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# A Sharp Generalized Ostrowski-Grüss Inequality*

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**Abstract**

The main purpose of this paper is to use a variant of Grüss inequality to obtain a sharp generalized Ostrowski-Grüss inequality for absolutely continuous functions whose derivative is bounded both above and below almost everywhere. Thus we provide improvement and generalization of some previous results.

**Keywords and Phrases:** *Grüss inequality, Ostrowski-Grüss inequality, Absolutely continuous, Sharp bound.*

## 1. Introduction

In 1935, G.Grüss (see for example [**6, p.296**]), proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

**Theorem A.** *Let* $h, g : [a, b] \to \mathbf{R}$ *be two integrable functions such that* $\phi \leq h(t) \leq \Phi$ *and* $\gamma \leq g(t) \leq \Gamma$ *for all* $t \in [a, b]$*, where* $\phi$*,* $\Phi$*,* $\gamma$*,* $\Gamma$ *are real numbers. Then we have*

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$$|T(h,g)| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{1}$$

*where*

$$T(h,g) = \frac{1}{b-a}\int_a^b h(t)g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \tag{2}$$

*and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.*

From then on, (1) is well known in the literature as Grüss inequality.

A premature Grüss inequality originated from Grüss' work (see also [**6, p.296**]) is embodied in the following theorem which was considered and applied for the first time in the paper [**5**] by M.Matić, J.Pečarić and N.Ujević in 2000.

**Theorem B.** *Let $h, g : [a,b] \to \mathbf{R}$ be two integrable functions such that $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, where $\gamma, \Gamma \in \mathbf{R}$. Then we have*

$$|T(h,g)| \leq \frac{\Gamma - \gamma}{2}[T(h,h)]^{\frac{1}{2}}, \tag{3}$$

*where $T(h,g)$ is as defined in (2).*

In 2002, X.L.Cheng and J.Sun [**3**] have got the following variant of the Grüss inequality.

**Theorem C.** *Let $h, g : [a,b] \to \mathbf{R}$ be two integrable functions such that $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, where $\gamma, \Gamma \in \mathbf{R}$. Then*

$$|T(h,g)| \leq \frac{\Gamma - \gamma}{2(b-a)}\int_a^b |h(t) - \frac{1}{b-a}\int_a^b h(u)\,du|dt, \tag{4}$$

*where $T(h,g)$ is as defined in (2).*

It is not difficult to find that the premature Grüss inequality (3) provides a sharper bound than the Grüss inequality (1) and the variant of Grüss inequality (4) provides a sharper bound than the premature Grüss inequality (3).

In [**1**], Theorem A and Theorem B have been used to provide a generalized Ostrowski-Grüss inequality with different bounds. In this paper, we will use Theorem C to give a sharp generalized Ostrowski-Grüss inequality for absolutely continuous functions whose derivative is bounded above and below almost everywhere. Some sharp integral inequalities of midpoint, trapezoidal and Simpson type are obtained or recaptured as particular cases.

## 2. The Results

**Theorem.** *Let $f : [a,b] \to \mathbf{R}$ be a function which is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in \mathbf{R}$ such that $\gamma \leq f'(t) \leq \Gamma$ a.e. on $[a,b]$. Then for all $x \in [a,b]$ we have*

$$\begin{aligned} &|\int_a^b f(t)\,dt - (b-a)\{(1-\theta)f(x) + \theta[(\tfrac{x-a}{b-a})f(a) + (\tfrac{b-x}{b-a})f(b)]\} \\ &+(b-a)(1-2\theta)(x-\tfrac{a+b}{2})S| \leq \tfrac{\Gamma-\gamma}{2}I(\theta,x), \end{aligned} \tag{5}$$

*where*

$$S = \frac{f(b)-f(a)}{b-a},$$

*and*

$$I(\theta,x) = \begin{cases} [\frac{a+b}{2} - (1-\theta)a - \theta x]^2, & a \leq x \leq \frac{a+(1-2\theta)b}{2(1-\theta)}, \\ [\frac{1}{4} + (\theta-\frac{1}{2})^2][(x-a)^2 + (b-x)^2], & \frac{a+(1-2\theta)b}{2(1-\theta)} < x < \frac{(1-2\theta)a+b}{2(1-\theta)}, \\ [\theta x + (1-\theta)b - \frac{a+b}{2}]^2, & \frac{(1-2\theta)a+b}{2(1-\theta)} \leq x \leq b \end{cases} \tag{6}$$

*for $0 \leq \theta \leq \frac{1}{2}$, and*

$$I(\theta,x) = \begin{cases} [\frac{a+b}{2} - \theta a - (1-\theta)x]^2, & a \leq x \leq \frac{a+(2\theta-1)b}{2\theta}, \\ [\frac{1}{4} + (\theta-\frac{1}{2})^2][(x-a)^2 + (b-x)^2], & \frac{a+(2\theta-1)b}{2\theta} < x < \frac{(2\theta-1)a+b}{2\theta}, \\ [(1-\theta)x + \theta b - \frac{a+b}{2}]^2, & \frac{(2\theta-1)a+b}{2\theta} \leq x \leq b \end{cases} \tag{7}$$

*for $\frac{1}{2} < \theta \leq 1$.*

**Proof.** Integrating by parts produces the identity

$$\int_a^b K(x,t)f'(t)\,dt = (1-\theta)(b-a)f(x)+\theta(x-a)f(a)+\theta(b-x)f(b)-\int_a^b f(t)\,dt, \tag{8}$$

where

$$K(x,t) = \begin{cases} t-[\theta x+(1-\theta)a], & t\in[a,x], \\ t-[\theta x+(1-\theta)b], & t\in(x,b]. \end{cases} \tag{9}$$

Moreover

$$\frac{1}{b-a}\int_a^b K(x,t)\,dt = (1-2\theta)(x-\frac{a+b}{2}). \tag{10}$$

Applying the variant of Grüss inequality (4) by associating $g(t)$ with $f'(t)$ and $h(t)$ with $K(x,t)$ and multiply through by (b-a) gives

$$\begin{array}{rl} & |\int_a^b K(x,t)f'(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)\,dt\int_a^b f'(t)\,dt| \\ \leq & \frac{\Gamma-\gamma}{2}\int_a^b |K(x,t)-\frac{1}{b-a}\int_a^b K(x,u)\,du|\,dt. \end{array}$$

Then for any fixed $x\in[a,b]$ we can derive from (8), (9) and (10) that

$$\begin{array}{l} |\int_a^b f(t)\,dt-(b-a)\{(1-\theta)f(x)+\theta[(\frac{x-a}{b-a})f(a)+(\frac{b-x}{b-a})f(b)]\} \\ +(b-a)(1-2\theta)(x-\frac{a+b}{2})S| \leq \frac{\Gamma-\gamma}{2}I(\theta,x), \end{array} \tag{11}$$

where

$$\begin{array}{rcl} I(\theta,x) & = & \int_a^x |t-[\theta x+(1-\theta)a]-(1-2\theta)(x-\frac{a+b}{2})|\,dt \\ & & +\int_x^b |t-[\theta x+(1-\theta)b]-(1-2\theta)(x-\frac{a+b}{2})|\,dt \\ & = & \int_a^x |t-[(1-\theta)x+\theta b-\frac{b-a}{2}]|\,dt+\int_x^b |t-[\theta a+(1-\theta)x+\frac{b-a}{2}]|\,dt. \end{array}$$

The last two integrals can be calculated as follows:

For brevity, we put

$$p_1(t) := t-[(1-\theta)x+\theta b-\frac{b-a}{2}],\ \ t\in[a,x],$$

$$p_2(t) := t-[\theta a+(1-\theta)x+\frac{b-a}{2}],\ \ t\in[x,b]$$

and denote $t_1 = (1-\theta)x + \theta b - \frac{b-a}{2}$, $t_2 = \theta a + (1-\theta)x + \frac{b-a}{2}$.

It is clear that both $p_1(t)$ and $p_2(t)$ are strictly increasing on $[a,x]$ and $[x,b]$ respectively. Moreover, we have

$$p_1(a) = (1-\theta)(b-x) - \frac{b-a}{2},\ \ p_1(x) = \frac{b-a}{2} - \theta(b-x);$$

$$p_2(x) = \theta(x-a) - \frac{b-a}{2},\ \ p_2(b) = \frac{b-a}{2} - (1-\theta)(x-a).$$

For $0 \le \theta \le \frac{1}{2}$, it is immediate that $p_1(x) > 0$ and $p_2(x) < 0$. Meanwhile, $p_1(a) \ge 0$ if $x \in [a, \frac{a+(1-2\theta)b}{2(1-\theta)}]$ and $p_1(a) < 0$ if $x \in (\frac{a+(1-2\theta)b}{2(1-\theta)}, b]$, $p_2(b) \le 0$ if $x \in [\frac{(1-2\theta)a+b}{2(1-\theta)}, b]$ and $p_2(b) > 0$ if $x \in [a, \frac{(1-2\theta)a+b}{2(1-\theta)})$. Notice that $\frac{a+(1-2\theta)b}{2(1-\theta)} \le \frac{(1-2\theta)a+b}{2(1-\theta)}$, there are three possible cases to be determined.

(i) In case $x \in [a, \frac{a+(1-2\theta)b}{2(1-\theta)}]$, $p_1(t) \ge 0$ for $t \in [a,x]$ and $p_2(b) > 0$ with $t_2 \in (x,b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta,x) &= \textstyle\int_a^x (t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\ &= \tfrac{(1-2\theta)(x-a)(b-x)}{2} + \theta(\theta-1)(x-a)^2 + \tfrac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(b-x) + (\tfrac{1}{2}-\theta)(x-a)]^2 = [\tfrac{a+b}{2} - (1-\theta)a - \theta x]^2. \end{aligned} \tag{12}$$

(ii) In case $x \in (\frac{a+(1-2\theta)b}{2(1-\theta)}, \frac{(1-2\theta)a+b}{2(1-\theta)})$, $p_1(a) < 0$ with $t_1 \in (a,x)$ such that $p_1(t_1) = 0$ and $p_2(b) > 0$ with $t_2 \in (x,b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta,x) &= \textstyle\int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^{t_2}(t_2-t)\,dt + \int_{t_2}^b (t-t_2)\,dt \\ &= [\tfrac{1}{4} + (\theta-\tfrac{1}{2})^2][(x-a)^2 + (b-x)^2]. \end{aligned} \tag{13}$$

(iii) In case $x \in [\frac{(1-2\theta)a+b}{2(1-\theta)}, b]$, $p_1(a) < 0$ with $t_1 \in (a,x)$ such that $p_1(t_1) = 0$ and $p_2(t) \le 0$ for $t \in [x,b]$. We have

$$\begin{aligned} I(\theta,x) &= \textstyle\int_a^{t_1}(t_1-t)\,dt + \int_{t_1}^x (t-t_1)\,dt + \int_x^b (t_2-t)\,dt \\ &= \tfrac{(1-2\theta)(x-a)(b-x)}{2} + \theta(\theta-1)(b-x)^2 + \tfrac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(x-a) + (\tfrac{1}{2}-\theta)(b-x)]^2 = [\theta x + (1-\theta)b - \tfrac{a+b}{2}]^2. \end{aligned} \tag{14}$$

For $\frac{1}{2} < \theta \le 1$, it is immediate that $p_1(a) < 0$ and $p_2(b) > 0$. Meanwhile, $p_1(x) \le 0$ if $x \in [a, \frac{a+(2\theta-1)b}{2\theta}]$ and $p_1(x) > 0$ if $x \in (\frac{a+(2\theta-1)b}{2\theta}, b]$, $p_2(x) \ge 0$ if

$x \in [\frac{(2\theta-1)a+b}{2\theta}, b]$ and $p_2(x) < 0$ if $x \in [a, \frac{(2\theta-1)a+b}{2\theta})$. Notice that $\frac{a+(2\theta-1)b}{2\theta} \leq \frac{(2\theta-1)a+b}{2\theta}$, there are three possible cases to be determined.

(iv) In case $x \in [a, \frac{a+(2\theta-1)b}{2\theta}]$, $p_1(t) \leq 0$ for $t \in [a, x]$ and $p_2(x) < 0$ with $t_2 \in (x, b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta, x) &= \int_a^x (t_1 - t)\, dt + \int_x^{t_2} (t_2 - t)\, dt + \int_{t_2}^b (t - t_2)\, dt \\ &= \frac{(2\theta-1)(x-a)(b-x)}{2} + \theta(\theta-1)(x-a)^2 + \frac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(b-x) + (\theta - \tfrac{1}{2})(x-a)]^2 = [\tfrac{a+b}{2} - \theta a - (1-\theta)x]^2. \end{aligned} \tag{15}$$

(v) In case $x \in (\frac{a+(2\theta-1)b}{2\theta}, \frac{(2\theta-1)a+b}{2\theta})$, $p_1(x) > 0$ with $t_1 \in (a, x)$ such that $p_1(t_1) = 0$ and $p_2(x) < 0$ with $t_2 \in (x, b)$ such that $p_2(t_2) = 0$. We have

$$\begin{aligned} I(\theta, x) &= \int_a^{t_1} (t_1 - t)\, dt + \int_{t_1}^x (t - t_1)\, dt + \int_x^{t_2} (t_2 - t)\, dt + \int_{t_2}^b (t - t_2)\, dt \\ &= [\tfrac{1}{4} + (\theta - \tfrac{1}{2})^2][(x-a)^2 + (b-x)^2]. \end{aligned} \tag{16}$$

(vi) In case $x \in [\frac{(2\theta-1)a+b}{2\theta}, b]$, $p_1(x) > 0$ with $t_1 \in (a, x)$ such that $p_1(t_1) = 0$ and $p_2(t) \geq 0$ for $t \in [x, b]$. We have

$$\begin{aligned} I(\theta, x) &= \int_a^{t_1} (t_1 - t)\, dt + \int_{t_1}^x (t - t_1)\, dt + \int_x^b (t - t_2)\, dt \\ &= \frac{(2\theta-1)(x-a)(b-x)}{2} + \theta(\theta-1)(b-x)^2 + \frac{(x-a)^2+(b-x)^2}{4} \\ &= [\tfrac{1}{2}(x-a) + (\theta - \tfrac{1}{2})(b-x)]^2 = [(1-\theta)x + \theta b - \tfrac{a+b}{2}]^2. \end{aligned} \tag{17}$$

Consequently, the inequality (5) with (6) and (7) follows from (11), (12), (13), (14), (15), (16) and (17).

The proof is completed.

**Remark.** It is not difficult to prove that the inequality (5) with (6) and (7) is sharp in the sense that we can construct the function $f$ to attain the equality in (5) with (6) and (7). Indeed, if $0 \leq \theta \leq \frac{1}{2}$ then we may choose $f$ such that

$$f(t) = \begin{cases} \Gamma(t-a), & a \leq t < x, \\ \gamma(t-x) + (x-a)\Gamma, & x \leq t < t_2, \\ \Gamma(t - t_2 + x - a) + (t_2 - x)\gamma, & t_2 \leq t \leq b \end{cases}$$

for any $x \in [a, \frac{a+(1-2\theta)b}{2(1-\theta)}]$, and

$$f(t)=\begin{cases}\gamma(t-a), & a\le t<t_1,\\ \Gamma(t-t_1)+(t_1-a)\gamma, & t_1\le t<x,\\ \gamma(t-x+t_1-a)+(x-t_1)\Gamma, & x\le t<t_2,\\ \Gamma(t-t_2+x-t_1)+(t_2-x+t_1-a)\gamma, & t_2\le t\le b\end{cases}$$

for any $x\in(\frac{a+(1-2\theta)b}{2(1-\theta)},\frac{(1-2\theta)a+b}{2(1-\theta)})$, and

$$f(t)=\begin{cases}\gamma(t-a), & a\le t<t_1,\\ \Gamma(t-t_1)+(t_1-a)\gamma, & t_1\le t<x,\\ \gamma(t-x+t_1-a)+(x-t_1)\Gamma, & x\le t\le b\end{cases}$$

for any $x\in[\frac{(1-2\theta)a+b}{2(1-\theta)},b]$, and if $\frac{1}{2}<\theta\le 1$ then we may choose $f$ such that

$$f(t)=\begin{cases}\gamma(t-a), & a\le t<t_2,\\ \Gamma(t-t_2)+(t_2-a)\gamma, & t_2\le t\le b\end{cases}$$

for any $x\in[a,\frac{a+(2\theta-1)b}{2\theta}]$, and

$$f(t)=\begin{cases}\gamma(t-a), & a\le t<t_1,\\ \Gamma(t-t_1)+(t_1-a)\gamma, & t_1\le t<x,\\ \gamma(t-x+t_1-a)+(x-t_1)\Gamma, & x\le t<t_2,\\ \Gamma(t-t_2+x-t_1)+(t_2-x+t_1-a)\gamma, & t_2\le t\le b\end{cases}$$

for any $x\in(\frac{a+(2\theta-1)b}{2\theta},\frac{(2\theta-1)a+b}{2\theta})$, and

$$f(t)=\begin{cases}\gamma(t-a), & a\le t<t_1,\\ \Gamma(t-t_1)+(t_1-a)\gamma, & t_1\le t\le b\end{cases}$$

for any $x\in[\frac{(2\theta-1)a+b}{2\theta},b]$.

It is clear that the above all $f(t)$ are absolutely continuous on $[a,b]$.

**Corollary 1.** *Let the assumptions of Theorem hold. Then for all $x\in[a,b]$, we have*

$$\left|\int_a^b f(t)\,dt-(b-a)[f(x)-(x-\frac{a+b}{2})S]\right|\le\frac{(\Gamma-\gamma)(b-a)^2}{8}. \tag{18}$$

**Proof.** Letting $\theta = 0$ in (5) readily produces the result (18) from (6) on noting that $I(0, x) = \frac{(b-a)^2}{4}$.

It should be noted that (18) is a sharp perturbed Ostrowski inequality with a uniform bound independent of $x$, and in particular, if we choose in (18), $x = \frac{a+b}{2}$, we get a sharp midpoint inequality

$$|\int_a^b f(t)\,dt - (b-a)f(\frac{a+b}{2})| \le \frac{(\Gamma-\gamma)(b-a)^2}{8}.$$

**Corollary 2.** *Let the assumptions of Theorem hold. Then for all $x \in [a, b]$, we have*

$$\begin{array}{l} |\int_a^b f(t)\,dt - (b-a)[(\frac{x-a}{b-a})f(a) + (\frac{b-x}{b-a})f(b) \\ -(x - \frac{a+b}{2})S]| \le \frac{(\Gamma-\gamma)(b-a)^2}{8}. \end{array} \tag{19}$$

**Proof.** Letting $\theta = 1$ in (5) readily produces the result (19) from (7) on noting that $I(1, x) = \frac{(b-a)^2}{4}$.

It should be noted that (19) is a sharp perturbed generalized trapezoidal inequality with a uniform bound independent of $x$, and in particular, if we choose in (19), $x = \frac{a+b}{2}$, we get a sharp trapezoid inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)]| \le \frac{(\Gamma-\gamma)(b-a)^2}{8}.$$

**Corollary 3.** *Let the assumptions of Theorem hold. Then for all $x \in [a, b]$, we have*

$$\begin{array}{l} |\int_a^b f(t)\,dt - \frac{1}{2}[(b-a)f(x) + (x-a)f(a) + (b-x)f(b)]| \\ \le \frac{\Gamma-\gamma}{8}[(x-a)^2 + (b-x)^2]. \end{array} \tag{20}$$

**Proof.** Letting $\theta = \frac{1}{2}$ in (5) readily produces the result (20) from (6) on noting that $I(\frac{1}{2}, x) = \frac{1}{4}[(x-a)^2 + (b-x)^2]$.

It should be noted that we can find the inequality (20) in [**2**] and [**8**] with different proofs. However, we here have pointed out that the inequality (20) is sharp in the sense that we can find $f$ such that the equality in (20) holds.

**Corollary 4.** *Let the assumptions of Theorem hold. Then for any $\theta \in [0, 1]$, we have*

$$\begin{aligned}&|\int_a^b f(t)\,dt-(b-a)\{(1-\theta)f(\tfrac{a+b}{2})+\tfrac{\theta}{2}[f(a)+f(b)]\}|\\&\le\tfrac{(\Gamma-\gamma)(b-a)^2}{4}[\tfrac{1}{4}+(\theta-\tfrac{1}{2})^2].\end{aligned}\tag{21}$$

**Proof.** Letting $x=\frac{a+b}{2}$ in (5) readily produces the result (21) from (6) and (7) on noting that $I(\theta,\frac{a+b}{2})=\frac{(b-a)^2}{2}[\frac{1}{4}+(\theta-\frac{1}{2})^2]$.

It should be noted that taking $x=\frac{a+b}{2}$ in (20) or $\theta=\frac{1}{2}$ in (21) is equivalent to taking both these values in (5) which produces a sharp simple three point inequality as

$$|\int_a^b f(t)\,dt-\frac{b-a}{2}\{f(\frac{a+b}{2})+\frac{1}{2}[f(a)+f(b)]\}|\le\frac{(\Gamma-\gamma)(b-a)^2}{16}.\tag{22}$$

**Corollary 5.** *Let the assumptions of Theorem hold. Then for all $x\in[a,b]$, we have*

$$\begin{aligned}&|\int_a^b f(t)\,dt-\tfrac{b-a}{3}[2f(x)+(\tfrac{x-a}{b-a})f(a)+(\tfrac{b-x}{b-a})f(b)]\\&+\tfrac{b-a}{3}(x-\tfrac{a+b}{2})S|\le\tfrac{\Gamma-\gamma}{2}I(\tfrac{1}{3},x),\end{aligned}\tag{23}$$

*where*

$$I(\frac{1}{3},x)=\begin{cases}\frac{1}{36}[(x-a)+3(b-x)]^2, & a\le x\le\frac{3a+b}{4},\\ \frac{5}{18}[(x-a)^2+(b-x)^2], & \frac{3a+b}{4}<x<\frac{a+3b}{4},\\ \frac{1}{36}[3(x-a)+(b-x)]^2, & \frac{a+3b}{4}\le x\le b.\end{cases}\tag{24}$$

**Proof.** Letting $\theta=\frac{1}{3}$ in (5) readily produces the result (23) with (24).

It should be noted that (23) with (24) is a sharp generalized Simpson type inequality for unprescribed $x$, and in particular, if we choose in (23) and (24), $x=\frac{a+b}{2}$, we get a sharp Simpson inequality

$$|\int_a^b f(t)\,dt-\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]|\le\frac{5(\Gamma-\gamma)(b-a)^2}{72}.\tag{25}$$

It is interesting to note from (22) and (25) we can conclude that an average of the midpoint quadrature rule and trapezoidal quadrature rule has a better estimation of error than the well-known Simpson quadrature rule when we estimate the error in terms of the first derivative $f'$ of integrand $f$. The same conclusion can also be found in the previous papers [**1**], [**4**] and [**7**]. However, we here provide a generalization of the result in [**4**], and since both (22) and (25) are sharp, our assertion is more convincing than that stated in [**1**] and [**7**].

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