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On an Inequality Related to Hadamard*

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Abstract

In the present note we establish a new inequality which in special case, reduces to a part of Hadamard's inequality.

Keywords and Phrases: *Hadamard's inequality, R-convex function, Integral power mean, Logarithmic mean, Stolarsky mean.*

1. Introduction

The inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}$$

which hold for all convex functions $f:[a,b] \to R$ are known in the literature as Hadamard's inequalities.

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If f is a positive function on $[a,b]$ such that for all $x, y \in [a,b]$ and $\lambda \in [0,1]$

$$f(\lambda x+(1-\lambda)y) \le \begin{cases} (\lambda f(x)^r + (1-\lambda) f(y)^r)^{1/r} & , \text{ if } r \ne 0 \\ f(x)^{\lambda} f(y)^{(1-\lambda)} & , \text{ if } r = 0 \end{cases}$$

then we say that f is r-convex.

The definition of r-convex naturally complements the concept of r-concavity, in which the inequality is reversed (cf. Uhrin [5]) and which plays an important role in statistics.

We note that 0-convex function is simply log-convex function and 1-convex function is ordinary convex function.

We recall that：

(I) If f is positive integral function on $[a,b]$, then the integral power mean $M_p(f)$ of f is define by

$$M_p(f) = \begin{cases} \left[\dfrac{1}{b-a} \displaystyle\int_a^b f(x)^p \, dx \right]^{1/p} & , \ \text{p} \ne 0; \\ \exp\left[\dfrac{1}{b-a} \displaystyle\int_a^b \ln f(x) \, dx \right] & , \ p = 0. \end{cases}$$

(II) The extended logarithmic mean L_p of two distinct positive numbers a,b is define by

$$L_P(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p} & , \ p \ne -1, 0; \\ \dfrac{b-a}{\ln b - \ln a} & , \ p = -1; \\ \dfrac{1}{e} \left(\dfrac{b^b}{a^a} \right)^{\frac{1}{(b-a)}} & , \ p = 0. \end{cases}$$

and $L_P(a,a) = a$.

(III) The alternative extended logarithmic mean $F_r(a,b)$ of two distinct positive numbers a,b is define by

$$F_r(a,b)=\begin{cases}\dfrac{r}{r+1}\cdot\dfrac{b^{r+1}-a^{r+1}}{b^r-a^r} & , r\neq 0,-1;\\ \dfrac{b-a}{\ln b-\ln a} & , r=0;\\ xy\left(\dfrac{\ln b-\ln a}{b-a}\right) & , r=-1,\end{cases}$$

and $F_r(a,a)=a$.

(IV) The Stolarsky mean $E(x,y,r,s)$ (see [4]) of two distinct positive numbers x,y is define by

$$E(x,y,r,s)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{\frac{1}{(s-r)}} \quad ,r\neq s \text{ and } rs\neq 0;$$

$$E(x,y,r,0)=E(x,y,0,r)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{\frac{1}{r}} \quad ,r\neq 0;$$

$$E(x,y,r,r)=\left[\exp\left(\frac{-1}{r}\right)\right]\cdot\left(\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}\right) \quad ,r\neq 0;$$

$$E(x,y,0,0)=\sqrt{xy},$$

and

$$E(x,y,r,s)=x \text{ if } x=y>0.$$

If q is a positive integral function on $[a,b]$, then

(V) The power mean of f on $[a,b]$ with respect to q is defined by

$$M_p(f,q)=\begin{cases}\left[\dfrac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot f(x)^p\,dx\right]^{1/p} & , p\neq 0;\\ \exp\left[\dfrac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln f(x)dx\right] & , p=0,\end{cases}$$

and

$$M_p(f,1)=M_p(f).$$

(VI) The Stolarsky mean $E(a,b,r,s,q)$ of two distinct positive numbers a,b with respect to q is defined by

$$E(a,b,r,s,q)=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\frac{x-a}{b-a}b^r+\frac{b-x}{b-a}a^r\right]^{s-r/r}dx\right\}^{1/s-r}\quad ,r\neq s\ and\ rs\neq 0,$$

$$E(a,b,0,r,q)=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[b^{\frac{x-a}{b-a}}\cdot a^{\frac{b-x}{b-a}}\right]^{r}dx\right\}^{1/r}\quad ,\ r\neq 0,$$

$$E(a,b,r,0,q)=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\frac{x-a}{b-a}b^r+\frac{b-x}{b-a}a^r\right]^{-1}dx\right\}^{-1/r}\quad ,\ r\neq 0,$$

$$E(a,b,r,r,q)=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[\frac{x-a}{b-a}b^r+\frac{b-x}{b-a}a^r\right]^{1/r}dx\right\}\quad ,\ r\neq 0,$$

$$E(a,b,0,0,q)=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[b^{\frac{x-a}{b-a}}\cdot a^{\frac{b-x}{b-a}}\right]dx\right\}\quad ,\ r=0,p=0,$$

$$E(a,b,r,s,q)=a\ if\ a=b>0,$$

and

$$E(a,b,r,s,1)=E(a,b,r,s).$$

The following are extensions of Hadamard's inequality:
C. E. M. Pearce and J. Pečarić. ([2]) proved the following theorem.

Theorem A. *If* $f:[a,b]\to R$ *is positive, continuous and convex, then*

$$M_p(f)\le L_p(f(a),f(b))$$

while if f is positive r-concave, the inequality is reversed.

We note that if $p=1$, then

$$M_1(f) = \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

P. M. Gill, C. E. M. Pearce and J. Pečarić. ([1]) proved the following two theorems

Theorem B. *If $f:[a,b] \to R$ is positive r-convex, then*

$$\frac{1}{b-a}\int_a^b f(t)\ dt \le F_r(f(a), f(b)).$$

while if f is positive r-concave, the inequality is reversed.

We note that if $r = 1$ then

$$\frac{1}{b-a}\int_a^b f(t)\ dt \le F_1(f(a), f(b)) = \frac{f(a)+f(b)}{2}.$$

Theorem C. *If f is a positive r-convex function on $[a,b]$, then*

$$M_p(f) \le E(f(a), f(b), r, p+r)$$

while if f is r-concave, the inequality is reversed.

We note that Theorem C reduces to Theorem A and Theorem B when $r = 1$ *and* $p = 1$, respectively.

The main purpose of this note is to establish a Generalizations of **Theorem C.**

2. Main Results

Theorem *If f is positive r-convex function on $[a,b]$, q is a positive integrable function and $G(t,q):[0,1] \to R$ is defined by*

$$G(t,q)=\begin{cases}\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\frac{b-x}{b-a}f\left(ta+(1-t)x\right)^r+\frac{x-a}{b-a}f\left(tb+(1-t)x\right)^r\right]^{p/r}dx\right\}^{1/p}, & r\neq 0,\ p\neq 0;\\ \left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[f\left(ta+(1-t)x\right)^{\frac{b-x}{b-a}}+f\left(tb+(1-t)x\right)^{\frac{x-a}{b-a}}\right]^{p}dx\right\}^{1/p}, & r=0,\ p\neq 0;\\ \exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[\frac{b-x}{b-a}f\left(ta+(1-t)x\right)^r+\frac{x-a}{b-a}f\left(tb+(1-t)x\right)^r\right]^{1/r}dx\right\}, & r\neq 0,\ p=0;\\ \exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[f\left(ta+(1-t)x\right)^{\frac{b-x}{b-a}}+f\left(tb+(1-t)x\right)^{\frac{x-a}{b-a}}\right]dx\right\}, & r=0,\ p=0,\end{cases}$$

then

(i) $G(t,q)$ is monotonically increasing on $[0,1]$,

(ii) $G(0,q)=M_P(f,q)$ *and* $G(1,q)=E(f(a),f(b),r,p+r,q)$.

Proof. Let $x\in[a,b]$ and $0\leq s<t\leq 1$ Then

$$sa+(1-s)x=\frac{[bt-as+sx-tx]}{t(b-a)}[ta+(1-t)x]+\frac{[as-at+tx-sx]}{t(b-a)}[tb+(1-t)x]\quad\ldots(1)$$

$$sb+(1-s)x=\frac{[bt-bs+sx-tx]}{t(b-a)}[ta+(1-t)x]+\frac{[bs-at+tx-sx]}{t(b-a)}[tb+(1-t)x]\quad\ldots(2)$$

Case 1. For $r\neq 0$ and $p\neq 0$, it follows from (1) , (2) and the r-convexity of f that

$$G(s,q)=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\frac{b-x}{b-a}f\left(sa+(1-s)x\right)^r+\frac{x-a}{b-a}f\left(sb+(1-s)x\right)^r\right]^{p/r}dx\right\}^{1/p}$$

$$\leq\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\begin{array}{l}\frac{b-x}{b-a}\left[\frac{[bt-as+sx-tx]}{t(b-a)}f\left(ta+(1-t)x\right)^r+\frac{[as-at+tx-sx]}{t(b-a)}f\left(tb+(1-t)x\right)^r\right]\\+\frac{x-a}{b-a}\left[\frac{[bt-bs+sx-tx]}{t(b-a)}f\left(ta+(1-t)x\right)^r+\frac{[bs-at+tx-sx]}{t(b-a)}f\left(ta+(1-t)x\right)^r\right]\end{array}\right]^{p/r}dx\right\}^{1/p}$$

$$=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\frac{b-x}{b-a}f\left(ta+(1-t)x\right)^r+\frac{x-a}{b-a}f\left(tb+(1-t)x\right)^r\right]^{p/r}dx\right\}^{1/p}$$
$$=G(t,q)$$

Case 2. If $r=0$ *and* $p\neq 0$, then f is log-convex , it follows from (1) and (2) that

$$G(s,q)=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[f\left(sa+(1-s)x\right)^{\frac{b-x}{b-a}}\cdot f\left(sb+(1-s)x\right)^{\frac{x-a}{b-a}}\right]^p dx\right\}^{1/p}$$

$$\leq\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\begin{array}{l}\left[f\left(ta+(1-t)x\right)^{\frac{[bt-as+sx-tx]}{t(b-a)}\cdot\frac{b-x}{b-a}}\cdot f\left(tb+(1-t)x\right)^{\frac{[as-at+tx-sx]}{t(b-a)}\cdot\frac{b-x}{b-a}}\right]\\\cdot\left[f\left(ta+(1-t)x\right)^{\frac{[bt-bs+sx-tx]}{t(b-a)}\cdot\frac{x-a}{b-a}}\cdot f\left(tb+(1-t)x\right)^{\frac{[bs-at+tx-sx]}{t(b-a)}\cdot\frac{x-a}{b-a}}\right]\end{array}\right]^p dx\right\}^{1/p}$$

$$=\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[f\left(ta+(1-t)x\right)^{\frac{b-x}{b-a}}\cdot f\left(tb+(1-t)x\right)^{\frac{x-a}{b-a}}\right]^p dx\right\}^{1/p}$$
$$=G(t,q)$$

Case 3. If $r \neq 0$ and $p = 0$, using (1) and (2), we have

$$G(s,q)=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[\frac{b-x}{b-a}f\left(sa+(1-s)x\right)^r+\frac{x-a}{b-a}f\left(sb+(1-s)x\right)^r\right]^{1/r}dx\right\}$$

$$\leq\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[\begin{array}{l}\frac{b-x}{b-a}\left[\frac{[bt-as+sx-tx]}{t(b-a)}f\left(ta+(1-t)x\right)^r+\frac{[as-at+tx-sx]}{t(b-a)}f\left(tb+(1-t)x\right)^r\right]\\+\frac{x-a}{b-a}\left[\frac{[bt-bs+sx-tx]}{t(b-a)}f\left(ta+(1-t)x\right)^r+\frac{[bs-at+tx-sx]}{t(b-a)}f\left(ta+(1-t)x\right)^r\right]\end{array}\right]^{1/r}dx\right\}$$

$$=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\left[\frac{b-x}{b-a}f\left(ta+(1-t)x\right)^r+\frac{x-a}{b-a}f\left(tb+(1-t)x\right)^r\right]^{1/r}dx\right\}$$

$$=G(t,q)$$

Case 4. If $r = 0$ and $p \neq 0$, using (1) and (2), we have

$$G(s,q)=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[f\left(sa+(1-s)x\right)^{\frac{b-x}{b-a}}\cdot f\left(sb+(1-s)x\right)^{\frac{x-a}{b-a}}\right]dx\right\}$$

$$\leq\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[\begin{array}{l}\left[f\left(ta+(1-t)x\right)^{\frac{[bt-as+sx-tx]}{t(b-a)}\cdot\frac{b-x}{b-a}}\cdot f\left(tb+(1-t)x\right)^{\frac{[as-at+tx-sx]}{t(b-a)}\cdot\frac{b-x}{b-a}}\right]\\\cdot\left[f\left(ta+(1-t)x\right)^{\frac{[bt-bs+sx-tx]}{t(b-a)}\cdot\frac{x-a}{b-a}}\cdot f\left(tb+(1-t)x\right)^{\frac{[bs-at+tx-sx]}{t(b-a)}\cdot\frac{x-a}{b-a}}\right]\end{array}\right]dx\right\}$$

$$=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[f\left(ta+(1-t)x\right)^{\frac{b-x}{b-a}}\cdot f\left(tb+(1-t)x\right)^{\frac{x-a}{b-a}}\right]dx\right\}$$

$$=G(t,q)$$

This completes the proof of (i).

To prove (ii) we observe first that

$$G(0,q) = M_p(f,q).$$

To prove $G(1,q) = E(f(a), f(b), r, p+r, q)$, suppose first $f(a) = f(b)$. Then it is obviously that $G(1,q) = E(f(a), f(b), r, p+r, q)$, so that we may assume $f(a) \neq f(b)$.

Case 1. If $r \neq 0$ *and* $p \neq 0$, then we have

(i) If $r + p \neq 0$, then

$$G(1,q) = \left\{ \frac{1}{\int_a^b q(x)dx} \int_a^b q(x) \cdot \left[\frac{b-x}{b-a} f(a)^r + \frac{x-a}{b-a} f(b)^r \right]^{p/r} dx \right\}^{1/p}$$

$$= E\left(f(a), f(b), r, p+r, q\right)$$

(ii) If $r + p = 0$, then

$$G(1,q) = \left\{ \frac{1}{\int_a^b q(x)dx} \int_a^b q(x) \cdot \left[\frac{b-x}{b-a} f(a)^r + \frac{x-a}{b-a} f(b)^r \right]^{-1} dx \right\}^{-1/r}$$

$$= E\left(f(a), f(b), r, 0, q\right)$$

$$= E\left(f(a), f(b), r, r+p, q\right).$$

Case 2. If $r = 0$ *and* $p \neq 0$, then

$$G(1,q) = \left\{ \frac{1}{\int_a^b q(x)dx} \int_a^b q(x) \cdot \left[f(a)^{\frac{b-x}{b-a}} + f(b)^{\frac{x-a}{b-a}} \right]^p dx \right\}^{1/p}$$

$$= E\left(f(a), f(b), 0, p, q\right)$$

$$= E\left(f(a), f(b), r, p+r, q\right).$$

Case 3. If $r \neq 0$ *and* $p = 0$, then

$$G(1,q)=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[\frac{b-x}{b-a}f(a)^r+\frac{x-a}{b-a}f(b)^r\right]^{1/r}dx\right\}$$
$$=E\left(f(a),f(b),r,r,q\right)$$
$$=E\left(f(a),f(b),r,p+r,q\right).$$

Case 4. If $r=0$ *and* $p=0$, then

$$G(1,q)=\exp\left\{\frac{1}{\int_a^b q(x)dx}\int_a^b q(x)\cdot\ln\left[f(a)^{\frac{b-x}{b-a}}+f(b)^{\frac{x-a}{b-a}}\right]dx\right\}$$
$$=E\left(f(a),f(b),0,0,q\right)$$
$$=E\left(f(a),f(b),r,p+r,q\right).$$

This completes the proof of (ii).

Remark 1. *If f is a positive r-concave function, then* $G(t,q)$ *is monotonically decreasing on* $[0,1]$.

Remark 2. *If* $q(x)\equiv 1$, *then our Theorem reduces to Theorem C.*

Remark 3. *If* $q(x)\equiv 1$, *then our Theorem reduces to a theorem proved by Yang and Hwang [6].*

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