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A Certain Class of Analytic Functions Associated with Fractional Derivative Operators*

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Abstract

The present paper aims at a systematic investigation of a new class $S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$ of analytic functions in open unit disk $\mathcal{U}$. We have derived several results like characterization property, distortion theorem and other interesting properties of the same class.

Keywords and Phrases: *Analytic functions, Distortion theorem, Hadamard product.*

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0;\ n \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk $\mathcal{U} := \{z : |z| < 1\}$.

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We introduce a subclass $S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$ of analytic functions $f(z)$ belonging to $\mathcal{A}$ and satisfying the condition

$$\left|\frac{\Delta_{z,m}^{\lambda,\mu,\eta}f(z)-1}{(B+(A-B)(1-\alpha))-B\Delta_{z,m}^{\lambda,\mu,\eta}f(z)}\right|<\beta \quad (z\in\mathcal{U}), \tag{1.2}$$

where

$0\le\lambda<1, 0\le\alpha<1,\ \mu<1,\ 0<\beta\le 1,\ -1\le B<A\le 1,\ 0<A\le 1,\ m\in\mathbb{N}$

and

$$\eta>\max(\lambda,\mu)-1, \tag{1.3}$$

where the function $\Delta_{z,m}^{\lambda,\mu,\eta}f(z)$ is defined by

$$\Delta_{z,m}^{\lambda,\mu,\eta}f(z)=L(\lambda,\mu,\eta,m)z^{\frac{\mu}{m}-1}D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta}f(z), \tag{1.4}$$

such that $0\le\lambda<1$, $\mu<1$, $m\in\mathbb{N}$, $\eta>\max\{\lambda,\mu\}-1$ and

$$L(\lambda,\mu,\eta,m)=\frac{\Gamma(1-\mu+m)\Gamma(1+\eta-\lambda+m)}{\Gamma(1+m)\Gamma(1+\eta-\mu+m)} \tag{1.5}$$

where the operator $D_{0,z,\frac{1}{m}}^{\lambda,\mu,\eta}f(z)$ is modified fractional derivative operator of Saigo et al. [9] and is defined as follows.

Definition 1. *For* $0\le\alpha<1,\beta,\eta\in\mathbb{R}$ *and* $m\in\mathbb{N}$

$$D_{0,z,m}^{\alpha,\beta,\eta}f(z)=\frac{d}{dz}\left\{\frac{z^{-m(\beta-\alpha)}}{\Gamma(1-\alpha)}\int_0^z(z^m-t^m)^{-\alpha}f(t)F\left(\beta-\alpha,1-\eta,1-\alpha,1-\frac{z^m}{t^m}\right)d(t^m)\right\}. \tag{1.6}$$

The function $f(z)$ is analytic in simply connected domain of the z-plane containing the origin, with the order $f(z)=O(|z|^{\epsilon}), z\longrightarrow 0$. where $\epsilon>\max\{0,m(\beta-\eta)\}-m$. The multiplicity of $(z^m-t^m)^{-\alpha}$ in (1.6) is removed by requiring $\log(z^m-t^m)$ to be real when $(z^m-t^m)>0$, and assumed to be well defined in the unit disk.

The operator defined by (1.6) include the well known Riemann- Liouvlle and Erdelyi-Kober operator of fractional calculus. The theory of fractional calculus has recently found interesting applications in the theory of analytic functions. The classical definition of Riemann- Liouville in fractional calculus operator [13,14] and their various generalization have fruitfully been applied in obtaining , for example, characterization properties, coefficient estimates, distortion inequalities and convolution structure for various subclasses of analytic functions . We investigate here several basic properties and characteristics of a general subclass $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$. These include for example, sharp bounds on coefficients, distortion theorem and result concern with Hadamard product. We note that our subclass generalizes several classes studied earlier by Gupta and Jain [3], Raina and Bolia [7,8], Bhatt and Raina [1] and Srivastava and Aouf [11] and others ([4], [5], [6]) as well as several works presented in the recent research monographs [13] and [14].

2. Characterization

Before stating and proving our main result , we need the following result to be used in the sequel.

Lemma 1. *If* $0 \leq \alpha < 1$, $m \in \mathbb{N}$, $\beta, \eta \in \mathbb{R}$ *and* $k > \max\{0, m(\beta - \eta)\} - m$, *then*

$$D^{\alpha,\beta,n}_{0,z,\frac{1}{m}} z^k = \frac{\Gamma(1+\frac{k}{m})\Gamma(1+\eta-\beta+\frac{k}{m})}{\Gamma(1-\beta+\frac{k}{m})\Gamma(1+\eta-\alpha+\frac{k}{m})} z^{k-m\beta}. \tag{2.1}$$

We investigate characterization property of the function $f(z)$ to be belonging to the class $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$, thereby obtaining coefficient bounds.

Theorem 1. *A function defined by (1.1) is in the class* $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$ *if and only if*

$$\sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, \eta, m)(1 - B\beta)a_n \leq (A - B)\beta(1 - \alpha) \tag{2.2}$$

where

$$\Phi_n(\lambda, \mu, \eta, m) = L(\lambda, \mu, \eta, m)\ M(\lambda, \mu, \eta, m, n) \tag{2.3}$$

with $L(\lambda, \mu, \eta, m)$ *defined by (1.5) and*

$$M(\lambda, \mu, \eta, m, n) = \frac{\Gamma(1+nm)\Gamma(1+\eta-\mu+nm)}{\Gamma(1+\eta-\lambda+nm)\Gamma(1-\mu+nm)}. \tag{2.4}$$

under the conditions given by (1.3). The result (2.2) is sharp.

Proof. Suppose that (2.2) holds true, and let $|z| = 1$. Then using (1.4), (1.5) and (2.1), we have

$$\begin{aligned} & \left|\Delta_{z,m}^{\lambda,\mu,\eta} f(z) - 1\right| - \beta \left|(B + (A-B)(1-\alpha)) - B\Delta_{z,m}^{\lambda,\mu,\eta} f(z)\right| \\ = & \left| -\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^{n-1} \right| \\ & -\beta \left| ((A-B)(1-\alpha)) + B\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^{n-1} \right| \\ \leq & \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)(1-\beta B) a_n - (A-B)(1-\alpha) \leq 0, \end{aligned}$$

by hypothesis, where $\Phi_n(\lambda,\mu,\eta,m)$ is given by (2.3). Therefore, it follows that $f(z) \in S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$.

Conversely, suppose that $f(z)$ defined by (1.1) be such that $f \in S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$. Then, in view (1.2), we have

$$\begin{aligned} & \left| \frac{\Delta_{z,m}^{\lambda,\mu,\eta} f(z) - 1}{(B + (A-B)(1-\alpha)) - B\Delta_{z,m}^{\lambda,\mu,\eta} f(z)} \right| \\ = & \frac{\left|\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^{n-1}\right|}{\left|((A-B)(1-\alpha)) + B\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^{n-1}\right|} < \beta \quad (z \in \mathcal{U}). \end{aligned}$$

Since $\Re(z) \leq |z|$, for all z, we get

$$\Re\left(\frac{\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^{n-1}}{((A-B)(1-\alpha)) + B\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n z^{n-1}} \right) < \beta. \tag{2.5}$$

Now choosing the value of z on real axis, simplifying and letting $z \longrightarrow 1^-$ through the real values , we get

$$\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n \leq \beta(A-B)(1-\alpha) + B\beta \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m) a_n \tag{2.6}$$

which yields (2.2). We also note that the assertion (2.2) is sharp and extremal function is given by

$$f(z) = z - \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_n(\lambda,\mu,\eta,m)} z^n, \quad (n \geq 2). \tag{2.7}$$

$\square$

Remark 1. *If $\mu = \lambda = m = 1$, then in view of (1.4), (1.5) and (1.6), we have,*

$$\Delta_{z,1}^{1,1,n} f(z) = f'(z) \tag{2.8}$$

and also we note that for $B = -1$, and $A = 1$, we get the class studied by Bhatt and Raina as follows

$$S_{\lambda,\mu,\eta}(\alpha, \beta, 1, -1, m) = S_{\lambda,\mu,\eta}(\alpha, \beta, m). \tag{2.9}$$

Corollary 1. *Let the function defined by (1.1) belongs to the class $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$. Then,*

$$a_n \leq \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_n(\lambda, \mu, \eta, m)}, \quad (\forall n \geq 2). \tag{2.10}$$

where $\Phi_n(\lambda, \mu, \eta, m)$ is given by (2.3).

Remark 2. *From (2.10) , we express*

$$a_n \leq \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_n(\lambda, \mu, \eta, m)} = K \frac{\Gamma(1+\eta-\lambda+nm)\Gamma(1-\mu+nm)}{\Gamma(1+nm)\Gamma(1+\eta-\mu+nm)}$$

where

$$K = \frac{(A-B)(1-\alpha)\beta \ \Gamma(1+m)\Gamma(1+\eta-\mu+m)}{(1-B\beta) \ \Gamma(1+\eta-\lambda+m)\Gamma(1-\mu+m)} \leq 1,$$

which is observed to be true for

$$0 \leq \lambda \leq \mu < 1, \ 0 \leq \alpha < 1, \ 0 < \beta \leq 1, \ -1 \leq B < A \leq 1, \ 0 < A \leq 1, \ m \in \mathbb{N}, \eta \in \mathbb{R}^+.$$

Using the asymptotic for the ratio of gamma function. For finite large n , we note that

$$\frac{\Gamma(1+\eta-\lambda+nm)\Gamma(1-\mu+nm)}{\Gamma(1+nm)\Gamma(1+\eta-\mu+nm)} \approx (mn)^{-\lambda} \leq n \quad (0 \leq \lambda < 1).$$

The assertion (2.10) of corollary 1 therefore satisfies

$$a_n \leq \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_n(\lambda, \mu, \eta, m)} \leq n, \quad (\forall n \geq 2), \tag{2.11}$$

for

$$0 \leq \lambda \leq \mu < 1, \ 0 \leq \alpha < 1, \ 0 < \beta \leq 1, \ -1 \leq B < A \leq 1, \ 0 < A \leq 1, \ m \in \mathbb{N}, \eta \in \mathbb{R}^+.$$

Thus, if T denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} C_n z^n, \quad (z \in \mathcal{U}) \tag{2.12}$$

that are analytic and univalent in $\mathcal{U}$ *, then there do exist function* $f \in S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$, *with*

$$0 \le \lambda \le \mu < 1,\ 0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le B < A \le 1,\ 0 < A \le 1,\ m \in \mathbb{N},\ \eta \in \mathbb{R}^+,$$

not necessarily in the class T, for which the celebrated Bieberbach conjecture (de Branges Theorem)

$$|C_n| \le n \quad (n \ge 2) \tag{2.13}$$

holds true [2].

3. Distortion Theorem

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$. Then*

$$f(z) \ge |z| - \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2 \tag{3.1}$$

and

$$f(z) \le |z| + \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2 \tag{3.2}$$

for $z \in \mathcal{U}$, where $\Phi_2(\lambda,\mu,\eta,m)$ is given by (2.3) holds under the conditions given by (1.3).

Proof. *If $f \in S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$, then by virtue of Theorem 1, we have*

$$\Phi_2(\lambda,\mu,\eta,m)(1-B\beta)\sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)a_n(1-B\beta) \le (A-B)(1-\alpha)\beta. \tag{3.3}$$

This yields

$$\sum_{n=2}^{\infty} a_n \le \frac{(A-B)(1-\alpha)\beta}{\Phi_2(\lambda,\mu,\eta,m)(1-B\beta)}. \tag{3.4}$$

Now,

$$|f(z)| \ge |z| - |z|^2\sum_{n=2}^{\infty} a_n \ge |z| - \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2. \tag{3.5}$$

Also,

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{(A-B)(1-\alpha)\beta}{(1-B\beta)\Phi_2(\lambda,\mu,\eta,m)}|z|^2, \tag{3.6}$$

which proves the assertion (3.1) and (3.2). □

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$. Then*

$$\left|D^{\lambda,\mu,\eta}_{0,z,\frac{1}{m}} f(z)\right| \geq \frac{|z|^{1-\frac{\mu}{m}}}{L(\lambda,\mu,\eta,m)}\left[1 - \frac{(A-B)\beta(1-\alpha)}{1-\beta B}|z|\right], \tag{3.7}$$

$$\left|D^{\lambda,\mu,\eta}_{0,z,\frac{1}{m}} f(z)\right| \leq \frac{|z|^{1-\frac{\mu}{m}}}{L(\lambda,\mu,\eta,m)}\left[1 + \frac{(A-B)\beta(1-\alpha)}{1-\beta B}|z|\right], \tag{3.8}$$

for $z \in \mathcal{U}$, if $\mu \leq m$, and $z \in \mathcal{U}\setminus\{0\}$ if $\mu > m$, where $L(\lambda,\mu,\eta,m)$ is given by (1.5), under the condition given by (1.3).

Proof. Using (1.1),(1.5) and (2.1) , we observe that

$$\begin{aligned}
\left|L(\lambda,\mu,\eta,m)z^{\frac{\mu}{m}} D^{\lambda,\mu,\eta}_{0,z,\frac{1}{m}} f(z)\right| &= \left|z - \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)a_n z^n\right| \\
&\geq |z| - \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)a_n|z|^n \\
&\geq |z| - |z|^2 \sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)a_n \\
&\geq |z| - |z|^2 \frac{(A-B)\beta(1-\alpha)}{(1-\beta B)}
\end{aligned}$$

because $f \in S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$, by hypothesis. Thus, the assertion (3.7) is proved. The assertion (3.8) can be proved in similar manner. □

Corollary 2. *Under the hypothesis of Theorem 2,$f(z)$ is included in a disk with its centre at the origin and radius r given by*

$$r = 1 + \frac{(A-B)\beta(1-\alpha)}{(1-\beta B)\Phi_n(\lambda,\mu,\eta,m)}. \tag{3.9}$$

Corollary 3. *Under the hypothesis of Theorem 3, $D^{\lambda,\mu,\eta}_{0,z,\frac{1}{m}} f(z)$ is included in a disk with its centre at the origin and radius R given by*

$$R = \frac{1}{L(\lambda,\mu,\eta,m)}\left[1 + \frac{(A-B)\beta(1-\alpha)}{(1-\beta B)}\right]. \tag{3.10}$$

4. Properties of the class $S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$

We next study some interesting properties of the class.

Theorem 4. *Let* $0 \le \lambda < 1, \mu < 1,\ 0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le B < A \le 1,\ 0 < A \le 1,\ 0 \le \alpha' < 1,\ 0 < \beta' \le 1,\ -1 \le B' < A' \le 1,\ 0 < A' \le 1,\ m \in \mathbb{N},\ \eta > \max\{\lambda,\mu\} - 1$. *Then*

$$S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m) = S_{\lambda,\mu,\eta}(\alpha',\beta',A',B',m), \tag{4.1}$$

if and only if

$$\frac{(A-B)\beta(1-\alpha)}{(1-\beta B)} = \frac{(A'-B')\beta'(1-\alpha')}{(1-\beta' B')}. \tag{4.2}$$

Proof. First assume that $f \in S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$, and let the condition (4.2) hold true. By using assertion (2.2) of Theorem 1, we have then

$$\sum_{n=2}^{\infty} \Phi_n(\lambda,\mu,\eta,m)a_n \le \frac{(A-B)\beta(1-\alpha)}{(1-\beta B)} = \frac{(A'-B')\beta(1-\alpha')}{(1-\beta' B')} \tag{4.3}$$

which readily shows that $f \in S_{\lambda,\mu,\eta}(\alpha',\beta',A',B',m)$, making use of Theorem 1 .

Reversing the above steps , we can establish the other part of the equivalence of (4.1). Conversely, the assertion (4.1) can easily be used to imply the condition (4.2) and this completes the proof of Theorem 4. □

Remark 3. *For* $0 \le \lambda < 1, \mu < 1,\ 0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le B < A \le 1,\ 0 < A \le 1,\ \ m \in \mathbb{N},\ \eta > \max\{\lambda,\mu\} - 1$, *it follows that from (4.1) that*

$$S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m) = S_{\lambda,\mu,\eta}\left(\frac{1 - A\beta + (A-B)\alpha\beta}{(1-\beta B)}, 1, 1, -1, m\right). \tag{4.4}$$

Theorem 5. *Let* $0 \le \lambda < 1, \mu < 1,\ 0 \le \alpha_1 \le< \alpha_2 < 1,\ 0 < \beta \le 1,\ -1 \le B < A \le 1,\ 0 < A \le 1,\ \ m \in \mathbb{N},\ \eta > \max\{\lambda,\mu\} - 1$. *Then*

$$S_{\lambda,\mu,\eta}(\alpha_1,\beta,A,B,m) \supset S_{\lambda,\mu,\eta}(\alpha_2,\beta,A,B,m). \tag{4.5}$$

Proof. The result follows easily from Theorem 1. □

5. Results involving Hadamard Product

In this section we study interesting properties and theorems for the class of the

functions $S_{\lambda,\mu,\eta}(\alpha,\beta,A,B,m)$ involving the modified Hadamard product of several functions. Let $f(z)$ be defined by (1.1) and let

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0). \tag{5.1}$$

Then the modified Hadamard product of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{5.2}$$

The following result reveals an interesting property of the modified Hadamard product of several functions.

Theorem 6. *Let the functions $f_1(z)$, $f_2(z)$,$f_3(z), \cdots$, $f_r(z)$ defined by*

$$f_i(z) = z - \sum_{n=2}^{\infty} C_{n,i} z^n \quad (C_{n,i} \geq 0) \tag{5.3}$$

be in the class $S_{\lambda,\mu,\eta}(\alpha_i,\beta_i,A_i,B_i,m)$, $i = 1,2,3,\ldots r$, respectively. Also, let

$$\Phi_2(\lambda,\mu,\eta,m)\left(1 - \max_{1\leq i\leq r} \beta_i B_i\right) \geq 1. \tag{5.4}$$

Then

$$f_1 * f_2 * \cdots * f_r(z) \in S_{\lambda,\mu,\eta}(\prod_{i=1}^{r}\alpha_i, \prod_{i=1}^{r}\beta_i, \prod_{i=1}^{r}A_i, \prod_{i=1}^{r}B_i, m). \tag{5.5}$$

The result is sharp.

By hypothesis, $f_j(z) \in S_{\lambda,\,\mu,\,\eta}(\alpha_i,\,\beta_i,\,A_i,\,B_i,\,m)$, for all $i = 1,\,2,\,\cdots,r$ therefore, by Theorem 1, we have

$$\sum_{n=2}^{\infty} \Phi_n(\lambda,\,\mu,\,\eta,\,m)(1 - B_i\beta_i)C_{n,\,i} \leq (A_i - B_i)\beta_i(1-\alpha_i) \tag{5.6}$$

and

$$\sum_{n=2}^{\infty} C_{n,\,i} \leq \frac{(A_i - B_i)\beta_i(1-\alpha_i)}{\Phi_2(\lambda,\,\mu,\,\eta,\,m)(1-B_i\beta_i)},\ \text{ for all, }\ i = 1,\,2,\,\cdots. \tag{5.7}$$

For β_i satisfying $0 < \beta \leq 1,\ \ i = 1,\,2,\,\cdots$, we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Phi_n(\lambda,\,\mu,\,\eta,\,m)\left[1 - \prod_{i=1}^{r} B_i\beta_i\right]\prod_{i=1}^{r} C_{n,\,i} \\
\leq\ &\sum_{n=2}^{\infty} \Phi_n(\lambda,\,\mu,\,\eta,\,m)[1 - B_r\beta_r]\prod_{i=1}^{r} C_{n,\,i} \\
=\ &\sum_{n=2}^{\infty} \{\Phi_n(\lambda,\,\mu,\,\eta,\,m)[1 - B_r\beta_r]C_{n,\,r}\}\prod_{i=1}^{r} C_{n,\,i}.
\end{aligned}$$

Using (5.6) for any fixed $i = r$, and (5.7) for rest, it follows

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Phi_n(\lambda,\,\mu,\,\eta,\,m)\left[1 - \prod_{i=1}^{r} B_i\beta_i\right]\prod_{i=1}^{r} C_{n,\,i} \\
\leq\ &\frac{\beta_r(A_r - B_r)(1-\alpha_r)\prod_{i=1}^{r-1}\beta_i(A_i - B_i)(1-\alpha_i)}{\prod_{i=1}^{r-1}(1 - B_i\beta_i)\{\Phi_2(\lambda,\,\mu,\,\eta,\,m)\}^{r-1}} \\
\leq\ &\prod_{i=1}^{r}\beta_r\prod_{i=1}^{r}(A_r - B_r)\prod_{i=1}^{r}(1-\alpha_i)\left[\frac{1}{\Phi_n(\lambda,\,\mu,\,\eta,\,m)1 - \max\limits_{1\leq i\leq r}\beta_i B_i}\right]^{r-1} \\
\leq\ &\prod_{i=1}^{r}\beta_i\left[\prod_{i=1}^{r} A_i - \prod_{i=1}^{r} B_i\right]\left[\prod_{i=1}^{r}(1-\alpha_i)\right]
\end{aligned}$$

because in view of (5.4)

$$0 < \left[\frac{1}{\Phi_2(\lambda,\,\mu,\,\eta,\,m)[1 - \max_{(1\leq i\leq r)}(1 - \beta_i B_i)]}\right] \leq 1. \tag{5.8}$$

Hence with the aid of Theorem 1, the assertion (5.5) is proved. For $\alpha_i = \alpha,\ \beta_i = \beta,\ A_i = A,\ B_i = B, 1 = 1, 2, 3\cdots r$, Theorem 6 yields the following result. $\square$

Corollary 4. *Let the functions* $f_1(z),\ f_2(z), f_3(z), \cdots,\ f_r(z)$ *defined by (5.3) be in the same class* $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m),\ i = 1, 2, 3, \ldots r,$. *Also, let*

$$\Phi_2(\lambda, \mu, \eta, m)\,(1 - \max \beta B) \geq 1. \tag{5.9}$$

Then

$$f_1 * f_2 * f_3 \cdots * f_r(z) \in S_{\lambda,\mu,\eta}(\alpha^r, \beta^r, A^r, B^r, m). \tag{5.10}$$

Theorem 7. *Let the functions*$f_i(z)\,(i = 1, 2)$ *, defined by (5.3) be in the class* $S_{\lambda,\mu,\eta}(\alpha, \beta, A, B, m)$. *Then*

$$(f_1 * f_2)(z) \in S_{\lambda,\mu,\eta}(\sigma, \beta, A, B, m), \tag{5.11}$$

where

$$\sigma = \sigma(\alpha, \beta, A, B, \lambda, \mu, \eta, m) = 1 - \frac{(A - B)\beta(1 - \alpha)^2}{\Phi_2(\lambda, \mu, \eta, m)\,(1 - \beta B)}. \tag{5.12}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largestσ such that

$$\sum_{n=2}^{\infty} \frac{\Phi_n(\lambda, \mu, \eta, m)\,(1 - \beta B)}{(A - B)\beta(1 - \sigma)} C_{n,1}C_{n,2} \leq 1. \tag{5.13}$$

where σ is the function given by (5.13). By Cauchy-Schwarz inequality it follows from (2.2) of Theorem 1 that

$$\sum_{n=2}^{\infty} \frac{\Phi_n(\lambda, \mu, \eta, m)\,(1 - \beta B)}{(A - B)\beta(1 - \sigma)} \sqrt{C_{n,1}C_{n,2}} \leq 1. \tag{5.14}$$

Let us find σ such that

$$\sum_{n=2}^{\infty} \frac{\Phi_n(\lambda, \mu, \eta, m)\,(1 - \beta B)}{(A - B)\beta(1 - \sigma)} C_{n,1}C_{n,2} \leq \sum_{n=2}^{\infty} \frac{\Phi_n(\lambda, \mu, \eta, m)\,(1 - \beta B)}{(A - B)\beta(1 - \sigma)} \sqrt{C_{n,1}C_{n,2}} \tag{5.15}$$

which implies that

$$\sqrt{C_{n,1}C_{n,2}} \leq \frac{1 - \sigma}{1 - \alpha} \text{with } n \geq 2. \tag{5.16}$$

In view of (5.15) it is sufficient to find largest σ such that

$$\frac{(A-B)\beta(1-\alpha)}{\Phi_n(\lambda,\mu,\eta,m)\,(1-\beta B)} \leq \frac{1-\sigma}{1-\alpha}, \tag{5.17}$$

which yields

$$\sigma \leq 1 - \frac{(A-B)\beta(1-\alpha)^2}{\Phi_n(\lambda,\mu,\eta,m)\,(1-\beta B)}. \tag{5.18}$$

That is

$$\sigma \leq 1 - \frac{(A-B)\beta(1-\alpha)^2}{(1-\beta B)}\Theta_1(n). \tag{5.19}$$

where

$$\Theta_1(n) = \frac{1}{\Phi_n(\lambda,\mu,\eta,m)}. \tag{5.20}$$

Noting that $\Theta_1(n)$ is decreasing function of $n(n \geq 2)$ for a fixed λ, μ, η, m satisfying $0 \leq \lambda \leq \mu < 1$, $m \in \mathbb{N}$, and $\eta \in \mathbb{R}^+$ since we have for large

$$\frac{\Theta_1(n+1)}{\Theta_1(n)}\ \frac{(n+1)^{-\lambda}}{(n)^{-\lambda)}} = \left[1 - \frac{1}{n+1}\right]^{\lambda} \leq 1$$

for $n \geq 2$, $0 \leq \lambda < 1$ and under aforementioned constraints. Hence,

$$\sigma \leq \sigma(\alpha,\beta,A,B,\lambda,\mu,\eta,m) = 1 - \frac{(A-B)\beta(1-\alpha)^2}{(1-\beta B)}\Theta_1(2). \tag{5.21}$$

In view of (5.14), (5.18), (5.20), and (5.22) , the assertion (5.12) is hence proved. Lastly, by considering the function

$$f_i(z) = z - \frac{(A-B)\beta(1-\alpha)^2}{(1-\beta B)\,\Phi_2(\lambda,\mu,\eta,m)}z^2, \quad (i = 1,2), \tag{5.22}$$

it can be shown that the result is sharp. □

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