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Neighborhood Properties of Certain Classes of Analytic Functions Using the Dziok-Srivastava Operator*

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Abstract

The main object of this paper is to prove several inclusion relations associated with the (n, δ)-neighborhoods of various subclasses of starlike and convex functions of complex order, which are introduced here by means of the Dziok-Srivastava operator. Special cases of some of these inclusion relations are shown to yield known results.

Keywords and Phrases: *Analytic functions, Starlike functions, Convex functions, Dziok-Srivastava operator, (n, δ)-neighborhood, Inclusion relations, Identity function.*

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1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \ \ (a_k \geq 0;\ k \in \mathbb{N} - \{1\};\ n \in \mathbb{N} := \{1, 2, 3, \cdots\}), \quad (1.1)$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Following [10,13],we define the (n, δ)-neighborhood of a function $f(z) \in \mathcal{A}(n)$ by

$$N_{n,\delta}(f) := \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\} \quad (1.2)$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \quad (1.3)$$

A function $f(z) \in \mathcal{A}(n)$ is said to be starlike of complex order γ $(\gamma \in \mathbb{C} - \{0\})$, that is $f(z) \in \mathcal{S}_n^{\star}(\gamma)$, if it satisfies the inequality:

$$\mathrm{Re}\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0 \quad (z \in \Delta;\ \gamma \in \mathbb{C} - \{0\}). \quad (1.4)$$

Furthermore, a function $f(z) \in \mathcal{A}(n)$ is said to be convex of complex order γ $(\gamma \in \mathbb{C} - \{0\})$, that is $f(z) \in \mathcal{C}_n(\gamma)$, if it satisfies the inequality:

$$\mathrm{Re}\left(1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \Delta;\ \gamma \in \mathbb{C} - \{0\}). \quad (1.5)$$

The classes $\mathcal{S}_n^{\star}(\gamma)$ and $\mathcal{C}_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered by Nasr and Aouf [12] and Wiatrowski [16], respectively (see also [5] and [6]).

Let $\mathcal{S}_n(\gamma,\mu,\beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left|\frac{1}{\gamma}\left\{\frac{zf'(z)+\mu z^2 f''(z)}{\mu z f'(z)+(1-\mu)f(z)}-1\right\}\right|<\beta, \tag{1.6}$$

$$(z\in\Delta;\ \gamma\in\mathbb{C}-\{0\};\quad 0\le\mu\le 1;\quad 0<\beta\le 1).$$

Let $\mathcal{R}_n(\gamma,\mu,\beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left|\frac{1}{\gamma}\left\{f'(z)+\mu z f''(z)-1\right\}\right|<\beta, \tag{1.7}$$

$$(z\in\Delta;\ \gamma\in\mathbb{C}\setminus\{0\};\ 0\le\mu\le 1;\ 0<\beta\le 1).$$

Neighborhoods of the classes $\mathcal{S}_n(\gamma,\mu,\beta)$ and $\mathcal{R}_n(\gamma,\mu,\beta)$ were studied by Altintaş, Özkan and Srivastava[4]. Let $\mathcal{A}$ denote the class of functions of the form $f(z)=z-\sum_{k=2}^{\infty}a_k z^k$ which are analytic in the open unit disk $\Delta:=\{z:|z|<1\}$. Following the linear convolution operator introduced and investigated in a series of papers by Dziok and Srivastava (see, for details, [7], [8] and [9]), we define the Dziok-Srivastava operator for $f(z)$ belonging to $\mathcal{A}$ as follows:

For $c_i\in\mathbb{C}(i=1,\cdots,l)$ and $d_j\in\mathbb{C}-\{0,-1,-2,\cdots\}\,(j=1,\cdots,m)$, the generalised hypergeometric function is defined by

$${}_lF_m\,(c_1,\cdots,c_l;\ d_1,\cdots,d_m)=\sum_{k=0}^{\infty}\frac{(c_1)_k\cdots(c_l)_k}{(d_1)_k\cdots(d_m)_k}\times\frac{z^k}{k!},$$

$$(l\le m+1;\ l,m\in\mathbb{N}_0=\{0,1,2,\cdots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1)$$

for $n\in\mathbb{N}=\{1,2,\cdots\}$ and 1 when $n=0$.

Corresponding to the function

$$h(c_1,\cdots,c_l;\ d_1,\cdots,d_m;\ z)=z\ {}_lF_m(c_1,\cdots,c_l;\ d_1,\cdots,d_m),$$

the Dziok-Srivastava operator, $H_m^l(c_1, \cdots, c_l;\ d_1, \cdots, d_m)$ is defined by

$$H_m^l(c_1, \cdots, c_l;\ d_1, \cdots, d_m)f(z) = h(c_1, \cdots, c_l;\ d_1, \cdots, d_m;\ z) * f(z)$$

$$= z - \sum_{k=2}^{\infty} \frac{(c_1)_{k-1} \cdots (c_l)_{k-1}}{(d_1)_{k-1} \cdots (d_m)_{k-1}} \cdot \frac{a_k z^k}{(k-1)!}\ , \tag{1.8}$$

where "$*$" stands for convolution.

It is well known that

$$\begin{aligned} & c_1 H_m^l\,(c_1+1, \cdots, c_l;\ d_1, \cdots, d_m)\, f(z) \\ = \;& z\left[H_m^l(c_1, \cdots, c_l;\ d_1, \cdots, d_m) f(z)\right]' \\ & +(c_1-1) H_m^l(c_1, \cdots, c_l;\ d_1, \cdots, d_m) f(z). \end{aligned} \tag{1.9}$$

To make the notation simple, we write,

$$H_m^l[c_1]f(z) = H_m^l(c_1, \cdots, c_l;\ d_1, \cdots, d_m;\ z)f(z). \tag{1.10}$$

We can define the Dziok-Srivatsava operator for $f(z)$ belonging to the class $\mathcal{A}(n)$ as follows:

$$H_m^l[c_1]f(z) = z - \sum_{k=n+1}^{\infty} \frac{(c_1)_{k-1} \cdots (c_l)_{k-1}}{(d_1)_{k-1} \cdots (d_m)_{k-1}} \cdot \frac{a_k z^k}{(k-1)!}. \tag{1.11}$$

Let $\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ in $\mathcal{A}(n)$ which satisfy the inequality:

$$\left| \frac{1}{\gamma} \left\{ \frac{z\left(H_m^l[c_1]f(z)\right)' + \mu z^2 \left(H_m^l[c_1]f(z)\right)''}{\mu z \left(H_m^l[c_1]f(z)\right)' + (1-\mu)\left(H_m^l[c_1]f(z)\right)} - 1 \right\} \right| < \beta \tag{1.12}$$

$\big(z \in \Delta;\ \gamma \in \mathbb{C} - \{0\}\,;\ l \leq m+1; \quad 0 < \beta \leq 1; \mu \geq 0;\ c_i \in \mathbb{C},\ i = 1, 2, \cdots, l;$ $d_j \in \mathbb{C} - \{0, -1, -2, \cdots\},\ j = 1, 2, \cdots, m\big)$.

Also let $\mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ in $\mathcal{A}(n)$ which satisfy the inequality:

$$\left| \frac{1}{\gamma} \left\{ \left(H_m^l[c_1]f(z)\right)' + \mu z \left(H_m^l[c_1]f(z)\right)'' - 1 \right\} \right| < \beta \tag{1.13}$$

$(z \in \Delta;\ \gamma \in \mathbb{C} - \{0\}\,; \mu \geq 0; \quad 0 < \beta \leq 1; \quad l \leq m+1; \quad c_i \in \mathbb{C}, i = 1, 2, \cdots, l;$
$d_j \in \mathbb{C} - \{0, -1, -2, \cdots\},\ j = 1, \cdots, m)\,.$

For $c_1 = \lambda+1, c_2 = 1$ and $d_1 = 1$ with $l = 2$ and $m = 1$ the Dziok-Srivastava operator for $f(z)$ belong to $\mathcal{A}(n)$ reduces to the Ruscheweyh derivative operator $D^\lambda : \mathcal{A}(n) \to \mathcal{A}(n)$ which is defined as

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z)\ (\lambda > -1; \quad f(z) \in \mathcal{A}(n))$$

or equivalently, by

$$D^\lambda f(z) = z - \sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} a_k z^k\ \ (\lambda > -1; \quad f(z) \in \mathcal{A}(n))\,. \tag{1.14}$$

Here, and in what follows, we make use of the following standard notation:

$$\binom{p}{q} := \frac{p(p-1)\cdots(p-q+1)}{q!}\ (p \in \mathbb{C};\ q \in \mathbb{N}_0).$$

for a binomial coefficient. In particular, we have

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}\ (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

The above defined classes $\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$ and $\mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ with the above convention for c_1, c_2 and d_1 reduces to the classes $\mathcal{S}_n(\gamma, \mu, \beta, \lambda)$ and $\mathcal{R}_n(\gamma, \mu, \beta, \lambda)$ which are defined as follows:

Let $\mathcal{S}_n(\gamma, \mu, \beta, \lambda)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ in $\mathcal{A}(n)$ which satisfy the inequality:

$$\left| \frac{1}{\gamma} \left\{ \frac{z\left(D^\lambda f(z)\right)' + \mu z^2 \left(D^\lambda f(z)\right)''}{\mu z \left(D^\lambda f(z)\right)' + (1-\mu)\left(D^\lambda f(z)\right)} - 1 \right\} \right| < \beta \tag{1.15}$$

$$(z \in \Delta;\ \gamma \in \mathbb{C} - \{0\}\,; \quad 0 < \beta \leq 1; \mu \geq 0,\ \lambda > -1)\,.$$

Also, let $\mathcal{R}_n(\gamma, \mu, \beta, \lambda)$ denote the subclass of $\mathcal{A}(n)$ consisting of $f(z)$ in $\mathcal{A}(n)$ which satisfy the inequality:

$$\left| \frac{1}{\gamma} \left\{ \left(D^\lambda f(z)\right)' + \mu z \left(D^\lambda f(z)\right)'' - 1 \right\} \right| < \beta \tag{1.16}$$

$$(z \in \Delta;\ \gamma \in \mathbb{C} - \{0\}\,; \quad 0 < \beta \leq 1; \mu \geq 0,\ \lambda > -1)\,.$$

Various subclasses of $\mathcal{S}_n(\gamma, \mu, \beta, 0)$ and $\mathcal{R}_n(\gamma, \mu, \beta, 0)$ with $\gamma = 1$ were studied in many earlier works (cf., e.g., [2,15]).

Clearly we have,

$$\mathcal{S}_n(\gamma, 0, 1, 0) \subset \mathcal{S}_n^{\star}(\gamma) \text{ and } \mathcal{S}_n(\gamma, 0, 1, 1) \subset \mathcal{C}_n(\gamma)\ \ (n \in \mathbb{N}, \gamma \in \mathbb{C} - \{0\})\,.$$

The main object of the present paper is to investigate the (n, δ)-neighborhoods of the subclasses

$$\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j), \mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j) \text{ and } \mathcal{S}_n(\gamma, \mu, \beta, \lambda), \mathcal{R}_n(\gamma, \mu, \beta, \lambda).$$

of the class $\mathcal{A}(n)$ of normalised analytic functions in Δ with negative and missing coefficients.

Our definition of the function classes

$$\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j), \mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j) \text{ and } \mathcal{S}_n(\gamma, \mu, \beta, \lambda), \mathcal{R}_n(\gamma, \mu, \beta, \lambda)$$

are motivated essentially by two earlier investigations [4] and [11], in each of which closely- related subclasses can be found.In particular in our function classes $\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j), \mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ and $\mathcal{S}_n(\gamma, \mu, \beta, \lambda), \mathcal{R}_n(\gamma, \mu, \beta, \lambda)$ involving the inequality (1.12),(1.13),(1.15) and (1.16), we have relaxed the parametric constraint $0 \leq \mu \leq 1$, which was imposed earlier by Altintaş,Özkan and Srivastava[4, p.64 Equations (1.7) and (1.8)](see also Remark 3 below).

2. A Set of Inclusion Relations Involving $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we need the following Lemmas.

Lemma 1. Let the function $f(z) \in \mathcal{A}(n)$ be defined by (1.1), then $f(z)$ is in the class $\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$if and only if

$$\sum_{k=n+1}^{\infty} \frac{(c_1)_{k-1} \cdots (c_l)_{k-1}}{(d_1)_{k-1} \cdots (d_m)_{k-1}} \left[\left((k-1) + \beta|\gamma| \right) \left(1 + \mu(k-1) \right) \right] \frac{a_k}{(k-1)!} \leq \beta|\gamma|. \quad (2.1)$$

Proof. Let $f(z) \in \mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$. Then by appealing to condition (1.12), we readily get

$$\text{Re}\ \left\{ \frac{z\left(H_m^l[c_1]f(z)\right)' + \mu z^2 \left(H_m^l[c_1]f(z)\right)''}{\mu z\left(H_m^l[c_1]f(z)\right)' + (1-\mu)\left(H_m^l[c_1]f(z)\right)} - 1 \right\} > -\beta|\gamma|,\ \ (z \in \Delta) \quad (2.2)$$

or equivalently,

$$\mathrm{Re}\left\{\frac{-\sum_{k=n+1}^{\infty}\frac{(c_1)_{k-1}\cdots(c_l)_{k-1}}{(d_1)_{k-1}\cdots(d_m)_{k-1}}\left[(k-1)(1+\mu(k-1))\right]\frac{a_k z^k}{(k-1)!}}{z-\sum_{k=n+1}^{\infty}\frac{(c_1)_{k-1}\cdots(c_l)_{k-1}}{(d_1)_{k-1}\cdots(d_m)_{k-1}}[1+\mu(k-1)]\frac{a_k z^k}{(k-1)!}}\right\}$$
$$> \quad -\beta|\gamma|, (z \in \Delta) \tag{2.3}$$

where we have made use of definition (1.1). Now choose values of z on the real axis and let $z \to 1^-$ through real values. Then the inequality (2.3) immediately yields the required condition(2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find that

$$\left|\frac{z\left(H_m^l[c_1]f(z)\right)' + \mu z^2\left(H_m^l[c_1]f(z)\right)''}{\mu z\left(H_m^l[c_1]f(z)\right)' + (1-\mu)\left(H_m^l[c_1]f(z)\right)} - 1\right|$$
$$= \left|\frac{\sum_{k=n+1}^{\infty}\frac{(c_1)_{k-1}\cdots(c_l)_{k-1}}{(d_1)_{k-1}\cdots(d_m)_{k-1}}\left[(k-1)(1+\mu(k-1))\right]\frac{a_k z^k}{(k-1)!}}{z-\sum_{k=n+1}^{\infty}\frac{(c_1)_{k-1}\cdots(c_l)_{k-1}}{(d_1)_{k-1}\cdots(d_m)_{k-1}}\left[1+\mu(k-1)\right]\frac{a_k z^k}{(k-1)!}}\right|$$
$$\leq \frac{\beta|\gamma|\left\{1-\sum_{k=n+1}^{\infty}\frac{(c_1)_{k-1}\cdots(c_l)_{k-1}}{(d_1)_{k-1}\cdots(d_m)_{k-1}}\left[1+\mu(k-1)\right]\frac{a_k}{(k-1)!}\right\}}{1-\sum_{k=n+1}^{\infty}\frac{(c_1)_{k-1}\cdots(c_l)_{k-1}}{(d_1)_{k-1}\cdots(d_m)_{k-1}}\left[1+\mu(k-1)\right]\frac{a_k}{(k-1)!}}$$
$$\leq \quad \beta|\gamma| \ . \tag{2.4}$$

Hence, by maximum modulus theorem, we have $f(z) \in \mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$, which evidently completes the proof of Lemma 1.

Similarly, we can prove the following.

Lemma 2. Let the function $f(z) \in \mathcal{A}(n)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{(c_1)_{k-1} \cdots (c_l)_{k-1}}{(d_1)_{k-1} \cdots (d_m)_{k-1}} \left[k(1+\mu(k-1))\right] \frac{a_k}{(k-1)!} \leq \beta|\gamma|. \tag{2.5}$$

Now, for $c_1 = \lambda + 1, c_2 = 1$ and $d_1 = 1$ with $l = 2$ and $m = 1$ in the above lemmas, we get the corresponding results for the subclasses $\mathcal{S}_n(\gamma, \mu, \beta, \lambda)$ and $\mathcal{R}_n(\gamma, \mu, \beta, \lambda)$.

Lemma 3. Let the function $f(z) \in \mathcal{A}(n)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{S}_n(\gamma, \mu, \beta, \lambda)$ if and only if

$$\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} \left[\left((k-1)+\beta|\gamma|\right)\left(1+\mu(k-1)\right)\right] a_k \leq \beta|\gamma|. \tag{2.6}$$

Lemma 4. Let the function $f(z) \in \mathcal{A}(n)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{R}_n(\gamma, \mu, \beta, \lambda)$ if and only if

$$\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} \left[k\left(1+\mu(k-1)\right)\right] a_k \leq \beta|\gamma|. \tag{2.7}$$

Remark 1. A special case of Lemma 1 and Lemma 2 when $l = 2, m = 1, c_1 = c_2 = d_1 = 1$ was given by Altintaş et. al[4, p.64,Lemma 1 and p.65,Lemma 2].

Remark 2. A special case of Lemma 3 when $\mu = 0$ was given recently by Murugusundaramoorthy and Srivastava [11]. Also a special case of Lemma 3 when $n = 1, \gamma = 1, \mu = 0$ and $\beta = 1 - \alpha$ $(0 \leq \alpha < 1)$ was given earlier by Ahuja [1]. Furthermore, if, in Lemma 3 with $n = 1, \gamma = 1, \mu = 0, \beta = 1 - \alpha$ $(0 \leq \alpha < 1)$, we set $\lambda = 0$ and $\lambda = 1$, we shall obtain the familiar results of Silverman [14].

Our first inclusion relations involving $N_{n,\delta}(e)$ are given by Theorems below.

Theorem 1. *If*

$$\delta := \frac{(n+1)\beta|\gamma|}{\left[\dfrac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\dfrac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right]}\ , \tag{2.8}$$

then

$$\mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j) \subset N_{n,\delta}(e). \tag{2.9}$$

Proof. For a function $f(z) \in \mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$, lemma 1 immediately yields

$$\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right] \left[\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right] \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right] \left[\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right]}. \tag{2.10}$$

On the other hand, we also find from (2.1) and (2.10) that

$$\begin{aligned}
&\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right] \left[\frac{1+\mu n}{n!}\right] \sum_{k=n+1}^{\infty} k a_k \\
\leq\ & \beta|\gamma| + (1-\beta|\gamma|) \left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right] \left[\frac{1+\mu n}{n!}\right] \sum_{k=n+1}^{\infty} a_k \\
\leq\ & \beta|\gamma| + (1-\beta|\gamma|) \left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right] \left[\frac{1+\mu n}{n!}\right] \\
&\left[\frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_l)_n \cdots (d_m)_n}\right] \left[\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right]}\right] \\
\leq\ & \frac{(n+1)\beta|\gamma|}{n+\beta|\gamma|} \ (|\gamma| < 1),
\end{aligned}$$

that is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right] \left[\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right]} := \delta. \tag{2.11}$$

which, in view of definition (1.3), proves Theorem 1.

Similarly by applying Lemma 2 instead of Lemma 1, we now prove Theorem 2 below.

Theorem 2. *If*

$$\delta := \frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{1+\mu n}{n!}\right]}, \tag{2.12}$$

then

$$\mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j) \subset N_{n,\delta}(e). \tag{2.13}$$

Proof. Suppose that a function $f(z) \in \mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ is of the form (1.1). Then we find from the assertion (2.5) of Lemma 2 that

$$\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{(n+1)(1+\mu n)}{n!}\right]\sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|,$$

which yields the following inequality:

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{(n+1)(1+\mu n)}{n!}\right]}. \tag{2.14}$$

Making use of (2.5) in conjunction with (2.14), we have

$$\begin{aligned}
&\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\frac{\mu(n+1)}{n!}\sum_{k=n+1}^{\infty} k a_k \\
\leq\ & \beta|\gamma| + \left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{(\mu-1)(n+1)}{n!}\right]\sum_{k=n+1}^{\infty} a_k \\
\leq\ & \beta|\gamma| + \left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{(\mu-1)(n+1)}{n!}\right] \\
&\left[\frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{(n+1)(1+\mu n)}{n!}\right]}\right]
\end{aligned}$$

that is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right]\left[\frac{1+\mu n}{n!}\right]} := \delta \tag{2.15}$$

which in light of definition (1.3), completes the proof of Theorem 2.

Now for $c_1 = \lambda + 1,\ c_2 = 1$ and $d_1 = 1$ with $l = 2$ and $m = 1$ in the above theorems, we get the corresponding results for the subclasses $\mathcal{S}_n(\gamma, \mu, \beta, \lambda)$ and $\mathcal{R}_n(\gamma, \mu, \beta, lambda)$.

Theorem 3. *If*

$$\delta := \frac{(n+1)\beta|\gamma|}{\binom{\lambda+n}{n}(1+\mu n)(n+\beta|\gamma|)}, \tag{2.16}$$

then

$$\mathcal{S}_n(\gamma, \mu, \beta, \lambda) \subset N_{n,\delta}(e). \tag{2.17}$$

Theorem 4. *If*

$$\delta := \frac{\beta|\gamma|}{\binom{\lambda+n}{n}(1+\mu n)}, \tag{2.18}$$

then

$$\mathcal{R}_n(\gamma, \mu, \beta, \lambda) \subset N_{n,\delta}(e). \tag{2.19}$$

Remark 3. A special case of Theorem 1 and Theorem 2 when $l = 2, m = 1, c_1 = c_2 = d_1 = 1$ was given by Altintaş et. al[4, p.65,Theorem 1 and p.66,Theorem 2].Incidentally as we indicated in section 1 above, the condition of $\mu > 1$ is needed in the proof of one of these known results [4, p.66, Theorem 2].This implies that the constraint $0 \leq \mu \leq 1$ in [4, p.64 Equations (1.7) and (1.8)] should be replaced by a less stringent constraint $\mu \geq 0$.

Remark 4. A special case of Theorem 3 when $\mu = 0$ was given recently by Murugusundaramoorthy and Srivastava [11].Also a special case of Theorem 3 when $\lambda = 0, \mu = 0, \gamma = 1 - \alpha$ $(0 \leq \alpha < 1)$ *and* $\beta = 1$ was proven by Altintaş and Owa [3].

3. Neighborhoods for The Classes $\mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j)$ *and* $\mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j)$

In this section, we determine the neighborhood for each of the classes $\mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j), \mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j)$ and $\mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda), \mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda)$ which we define as follows.

A function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j)$, if there exists a function $g(z) \in \mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \alpha \ (z \in \Delta; \quad 0 \leq \alpha < 1). \tag{3.1}$$

Also, a function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j)$, if there exists a function $g(z) \in \mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ such that the inequality (3.1) holds true.

Analogously, we can define the classes $\mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda)$ and $\mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda)$.

Theorem 5. *If* $g(z) \in \mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$ *and*

$$\alpha = 1 - \frac{\delta \left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n} \right] \left[\frac{(1 + \mu n)(n + \beta|\gamma|)}{n!} \right]}{(n+1) \left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n} \left(\frac{(1 + \mu n)(n + \beta|\gamma|)}{n!} \right) - \beta|\gamma| \right]}, \tag{3.2}$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j). \tag{3.3}$$

Proof. Suppose that $f(z) \in N_{n,\delta}(g)$. We then find from (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \tag{3.4}$$

which readily implies the coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1} \ (n \in \mathbb{N}). \tag{3.5}$$

Next, since $g(z) \in \mathcal{S}_n(\gamma, \mu, \beta, c_i, d_j)$, we have from (2.14)

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{\left[\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\left(\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right)\right]} \tag{3.6}$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \\
&\leq \frac{\delta}{n+1}\left[\frac{\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\left(\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right)}{\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\left(\frac{(1+\mu n)(n+\beta|\gamma|)}{n!}\right) - \beta|\gamma|}\right] \\
&= 1 - \alpha, \qquad (3.7)
\end{aligned}$$

provided that α is given precisely by (3.2).

Thus, by definition, $f(z) \in \mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j)$ for α given by (3.2), which evidently completes the proof of Theorem 5.

The proof of Theorem 6 below is much similar to that of Theorem 5 and so the details are omitted.

Theorem 6. *If* $g(z) \in \mathcal{R}_n(\gamma, \mu, \beta, c_i, d_j)$ *and*

$$\alpha = 1 - \frac{\delta\left(\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right)\left(\frac{(1+\mu n)}{n!}\right)}{\left(\frac{(c_1)_n \cdots (c_l)_n}{(d_1)_n \cdots (d_m)_n}\right)\left(\frac{(n+1)(1+\mu n)}{n!}\right) - \beta|\gamma|}\ , \tag{3.8}$$

then

$$N_{n,\delta}(g) \subset \mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, c_i, d_j). \tag{3.9}$$

For the values of $c_1 = \lambda + 1, c_2 = 1$ and $d_1 = 1$ with $l = 2$ and $m = 1$ in Theorem 5 and Theorem 6, we get the corresponding results for the subclasses $\mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda)$ and $\mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda)$.

Theorem 7. *If $g(z) \in \mathcal{S}_n(\gamma, \mu, \beta, \lambda)$ and*

$$\alpha = 1 - \frac{\delta\binom{\lambda+n}{n}(1+\mu n)(n+\beta|\gamma|)}{(n+1)\left[\binom{\lambda+n}{n}(1+\mu n)(n+\beta|\gamma|) - \beta|\gamma|\right]} \tag{3.10}$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda). \tag{3.11}$$

Theorem 8. *If $g(z) \in \mathcal{R}_n(\gamma, \mu, \beta, \lambda)$ and*

$$\alpha = 1 - \frac{\delta\binom{\lambda+n}{n}(1+\mu n)}{\binom{\lambda+n}{n}(n+1)(1+\mu n) - \beta|\gamma|}\ , \tag{3.12}$$

then

$$N_{n,\delta}(g) \subset \mathcal{R}_n^{(\alpha)}(\gamma, \mu, \beta, \lambda). \tag{3.13}$$

Remark 5. A special case of Theorem 5 and Theorem 6 when $l = 2, m = 1, c_1 = c_2 = d_1 = 1$ was given by Altintaş et. al[4, p.67, Theorem 3 and p.67, Theorem 4].

Remark 6. A special case of Theorem 7 when $\mu = 0$ was given by Murugasundaramoorthy and Srivastava [11].

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