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A Note on An Ostrowski Type Inequality*

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Abstract

Calculate a definite integral by transforming it to a double integral and then changing the order of integration. Thus provide a neat error bound of an Ostrowski type inequality in the literature.

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As the first main result in [1], G.A. Anastassiou presented the following Ostrowski type inequality.

Theorem A. Let $f : [a, b] \to \mathbf{R}$ be 3-times differentiable on (a, b) whose third derivative $f''' : (a, b) \to \mathbf{R}$ is bounded on (a, b), i.e., $\|f'''\|_\infty := \sup_{t\in(a,b)}|f'''(t)| < +\infty$. Then we have the inequality

$$\begin{aligned} & |f(x) - \tfrac{1}{b-a}\int_a^b f(t)\,dt - \tfrac{f(b)-f(a)}{b-a}(x - \tfrac{a+b}{2}) \\ - & \tfrac{f'(b)-f'(a)}{2(b-a)}[(x - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}]| \\ \leq & \tfrac{\|f'''\|_\infty}{(b-a)^3} \cdot A(x), \end{aligned} \tag{1}$$

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for every $x \in [a, b]$, where

$$\begin{array}{rcl} A(x) & := & abx^4 - \frac{1}{3}a^2b^3x + \frac{1}{3}a^3bx^2 - ab^2x^3 - \frac{1}{3}a^3b^2x + \frac{1}{3}ab^3x^2 + a^2b^2x^2 \\ & & -a^2bx^3 - \frac{1}{2}ax^5 - \frac{1}{2}bx^5 + \frac{1}{6}x^6 + \frac{3}{4}a^2x^4 + \frac{3}{4}b^2x^4 + \frac{1}{3}b^2a^4 \\ & & -\frac{2}{3}a^3x^3 - \frac{2}{3}b^3x^3 - \frac{1}{3}b^3a^3 + \frac{5}{12}a^4x^2 + \frac{5}{12}b^4x^2 + \frac{1}{3}b^4a^2 \\ & & -\frac{2}{15}ba^5 - \frac{2}{15}ab^5 - \frac{1}{6}a^5x - \frac{1}{6}b^5x + \frac{a^6}{20} + \frac{b^6}{20}. \end{array} \tag{2}$$

Inequality (1) is attained by

$$f(x) = (x-a)^3 + (b-x)^3,$$

in that case both sides of the inequality equal zero.

Inequality (1) with (2) in [1] was proved by repeatedly using the Montgomery idenity ([2], Ch. XVIII, p.565)

$$f(x) = \frac{1}{b-a}\int_a^b f(s)\,ds + \int_a^b P(x,s)f'(s)\,ds,$$

in the following way as

$$\begin{array}{rl} & f(x) - \frac{1}{b-a}\int_a^b f(s_1)\,ds_1 - \frac{f(b)-f(a)}{b-a}\int_a^b P(x,s_1)\,ds_1 \\ - & \frac{f'(b)-f'(a)}{b-a}\int_a^b\int_a^b P(x,s_1)P(s_1,s_2)\,ds_1\,ds_2 \\ = & \int_a^b\int_a^b\int_a^b P(x,s_1)P(s_1,s_2)P(s_2,s_3)f'''(s_3)\,ds_1\,ds_2\,ds_3 \end{array}$$

where $P(x.s)$ is the Peano kernel defined by

$$P(x,s) = \begin{cases} \frac{s-a}{b-a}, & a \le s \le x, \\ \frac{s-b}{b-a}, & x < s \le b. \end{cases}$$

Thus finally has led to the inequality

$$\begin{array}{rl} & |f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}(x - \frac{a+b}{2}) \\ - & \frac{f'(b)-f'(a)}{2(b-a)}[(x-\frac{a+b}{2})^2 - \frac{(b-a)^2}{12}]| \\ \le & \frac{\|f'''\|_\infty}{(b-a)^3}\cdot A(x), \end{array}$$

for every $x \in [a, b]$, where

$$\begin{array}{rcl} A(x) & := & \frac{1}{12}\int_a^x (s_1-a)[6(s_1-\frac{a+b}{2})^4 + 3(b-a)^2(s_1-\frac{a+b}{2})^2 + \frac{7}{8}(b-a)^4]\,ds_1 \\ & & +\frac{1}{12}\int_x^b (b-s_1)[6(s_1-\frac{a+b}{2})^4 + 3(b-a)^2(s_1-\frac{a+b}{2})^2 + \frac{7}{8}(b-a)^4]\,ds_1 \end{array} \tag{3}$$

and the integral in (3) was calculated by use of Mathematica-4 and Maple-6 to get (2).

In this note, we would like to point out that the integral in (3) can be easily calculated by transforming the definite integral to double integral and interchanging the order of integration as follows:

$$\begin{aligned}
& \tfrac{1}{12}\int_a^x (s_1-a)[6(s_1-\tfrac{a+b}{2})^4+3(b-a)^2(s_1-\tfrac{a+b}{2})^2+\tfrac{7}{8}(b-a)^4]\,ds_1 \\
& +\tfrac{1}{12}\int_x^b (b-s_1)[6(s_1-\tfrac{a+b}{2})^4+3(b-a)^2(s_1-\tfrac{a+b}{2})^2+\tfrac{7}{8}(b-a)^4]\,ds_1 \\
= \; & \tfrac{1}{12}\int_a^x\int_a^{s_1} dt[6(s_1-\tfrac{a+b}{2})^4+3(b-a)^2(s_1-\tfrac{a+b}{2})^2+\tfrac{7}{8}(b-a)^4]\,ds_1 \\
& +\tfrac{1}{12}\int_x^b\int_{s_1}^b dt[6(s_1-\tfrac{a+b}{2})^4+3(b-a)^2(s_1-\tfrac{a+b}{2})^2+\tfrac{7}{8}(b-a)^4]\,ds_1 \\
= \; & \tfrac{1}{12}\int_a^x\int_t^x [6(s_1-\tfrac{a+b}{2})^4+3(b-a)^2(s_1-\tfrac{a+b}{2})^2+\tfrac{7}{8}(b-a)^4]\,ds_1dt \\
& +\tfrac{1}{12}\int_x^b\int_x^t [6(s_1-\tfrac{a+b}{2})^4+3(b-a)^2(s_1-\tfrac{a+b}{2})^2+\tfrac{7}{8}(b-a)^4]\,ds_1dt \\
= \; & \tfrac{1}{6}[(x-\tfrac{a+b}{2})^6+\tfrac{3}{4}(b-a)^2(x-\tfrac{a+b}{2})^4+\tfrac{7}{16}(b-a)^4(x-\tfrac{a+b}{2})^2+\tfrac{41}{320}(b-a)^6].
\end{aligned}$$

Consequently, we have

Theorem B. *Let $f:[a,b]\to\mathbf{R}$ be 3-times differentiable on (a,b) whose third derivative $f''':(a,b)\to\mathbf{R}$ is bounded on (a,b), i.e., $\|f'''\|_\infty := \sup_{t\in(a,b)}|f'''(t)| < +\infty$. Then we have the inequality*

$$\begin{aligned}
& |f(x)-\tfrac{1}{b-a}\int_a^b f(t)\,dt-\tfrac{f(b)-f(a)}{b-a}(x-\tfrac{a+b}{2})-\tfrac{f'(b)-f'(a)}{2(b-a)}[(x-\tfrac{a+b}{2})^2-\tfrac{(b-a)^2}{12}]| \\
\le \; & \tfrac{1}{6}[\delta^6(x)+\tfrac{3}{4}\delta^4(x)+\tfrac{7}{16}\delta^2(x)+\tfrac{41}{320}](b-a)^3\|f'''\|_\infty \\
\le \; & \tfrac{1}{20}(b-a)^3\|f'''\|_\infty,
\end{aligned} \tag{4}$$

for every $x\in[a,b]$, where

$$\delta(x) := \frac{|x-\frac{a+b}{2}|}{b-a}. \tag{5}$$

The second inequality in (4) is obvious by the fact that

$$0\le\delta(x)\le\frac{1}{2}$$

for all $x\in[a,b]$.

Remark. *Compare Theorem B with Theorem A, we see that the bound of inequality (4) has a neat expression. If we take* $x = \frac{a+b}{2}$*, then we get a perturbed midpoint inequality as*

$$|\int_a^b f(t)\,dt - (b-a)f(\frac{a+b}{2}) - \frac{(b-a)^2}{24}[f'(b) - f'(a)]| \leq \frac{41}{1920}(b-a)^4\|f'''\|_\infty.$$

If we take $x = a$ *or* $x = b$*, then we get a perturbed trapezoid inequality as*

$$|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)]| \leq \frac{1}{20}(b-a)^4\|f'''\|_\infty.$$

References

[1] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited, *Monatshefte für Mathematik* **135** (2002), 175-189.

[2] D. S. Mitrinović, J.E.Pečarić and A. M. Fink, *Inequalities for Functions and Their Integral and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1944.