Tamsui Oxford Journal of Mathematical Sciences **24(1)** (2008) 1-5
Aletheia University

Banach Space Valued Mean Periodic Functions

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Received December 22, 2005, Accepted April 11, 2006.

Abstract

Here we give a necessary and sufficient condition for a Banach space to be separable.

Keywords and Phrases: *Mean periodic function, Vector valued measure.*

For a Banach space B over the complex field $\mathbb{C}$, let $C(\mathbb{R}, B)$ denote the set of all continuous functions defined on the real line $\mathbb{R}$ taking values in B with the compact convergence topology. When $B = \mathbb{C}$we write $C(\mathbb{R})$ for $C(\mathbb{R}, \mathbb{C})$. For a function ϕ in $C(\mathbb{R}, B)$ let $\tau(\phi)$ denote the closure in $C(\mathbb{R}, B)$ of the span of all translates of ϕ.

Definitiom 1. *A function ϕ in $C(\mathbb{R}, B)$ is said to be mean periodic if $\tau(\phi) \neq C(\mathbb{R}, B)$.*

We prove the following theorem.

Theorem *A Banach space B is separable if and only if not all functions in $C(\mathbb{R}, B)$ are mean periodic.*

Preliminaries *Let B^* be the dual of B and let B_s^* denote the space B^* along with the weak * topology on it. Let μ be a countably additive, regular, vector valued, Borel measure on $\mathbb{R}$ taking values in B_s^* . For a Borel set $b' \subseteq \mathbb{R}$, set $\|\mu(b')\| = \sup_{\|v\|\leq 1} | < v, \mu(b') > |$. This is the norm in the Banach space B^* . For a Borel set b in $\mathbb{R}$ let $P(b)$ be the set of all finite Borel partitions of b . If $\sup_{P(b)} \sum_{b'} \|\mu(b')\| < \infty$(where the sum is taken over $\{b'\}$ which form a finite Borel partition of b, for all Borel sets bin B, then μ is said to be of bounded variation.*

Let $M(\mathbb{R}, B_s^*)$ be the space of all vector valued Borel measures μ on $\mathbb{R}$ taking values in B_s^* with the following properties:

1. μ is countably additive and regular,
2. μ has compact support in $\mathbb{R}$,
3. μ is of bounded variation,
4. there exists a constant $C > 0$ such that for any Borel set b of $\mathbb{R}$, $\|\mu(b)\| \leq C$ where $\|.\|$ denotes the norm in the Banach space B^*.

Let $\mu \in M(\mathbb{R}, B_s^*)$ with support K. Let $b_1, b_2, \cdots b_n$ be a Borel partition of K and $v_1, v_2, \cdots v_n$ be arbitrary elements of B . Then $\phi = \sum_{i=1}^n v_i \chi_{b_i}$ is a simple function from $\mathbb{R}$ to B. Then $\int \phi d\mu$ is defined as

$$\int \phi d\mu = \sum_{i=1}^n < v_i, m(b_i) >$$

Now let $\phi : \mathbb{R} \longrightarrow B$ be any continuous function and let μ and K be as before. For any $\epsilon > 0$ we can get a finite Borel partition $\{b_i\}$ of K such that $\|\phi(x) - \phi(y)\| < \epsilon$ for all x, y in b_i for all i. Choose any point $x_i \in b_i$. Define the simple function $\phi_\epsilon = \sum \phi(x_i)\chi_{b_i}$. Then $\int \phi d\mu = \lim_{\epsilon \to 0} \int \phi_\epsilon d\mu$. This limit exists and is independent of the choice of the Borel partition $\{b_i\}$ and also of the choice of the points $\{x_i\}$ (see [2]).

For a function $f \in C(\mathbb{R})$ and a measure $\mu \in M(\mathbb{R}, B_s^*)$ we define the integral $\int f d\mu$ in a similar way: for a simple function $f = \sum_i y_i \chi_{b_i}$ where $y_i \in \mathbb{C}$ and $\{b_i\}$ is a Borel partition of K , the support of μ , define $\int f d\mu = \sum_i y_i \mu(b_i) \in B_s^*$. For $f \in C(\mathbb{R})$, $\int f d\mu = \lim_{\epsilon \to 0} \int f_\epsilon d\mu$, where f_ϵ are simple functions defined as before. Therefore $\int f d\mu$ is an element of B_s^*.

Singer's Theorem (see [2]): The dual of $C(\mathbb{R}, B)$ is the space $M(\mathbb{R}, B_s^*)$.

Using Hahn Banach Theorem it is easy to see that a function $\phi$is mean periodic if and only if there exists a nonzero measure μ in $M(\mathbb{R}, B_s^*)$ such that $\phi * \mu = 0$.

Proof of the theorem *Suppose B is not separable. Let ϕ be any function in $C(\mathbb{R}, B)$. Since $\mathbb{R}$ is separable, $\phi(\mathbb{R})$is separable. Let B_1 be the closure of the space generated by $\phi(\mathbb{R})$. It is easy to see that B_1 is also separable. Moreover, range of any function in $\tau(\phi)$ is contained in B_1 . Since B_1 is a proper subspace of B we can find a function in $C(\mathbb{R}, B)$ whose range is not contained in $B_{1.}$ Hence $\tau(\phi) \neq C(\mathbb{R}, B)$.*

Conversely, if B is separable we will show that there exists a function in $C(\mathbb{R}, B)$ which is not mean periodic. First we will take B to be l^1, the space of absolutely summable sequences in $\mathbb{C}$. For each $n \in N$, let $e_n = (0, \cdots 1, 0, \cdots)$ be the sequence in l^1 where 1 occurs only at the n th entry. For each $n \in N$ we define an element $\{a_j(n)\}_{j=1}^{\infty}$in l^1 and a sequence of real numbers $\{\lambda_j(n)\}_{j=1}^{\infty}$ such that:

1. $a_j(n)$ is nonzero for all j and n.
2. If we denote $\sum_{j=1}^{\infty} |a_j(n)| = a(n)$, then $\sum_{n=1}^{\infty} a(n) < \infty$ (that is, $\{a(n)\}_{n=1}^{\infty}$ is also in l^1.
3. $\lambda_j(n)$ are all different. that is, if $\lambda_j(n) = \lambda_i(m)$ then $j = i$ and $n = m$.
4. The sequence $\{\lambda_j(n)\}_{j=1}^{\infty}$ converges to a real number, say $\lambda(n)$.

For a real number $s \in \mathbb{R}$ define $f_n(s) = \sum_{j=1}^{\infty} a_j(n) e^{i\lambda_j(n)s}$. Then $f_n : \mathbb{R} \to \mathbb{C}$ is continuous. Put $\phi(s) = \sum_{n=1}^{\infty} f_n(s) e_n$. It is easy to see that $\phi : \mathbb{R} \to l^1$ is continuous. The series $\sum_{n=1}^{\infty} f_n e_n$ converges to the function ϕ uniformly on $\mathbb{R}$. We will show that for this function $\phi \in C(\mathbb{R}, l^1)$, $\tau(\phi) = C(\mathbb{R}, l^1)$.

Suppose not. Then by Singer's theorem there exists a nonzero measure $\mu \in M(\mathbb{R}, l_s^{\infty})$ such that $\phi * \mu = 0$. Let K be the compact support of μ.

$$0 = \phi * \mu = \sum_{n=1}^{\infty} (f_n e_n \star \mu) = \sum_{n=1}^{\infty} < e_n, f_n \star \mu >.$$

For the last equality refer to theorem III.2.6 in [2]. For $n \in N$and a Borel set b in $\mathbb{R}$, put $\mu_n(b) =< e_n, \mu(b) >$. Then μ_n is a countably additive, regular complex measure with compact support contained in K(see the proof of Singer's theorem in [2]). We can write $\mu(b) = (\mu_1(b), \mu_2(b), \cdots) \in l_s^\infty$. For any $f \in C(\mathbb{R})$ we have $< e_n, \int f d\mu >= \int f d\mu_n$: this is true for characteristic functions so also for simple functions. By the limiting process one can prove it for any continuous function on $\mathbb{R}$. So by (1) we get $\sum f_n \star \mu_n = 0$. That is,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} a_j(n)\hat{\mu_n}(\lambda_j(n))e^{i\lambda_j(n)s_0} = 0 \forall s_0 \in \mathbb{R}.$$

Since $\hat{\mu_n}(\lambda_j(n))$ are uniformly bounded by $\|\mu\|$ the left hand side is an almost periodic function. Because $a_j(n) \neq 0$ for all j, n, this implies $\hat{\mu_{n j}}(n)) = 0$. The zero's of the holomorphic function $\hat{\mu_n}$ have a limit point $\lambda(n)$. Therefore $\mu_n = 0$ for all n and hence $\mu = 0$, a contradiction.

Now let B be an arbitrary separable Banach space. Since B is separable we can find a countable set $\{h_n\}$ in B such that $\|h_n\| = 1$ for all $n$and the subspace H generated by $\{h_n\}$ is dense in B. Define a function $\psi : \mathbb{R} \to B$by $\psi(s) = \sum_{n=1}^{\infty} f_n(s)h_n$ where f_n 's are defined as before. Then this series converges in $B$and the function ψ is continuous. Let $g \in C(\mathbb{R})$ be any function. Then we claim that $gh_1 \in \tau(\psi)$:

Since $\tau(\phi) = C(\mathbb{R}, l^1)$ we know that ge_1 is a limit (in $C(\mathbb{R}, l^1)$) of a sequence of linear combinations of translates of ϕ. That is, $ge_1 = \lim_{m\to\infty} \Phi_m$ where for each $m \in N$, Φ_m is a finite sum $\Phi_m = \sum c_i \phi_{y_i}$. Define $\Psi_m = \sum c_i \psi_{y_i}$for each m. Then $gh_1 = \lim_{m\to\infty} \Psi_m$: Let $\epsilon > 0$ and C be a compact subset of $\mathbb{R}$. For any $s \in C$

$$\begin{aligned}
\|g(s)h_1 - \Psi_m(s)\| &= \|g(s)h_1 - \sum c_i \psi_{y_i}(s)\| \\
&= \|g(s)h_1 - \sum_{n=1}^{\infty}\sum_i c_i (f_n)_{y_i}(s) h_n\| \\
&\leq |g(s) - \sum_i c_i (f_1)_{y_i}(s) + \sum_{n\geq 2} |\sum_i c_i (f_n)_{y_i}(s)| \\
&= \|g(s)e_{1-}\Phi_m(s)\|_{l^1} \leq \epsilon
\end{aligned}$$

for all m sufficiently large.

Similarly we can prove that any finite sum $\sum g_i h_i$ in $C(\mathbb{R}) \otimes H$ is in $\tau(\psi)$. Since $\tau(\psi)$ is closed and $C(\mathbb{R}) \otimes H$is dense in $C(\mathbb{R}) \otimes B$, we get $C(\mathbb{R}) \otimes B \subseteq \tau(\psi)$. But $C(\mathbb{R}) \otimes B$is dense in $C(\mathbb{R}, B)$ (see [2]). Hence $\tau(\psi) = C(\mathbb{R}, B)$.

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