^2;q^2) = -q^{-1}\lambda\mu(\lambda^2;q^2)\sin(q\lambda x;q^2)+\lambda\nu(\lambda^2;q^2)\cos(q^{\frac{1}{2}}\lambda x;q^2)+\mathcal{O}(1;q^2)$$

and

$$D_{q,x}\theta(x,\lambda^2;q^2) = -\lambda q^{-1}\mu_1(\lambda^2;q^2)\sin(q\lambda x;q^2)+\lambda\nu_1(\lambda^2;q^2)\cos(q^{\frac{1}{2}}\lambda x;q^2)+\mathcal{O}(1;q^2),$$

then, the use of definition 3.3 leads to

$$\begin{aligned}
[\varphi,\theta]_q(x) &= \varphi(qx;q^2)D_{q,x}\theta(x;q^2) - \theta(qx;q^2)D_{q,x}\varphi(x;q^2)\\
&= \lambda\big[\nu(\lambda^2;q^2)\mu_1(\lambda^2;q^2) - \mu(\lambda^2;q^2)\nu_1(\lambda^2;q^2)\big]\big[\cos(q\lambda x;q^2)\cos(q^{\frac{1}{2}}\lambda x;q^2)\\
&+ q^{-\frac{3}{2}}\sin(q^{\frac{3}{2}}\lambda x;q^2)\sin(q\lambda x;q^2)\big] + \mathcal{O}(1;q^2).
\end{aligned}$$

In the other side, the fact that φ and θ are solutions of (E) and by proposition 3.4 we can deduce that

$$D_{q,x}\,[\varphi,\theta]_q(x) = 0.$$

Then from proposition 2.1

$$\begin{aligned}
[\varphi,\theta]_q(x) &= [\varphi,\theta]_q(0) = q^{\frac{1}{2}}\cos(q^{\frac{1}{2}}\alpha;q^2)\cos(q\alpha;q^2) + q^{-1}\sin(q\alpha;q^2)\sin(q^{\frac{3}{2}}\alpha;q^2)\\
&= q^{\frac{1}{2}}.
\end{aligned}$$

The result follows.

4. Asymptotic Behaviour of $j_\alpha(\lambda x;q^2)$

In this section, our objective is to establish; using the method of variation of constant given in the last section; the asymptotic expansion of $j_\alpha(\lambda x;q^2)$ when $\lambda \longrightarrow +\infty$.
We recall some properties given in [5]: For $\alpha > -\frac{1}{2}$, the q-Bessel function is defined by:

$$j_\alpha(x;q^2) = \Gamma_{q^2}(\alpha+1)\frac{q^\alpha(1+q)^\alpha}{x^\alpha}J_\alpha((1-q)x;q^2), \tag{4.1}$$

where $J_\alpha(x;q^2)$ is the q-Bessel Han Exton [15], defined by

$$J_\alpha(z;q) = \left(\frac{z}{1-q}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}} q^k}{\Gamma_q(k+1)\Gamma_q(\alpha+k+1)} \left(\frac{z}{1-q}\right)^{2k}. \tag{4.2}$$

This function satisfies the following relations

$$j_{-\frac{1}{2}}(x;q^2) = \cos(x;q^2) \tag{4.3}$$

$$j_{\frac{1}{2}}(x;q^2) = \frac{1}{x}\sin(x;q^2). \tag{4.4}$$

Proposition 4.1. *The function $j_\alpha(\lambda x;q^2)$; λ being complex; is the solution of the following q-difference problem:*

$$(E) = \begin{cases} \Delta_{q,\alpha} y(x) + \lambda^2 y(x) = 0 \\ y(0) = 1 \\ D_q y(0) = 0 \end{cases} \tag{4.5}$$

where $\Delta_{q,\alpha}$ is the q-Bessel operator, defined by

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_{q,x}\left[x^{2\alpha+1} D_{q,x} f\right](q^{-1}x), \tag{4.6}$$

$$= q^{2\alpha+1}\Delta_{q,x} f(x) + \frac{1-q^{2\alpha+1}}{(1-q)q^{-1}x} D_{q,x} f(q^{-1}x) \tag{4.7}$$

and

$$\Delta_{q,x} f(x) = (D_{q,x}^2 f)(q^{-1}x) \tag{4.8}$$

Corollary 4.2. *For $x \in \mathbb{R}_q$ and $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$, we have the following estimation*

$$|j_\alpha(x;q^2)| \leq \frac{1}{(q;q^2)_\infty^2}. \tag{4.9}$$

Now, we consider the following q-Bessel equation given by:

$$(E) \quad D_{q,x}^2 y(x,\lambda^2;q^2) + \frac{\lambda^2}{q^{2\alpha+1}} y(qx,\lambda^2;q^2) = -\frac{1-q^{2\alpha+1}}{q^{2\alpha+1}(1-q)x} D_{q,x} y(x,\lambda^2;q^2).$$

Let $y_h(x,\lambda^2;q^2)=\sum_{n\geq 0} a_n(\alpha,\lambda^2;q^2)x^n$ the homogeneous solution of (E_h) given by

$$(E_h)\quad D^2_{q,x}y(x,\lambda^2;q^2)+\frac{\lambda^2}{q^{2\alpha+1}}y(qx,\lambda^2;q^2)=0.$$

Then, we have

$$D^2_{q,x}y(x,\lambda^2;q^2)=\sum_{n\geq 2}\frac{(1-q^n)(1-q^{n-1})}{(1-q)^2}a_nx^{n-2}=\sum_{n\geq 0}\frac{(1-q^{n+2})(1-q^{n+1})}{(1-q)^2}a_{n+2}x^n.$$

By identification, we obtain the following recurrence relation:

$$\frac{(1-q^{n+2})(1-q^{n+1})}{(1-q)^2}a_{n+2}=-\frac{\lambda^2q^n}{q^{2\alpha+1}}a_n. \tag{4.10}$$

If we proceed in a similar way of proposition 2.3, we can deduce easily that the homogeneous solution $y_h(x,\lambda^2;q^2)$ is given by

$$y_h(x,\lambda^2;q^2)=a_0\cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2)+a_1\frac{q^{-\alpha-1}}{\lambda}\sin(q^{-\alpha}\lambda x;q^2), \tag{4.11}$$

where a_0,a_1 are constants in $\mathbb{R}$.
Now, we give a particular solution $y_p(x,\lambda^2;q^2)$ of (E) in the form

$$\begin{aligned} y_p(x,\lambda^2;q^2) &= a_\alpha(x,\lambda^2;q^2)\cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2) \\ &\quad +b_\alpha(x,\lambda^2;q^2)\frac{q^{-\alpha-1}}{\lambda}\sin(q^{-\alpha}\lambda x;q^2). \end{aligned} \tag{4.12}$$

The q-wronskian is given by

$$\begin{aligned} W(x,\lambda^2;q^2) &= D_{q,x}\left[\cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2)\right]\frac{q^{-\alpha-1}}{\lambda}\sin(q^{-\alpha}\lambda x;q^2) \\ &\quad -\cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2)D_{q,x}\left[\frac{q^{-\alpha-1}}{\lambda}\sin(q^{-\alpha}\lambda x;q^2)\right], \end{aligned}$$

thus, using corollary 2.4 we can show that

$$D_{q,x}W(x,\lambda^2;q^2)=0,$$

therefore

$$W(x,\lambda^2;q^2) = -q^{-2\alpha-1}\lambda. \tag{4.13}$$

Using the q-method of variation of constant (given in theorem 3.5) and proposition 2.1 we deduce that

$$a_\alpha(x,\lambda^2;q^2) = \frac{q^{-\alpha-1}}{\lambda}\int_x^{+\infty}\frac{1-q^{2\alpha+1}}{1-q}\frac{1}{t}D_{q,t}y(t,\lambda^2;q^2)\sin(q^{-\alpha+1}\lambda t;q^2)d_qt \tag{4.14}$$

$$b_\alpha(x,\lambda^2;q^2) = -\frac{1}{\lambda}\int_x^{+\infty}\frac{1-q^{2\alpha+1}}{1-q}\frac{1}{t}D_{q,t}y(t,\lambda^2;q^2)\cos(q^{-\alpha+\frac{1}{2}}\lambda t;q^2)d_qt \tag{4.15}$$

This leads to the following result

Proposition 4.3. *For $x,\lambda\in\mathbb{R}_{q,+}$, the solution $y(x,\lambda^2;q^2)$ of (E) is given by:*

$$\begin{aligned} y(x,\lambda^2;q^2) &= a\cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2) + \frac{b}{\lambda}q^{-\alpha-1}\sin(q^{-\alpha}\lambda x;q^2) \\ &+ \frac{q^{-\alpha-1}}{\lambda}\int_x^{+\infty}G_\alpha(t,x,\lambda;q^2)\frac{1-q^{2\alpha+1}}{1-q}\frac{1}{t}D_{q,t}y(t,\lambda^2;q^2)d_qt \end{aligned}$$

where

$$G_\alpha(t,x,\lambda;q^2) = \cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2)\sin(q^{-\alpha+1}\lambda t) - \cos(q^{-\alpha+\frac{1}{2}}\lambda t;q^2)\sin(q^{-\alpha}\lambda x;q^2) \tag{4.16}$$

On the other hand $j_\alpha(\lambda x;q^2)$ is the unique solution of (E) with initial conditions

$$\begin{aligned} j_\alpha(0;q^2) &= 1 \\ D_q j_\alpha(0;q^2) &= 0. \end{aligned}$$

Therefore, we can write $j_\alpha(\lambda x;q^2)$ as the following form

Theorem 4.4. *For $\lambda\in\mathbb{R}_{q,+}$ and $\alpha > -\frac{1}{2}$, we have*

1.

$$j_\alpha(\lambda x;q^2) = \cos(q^{-\alpha-\frac{1}{2}}\lambda x;q^2) + R_{q,\alpha}(x,\lambda^2), \tag{4.17}$$

where

$$R_{q,\alpha}(x,\lambda^2) = -\frac{1}{\lambda}\frac{1-q^{2\alpha+1}}{1-q}\int_x^{+\infty}G_\alpha(t,x,\lambda;q^2)\frac{1}{t}D_{q,t}j_\alpha(\lambda t;q^2)d_qt$$

2. Additionally, $R_{q,\alpha}(x,\lambda^2)$ *tend to* 0 *when* λ *tend to* $+\infty$.

Proof. To prove 2. it suffices to use (2.26), (2.27) and corollary4.2, then

$$\begin{aligned}
|R_{q,\alpha}(x,\lambda^2)| &\leq \frac{1}{\lambda}\frac{1-q^{2\alpha+1}}{1-q}\frac{2}{(q;q^2)^2_\infty}\int_x^{+\infty} \mid \frac{D_{q,t}j_\alpha(\lambda t;q^2)}{t} \mid d_qt \\
&\leq \frac{1-q^{2\alpha+1}}{1-q}\frac{2}{(q;q^2)^2_\infty}\int_{\lambda x}^{+\infty} \mid \frac{D_{q,t}j_\alpha(t;q^2)}{t} \mid d_qt \\
&\leq \frac{1-q^{2\alpha+1}}{1-q}\frac{2}{(q;q^2)^2_\infty}\int_{\lambda x}^{+\infty} \frac{|j_\alpha(t;q^2)-j_\alpha(qt;q^2)|}{(1-q)t^2}d_qt \\
&\leq \frac{1-q^{2\alpha+1}}{(1-q)^2}\Big[\frac{2}{(q;q^2)^2_\infty}\Big]^2\int_{\lambda x}^{+\infty} \frac{d_qt}{t^2} \\
&\leq \frac{C_q}{\lambda x} \longrightarrow 0 \quad ,\lambda \longrightarrow \infty
\end{aligned}$$

where

$$C_q = \frac{1-q^{2\alpha+1}}{q}\Big[\frac{2}{(1-q)(q;q^2)^2_\infty}\Big] \tag{4.18}$$

and the result follows immediately.

4.1 Application

We recall the q-equality of Weber integral study in [5].
For a,λ in $\mathbb{R}_{q,+}$ and $\alpha > -1$ we have

$$\frac{1}{A_\alpha}\int_0^\infty e(-a^2x^2;q^2)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx = \frac{1}{a^{2\alpha+2}}e(-\frac{\lambda^2}{a^2(1+q)^2};q^2), \tag{4.19}$$

where

$$A_\alpha = \int_0^\infty \frac{x^{2\alpha+1}}{(-(1-q^2)x^2;q^2)}d_qx \tag{4.20}$$

and if we take $a=\sqrt{t}$ we obtain

$$E_\alpha(t,\lambda^2;q^2) = \int_0^\infty e(-q^{-1}tx^2;q^2)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx \tag{4.21}$$

where

$$E_\alpha(t,\lambda^2;q^2) = \frac{A_\alpha}{t^{\alpha+1}}e(-\frac{\lambda^2}{(1+q)^2t};q^2). \tag{4.22}$$

Proposition 4.5. *For $\alpha > -1$ and $\lambda, t \in \mathbb{R}_{q,+}$, the heat kernel $E_\alpha(t, \lambda; q^2)$ has the following behaviour:*

$$E_\alpha(t, \lambda^2; q^2) = \int_0^\infty e(-q^{-1}tx^2; q^2)\cos(q^{-\alpha-\frac{1}{2}}\lambda x; q^2)x^{2\alpha+1}d_qx + \Theta_\alpha(\lambda, t; q^2) \tag{4.23}$$

where

$$\Theta_\alpha(\lambda, t; q^2) \longrightarrow 0 \quad , \lambda \to \infty. \tag{4.24}$$

Proof.the result follows by proposition 4.4 and the fact that

$$\begin{aligned} \mid \int_0^\infty e(-q^{-1}tx^2; q^2)R_q(\lambda^2, x)x^{2\alpha+1}d_qx \mid &\le \frac{C_q}{\lambda^2}\int_0^\infty e(-q^{-1}tx^2; q^2)x^{2\alpha}d_qx \\ &= \frac{C_q}{\lambda^2}(1-q)\sum_{-\infty}^{+\infty}\frac{q^{(2\alpha+1)k}}{(-q^{-1}(1-q^2)tq^{2k}; q^2)_\infty} \\ &= \frac{C_q}{\lambda^2}(1-q)\sum_{-\infty}^{+\infty}\frac{q^{2\beta k}}{(aq^{2k}; q^2)_\infty}, \end{aligned}$$

where $\beta = \alpha + \frac{1}{2}$ and $a = -q^{-1}(1-q^2)t$. The use of Ramanujan's sum (see [7]) leads to

$$\mid \int_0^\infty e(-q^{-1}tx^2; q^2)R_q(\lambda^2, x)x^{2\alpha+1}d_qx \mid \ \le \ \frac{C_q}{\lambda^2}B_\alpha(t; q^2) \longrightarrow 0 \quad , \lambda \to \infty,$$

where

$$B_\alpha(t; q^2) = \frac{(-q^{2\alpha}t(1-q^2), q^{2-\alpha}t^{-1}(1-q^2)^{-1}, q^2; q^2)_\infty}{(q^{2\alpha+1}, -q^{-1}t(1-q^2), -q^3t^{-1}(1-q^2)^{-1}; q^2)_\infty}. \tag{4.25}$$

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