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On a Second Order Mock Theta Functions*

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Abstract

In this paper we consider a second order mock theta function, recently given by Hikami. We define a generalized function and show it is a F_q-function and give its integral representation and multibasic expansion. We also show that this mock theta function can be obtained by performing a left half-shift transformation on a certain theta series.

Keywords and Phrases： *Mock theta functions, Basic hypergeometric series.*

1. Introduction

Recently Hikami [3] introduced the q-series $\mathfrak{D}_5[q]$,

$$\mathfrak{D}_5[q] = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^n}{(q;q^2)_{n+1}}, \quad \text{where} \quad q = e^{2\pi i \tau}$$

in connection with the quantum invariant of 3-manifold. Hikami [3] in his short note has shown the function $\mathfrak{D}_5[q]$ is a mock theta function and has called it of order two.

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He has given a transformation formula for $\mathfrak{D}_5[q]$, using the method of Watson [9]. Using the transformation formula [1]

$$\sum_{n=0}^{\infty}\frac{(\alpha;q^2)_n(\beta)_{2n}}{(q^2;q^2)_n(\gamma)_{2n}}z^n = \frac{(\beta)_\infty(\alpha z;q^2)_\infty}{(\gamma)_\infty(z;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma/\beta)_n(z;q^2)_n}{(q)_n(\alpha z;q^2)_n}\beta^n$$

$\mathfrak{D}_5[q]$ can be written as

$$\mathfrak{D}_5[q] = \frac{1}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}(q;q^2)_n^2 q^{2n}. \qquad (1)$$

Considering the importance of this second order mock theta function, we have studied the other properties of $\mathfrak{D}_5[q]$.

Mock theta functions are mysterious functions and not much is known about them. Ramanujan in his last letter to Hardy gave a list of 17 functions $F(q)$ and called them mock theta functions. He called them mock theta functions as they were not theta functions. He stated that as q radially approaches any point $e^{2\pi ir}$ (r rational) there is a theta function $\theta_r(q)$, such that $F(q)-\theta_r(q)=O(1)$. Moreover there is no single theta function which works for all r i.e. for every theta function $\theta(q)$, there is some root of unity r for which $F(q)-\theta(q)$ is unbounded as $q\to e^{2\pi ir}$ radially.

In Entry 12.45 of Andrews and Berndt [1], $\mathfrak{D}_5[q]$ has been written in the form of the Lerche's sum. We write $\mathfrak{D}_5[q]$ as a double summation series.

In section 4 we give a mild generalization of this mock theta function and show this generalized function belongs to the family of F_q-functions.

In sections 5 and 6 we give an integral representation and a bibasic expansion, respectively, for this generalized function.

Lastly, by performing a half left-shift transformation on a theta series, we get this mock theta function.

2. Notation

We shall use the following usual basic hypergeometric notations:

For $|q^k| < 1$,

$$(a;q^k)_n = (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)}),\ n\geq 1,$$

$$(a;q^k)_0 = 1,$$

$$(a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj}),$$

$$(a;q)_n = (a)_n.$$

3. Double Series Expansion

Theorem 1. $$(q^2;q^2)_\infty^2\, \mathscr{D}_5[q] = \sum_{n=0}^{\infty} \frac{q^{4n^2+4n}(1+q^{2n+1})}{(1-q^{2n+1})}\left[1+\sum_{j=-n}^{n}(-1)^j q^{-(3j^2+j)}\right]. \qquad (2)$$

For giving the double series representation we shall use the Bailey pair method.

Bailey's Lemma

If $\{\alpha_n\}$ and $\{\beta_n\}$ form a Bailey pair relative to a, then

$$\sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n}{\left(\frac{aq}{\rho_1}\right)_n\left(\frac{aq}{\rho_2}\right)_n} = \frac{(aq)_\infty\left(\frac{aq}{\rho_1\rho_2}\right)_\infty}{\left(\frac{aq}{\rho_1}\right)_\infty\left(\frac{aq}{\rho_2}\right)_\infty}\sum_{n=0}^{\infty}(\rho_1)_n(\rho_2)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n\ . \qquad (3)$$

Proof . Making $q\to q^2$ and letting $\rho_1=\rho_2=q$ and $a=q^2$ in (3), we have

$$\sum_{n=0}^{\infty}\frac{q^{2n}(q;q^2)_n^2}{(q^3;q^2)_n^2}\alpha_n = \frac{(q^4;q^2)_\infty(q^2;q^2)_\infty}{(q^3;q^2)_\infty^2}\sum_{n=0}^{\infty}q^{2n}(q;q^2)_n^2\beta_n\ . \qquad (4)$$

In the Bailey pair given in [2, (2.13) and (2.14), p 73], making $q\to q^2$ and taking $a=q^2$ and putting in (4), we have

$$\frac{(q^4;q^2)_\infty (q^2;q^2)_\infty}{(q^3;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{q^{2n}(q;q^2)_n^2}{(q^2b;q^2)_n (q^2c;q^2)_n}$$
$$= \sum_{n=0}^{\infty} \frac{q^{2n}(q;q^2)_n^2}{(q^3;q^2)_n^2} \frac{q^{2n^2}(bc)^n(1-q^{4n+2})(q^2/b;q^2)_n (q^2/c;q^2)_n}{(1-q^2)(q^2b;q^2)_n (q^2c;q^2)_n}$$
$$\times \sum_{j=0}^{n} \frac{(-1)^j(1-q^{4j})(q^2;q^2)_{j-1}(b;q^2)_j (c;q^2)_j}{q^{j^2-j}(bc)^j (q^2;q^2)_j \left(q^2/b;q^2\right)_j \left(q^2/c;q^2\right)_j} . \tag{5}$$

Letting $b,c \to \infty$ in (5) we get, after simple calculation,

$$(q^2;q^2)_\infty^2 \mathfrak{D}_5[q] = \sum_{n=0}^{\infty} \frac{q^{4n^2+4n}(1+q^{2n+1})}{(1-q^{2n+1})} \left[1+\sum_{j=-n}^{n}(-1)^j q^{-(3j^2+j)}\right]. \tag{6}$$

4. Generalized Function

We give mild generalization of $\mathfrak{D}_5[q]$. We define

$$\mathfrak{D}_5(z,\alpha) := \frac{1}{(q;q^2)_\infty (z)_\infty} \sum_{n=0}^{\infty} (z)_n (q;q^2)_n^2 q^{n(\alpha+1)} .$$

For $z=0$ and $\alpha=1$, this generalized function reduces to mock theta function $\mathfrak{D}_5[q]$.

F_q -Function

Truesdell [8] calls the functions which satisfy the F- equation

$$\frac{\partial}{\partial z} F(z,\alpha) = F(z,\alpha+1)$$

F -function.

The q-analogue of Truesdell’s definition: the functions which satisfy the q-difference equation

$$D_{q,z}\ F(z,\alpha) = F(z,\alpha+1)$$

where

$$zD_{q,z}\ F(z,\alpha) = F(z,\alpha) - F(zq,\alpha)$$

are called F_q-functions. We now show $\mathfrak{D}_5(\alpha, z)$ is a F_q-function.

$$z\, D_{q,z}\, \mathfrak{D}_5(z,\alpha) = \mathfrak{D}_5(z,\alpha) - \mathfrak{D}_5(zq,\alpha)$$

$$= \frac{1}{(q;q^2)_\infty^2} \left[\frac{1}{(z)_\infty} \sum_{n=0}^{\infty} (z)_n (q;q^2)_n^2 q^{n(\alpha+1)} - \frac{1}{(zq)_\infty} \sum_{n=0}^{\infty} (zq)_n (q;q^2)_n^2 q^{n(\alpha+1)} \right]$$

$$= \frac{1}{(q;q^2)_\infty^2 (z)_\infty} \left[\sum_{n=0}^{\infty} (z)_n (q;q^2)_n^2 q^{n(\alpha+1)} - \sum_{n=0}^{\infty} (z)_n (q;q^2)_n^2 q^{n(\alpha+1)} (1-zq^n) \right]$$

$$= \frac{1}{(q;q^2)_\infty^2 (z)_\infty} \left[z \sum_{n=0}^{\infty} (z)_n (q;q^2)_n^2 q^{n(\alpha+2)} \right]$$

$$= z\, \mathfrak{D}_5(z,\alpha+1), \tag{7}$$

which shows $\mathfrak{D}_5(z,\alpha)$ is a F_q-function.

5. Integral Representation

Theorem 2. $\mathscr{D}_5(q^z,\alpha) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \displaystyle\int_0^1 t^{z-1} (tq;q)_\infty\, \mathscr{D}_5(0,bt)\, d_q t$.

The theorem is an integral representation for the generalized function.
Thomae [4, p. 19] and Jackson [4, p.19] defined the q-integral as

$$\int_0^1 f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n .$$

Obviously the q-integration is a discrete process while Riemann integration is a limiting process. For q-integration the only requirements are the value of the *function* at the discrete set of points $\{q^r : r = 0,1,\cdots\}$ and the convergence of the series defining the q-integral.

Equations (1.10.14) and (1.11.7) [4, pp. 18-19] it is easily seen , in the limiting case,

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty \, d_q t\,. \tag{8}$$

Proof. By definition

$$(q;q^2)_\infty^2 \mathfrak{D}_5\,(z,\alpha) = \frac{1}{(z)_\infty}\sum_{n=0}^{\infty}(z)_n (q;q^2)_n^2 q^{n(\alpha+1)}\,.$$

Writing q^z for z and b for q^α, we have

$$(q;q^2)_\infty^2 \mathfrak{D}_5\,(q^z,\alpha) = \frac{1}{(q^z)_\infty}\sum_{n=0}^{\infty}(q^z)_n (q;q^2)_n^2 q^{n(\alpha+1)}$$

$$= \sum_{n=0}^{\infty}\frac{(q;q^2)_n^2 q^{n(\alpha+1)}}{(q^{n+z};q)_\infty}\,.$$

By (8)

$$(q;q^2)_\infty^2 \mathfrak{D}_5\,(q^z,\alpha) = \sum_{n=0}^{\infty}(q;q^2)_n^2 q^{n(\alpha+1)}\,\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{n+z-1}(tq;q)_\infty \, d_q t$$

$$= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{z-1}(tq;q)_\infty \sum_{n=0}^{\infty}(q;q^2)_n^2 q^n (bt)^n \, d_q t\,. \tag{9}$$

But

$$(q;q^2)_\infty^2 \mathfrak{D}_5\,(0,b) = \sum_{n=0}^{\infty}(q;q^2)_n^2 q^n b^n\,,$$

so

$$(q;q^2)_\infty^2 \mathfrak{D}_5\,(0,bt)\,. \tag{10}$$

Substituting (10) in (9), we have

$$\mathfrak{D}_5(q^z,\alpha) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{z-1}(tq;q)_\infty \, \mathfrak{D}_5\big(0,bt\big) d_q t\,, \tag{11}$$

which proves the theorem.

6. Bibasic expansion for $\mathfrak{D}_5(z,\alpha)$

We shall prove the following theorem which is a bibasic expansion of the generalized function.

Theorem 3. $$\mathfrak{D}_5(x,\alpha)=\sum_{k=0}^{\infty}(1-xq^{-k-1})(1-1/q)(q;q^2)_{k-1}^2(x;q)_{k-1}q^{k\alpha+3k}$$
$$\times\phi\left[\begin{matrix}xq^{k-1},q;q^{2k+1},q^{2k+2} \\ 0;0,0\end{matrix};q,q^2;q^{\alpha+1}\right].$$

The summation formula [4, p 71, (3.6.7)] is

$$\sum_{k=0}^{n}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_k q^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}.$$

Multiplying both sides by α_m and summing from 0 to ∞ and then applying Lemma 10 [7, p.57], we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_k q^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}\sum_{m=0}^{\infty}\alpha_{m+k}$$
$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_m(cq,aq/bc;q)_m}{(ap/c,bcp;p)_m(q,aq/b;q)_m}\alpha_m. \qquad (12)$$

Proof. Making $q\to q^2$ and taking $a=0$, $p=q$, $b=\dfrac{x}{q}$, $c=\dfrac{1}{q}$ in (12), we have

$$\sum_{m=0}^{\infty}\frac{(x;q)_m(q;q^2)_m}{(x/q;q)_m(q^2;q^2)_m}\alpha_m=\sum_{k=0}^{\infty}\frac{(1-xq^{-k-1})(1/q;q^2)_k q^{2k}}{(1-x/q)(q^2;q^2)_k}\sum_{m=0}^{\infty}\alpha_{m+k}. \qquad (13)$$

Take $\alpha_m=(q;q^2)_m(q^2;q^2)_m(x/q;q)_m q^{m(\alpha+1)}$ in (13),

$$(x)_\infty(q;q^2)_\infty^2\ \mathfrak{D}_5(x,\alpha)=\sum_{k=0}^{\infty}(1-xq^{-k-1})(1-1/q)(q;q^2)_{k-1}^2(x;q)_{k-1}q^{k\alpha+3k}$$

$$\times\phi\left[\begin{matrix} xq^{k-1}, q; q^{2k+1}, q^{2k+2} \\ 0;0,0 \end{matrix} ; q, q^2; q^{\alpha+1}\right]. \tag{14}$$

7. Relation Between Second Order Mock Theta $\mathfrak{D}_5(q)$ and the Other Second Order Mock Theta Function $B(q)$

We shall give an expansion of $\mathfrak{D}_5(q)$ in terms of the mock theta function $B(q)$. For this we shall require the following identity.

$$\sum_{r=0}^{p}\alpha_r\beta_r = \beta_{p+1}\sum_{r=0}^{p}\alpha_r + \sum_{m=0}^{p}(\beta_m - \beta_{m+1})\sum_{r=0}^{m}\alpha_r \; . \tag{15}$$

Proof. The proof is simple requiring a rearrangement of the series:

$$\sum_{m=0}^{p}(\beta_m - \beta_{m+1})\sum_{r=0}^{m}\alpha_r = (\beta_0-\beta_1)\alpha_0 + (\beta_1-\beta_2)\sum_{r=0}^{1}\alpha_r$$

$$+(\beta_2-\beta_3)\sum_{r=0}^{2}\alpha_r + \cdots + (\beta_p - \beta_{p+1})\sum_{r=0}^{p}\alpha_r$$

$$= \alpha_0\beta_0 + \alpha_1\beta_1 + \cdots + \alpha_p\beta_p - \beta_{p+1}(\alpha_0+\alpha_1+\cdots+\alpha_p)$$

$$= \sum_{r=0}^{p}\alpha_r\beta_r - \beta_{p+1}\sum_{r=0}^{p}\alpha_r \; .$$

The technique is that we choose α_r, β_r such that α_r is a mock theta function and $\alpha_r\beta_r$ is another mock theta function.

Relation between $\mathfrak{D}_5(q)$ and $B(q)$

Take $\alpha_r = q^r(-q;q^2)_r/(q^3;q^2)_r$ and $\beta_r = q^r(-q;q^2)_r(q^3;q^2)_r$ and put in (15), to get

$$\sum_{r=0}^{p}(-q;q^2)_r q^{2r} = q^{p+1}(-q;q^2)_{p+1}(q^3;q^2)_{p+1}\sum_{r=0}^{p}\frac{q^r(-q;q^2)_r}{(q^3;q^2)_r}$$
$$+\sum_{m=0}^{p}\left[q^m(-q;q^2)_m(q^3;q^2)_m - q^{m+1}(-q;q^2)_{m+1}(q^3;q^2)_{m+1}\right] .$$

Hence

$$(-q;q^2)_\infty^2 \mathfrak{D}_{5,p}(-q) = \frac{q^{p+1}}{1-q}(1-q^{2p+3})(q^2;q^4)_{p+1}B_p(q)\sum_{r=0}^{m}\frac{q^r(-q;q^2)_r}{(q^3;q^2)_r}$$
$$+\left[1-\frac{q^{p+1}}{1-q}(1-q^{2p+3})(q^2;q^4)_{p+1}\right]\sum_{m=0}^{\infty}B_m(q)\,, \qquad (16)$$

where $B(q)$ is a second order mock theta function [6] and $\mathfrak{D}_{5,p}(-q)$ and $B_m(q)$ are partial mock theta function. We take the partial sum from 0 to r and write by inserting suffix r to denote the partial sum.

8. Half-Shift Transformation On $\mathfrak{D}_5(q)$

Gordon and McIntosh [5] gave a method of constructing mock theta functions by performing left-shift transformation on certain q-series. Here we show this mock theta function $\mathfrak{D}_5(q)$ is obtained by using this method.

Take the series $\sum_{n=0}^{\infty}(q^2;q^2)_n^2 q^{2n}$.

We shall write the series

$$\sum_{n=0}^{\infty}(q^2;q^2)_n^2 q^{2n} = (q^2;q^2)_\infty^2 \sum_{n=0}^{\infty}\frac{q^{2n}}{(q^{2n+2};q^2)_\infty^2} = \sum_{n=0}^{\infty} a_n \quad \text{(say)},$$

where a_n is defined for all real n. We make a left half-shift and sum n over the positive half-integers $\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots,$ instead of the non-negative integers. Define

$b_n = a_{n-\frac{1}{2}}$.

So

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_{n-\frac{1}{2}} = (q^2;q^2)_\infty^2 \sum_{n=0}^{\infty}\frac{q^{2n-1}}{(q^{2n+1};q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}(q;q^2)_n^2 q^{2n-1}$$

$$= \frac{1}{q}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}(q;q^2)_n^2 q^{2n}$$

$$= \frac{1}{q}(q^2;q^2)_\infty \mathcal{D}_5(q).$$

Hence if we apply left half-shift transformation on the series

$$\frac{q}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(q^2;q^2)_n^2 q^{2n},$$

we have the mock theta function $\mathcal{D}_5(q)$.

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