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Polynomial Expansions for Solutions of Higher-order q-Bessel Heat Equation*

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Abstract

In this paper we give the q-analogue of the higher-order Bessel operators studied by M. I. Klyuchantsev [12] and A. Fitouhi, N. H. Mahmoud and S. A. Ould Ahmed Mahmoud [3]. Our objective is twofold. First, using the q-Jackson integral and the q-derivative, we aim at establishing some properties of this function with proofs similar to the classical case. Second our goal is to construct the associated q-Fourier transform and the q-analogue of the theory of the heat polynomials introduced by P. C. Rosenbloom and D. V. Widder [13]. Our operator for some value of the vector index generalize the q-j_α Bessel operator of the second order in [4] and a q-Third operator in [6].

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1. Introduction

The Bessel operator of r-order is defined on $(0,\infty)$ by

$$B_r u = u^{(r)} + \frac{a_1}{x} u^{(r-1)} + ... + \frac{a_{r-1}}{x^{r-1}} u^{(1)}, \tag{1}$$

where the coefficients a_k depend on the components α_k

$$\alpha_k \geq -1 + \frac{k}{r} \quad , \qquad k = 1, ..., r-1. \tag{2}$$

and

$$a_{r-k} = \frac{1}{(k-1)!} \sum_{j=1}^{k} (-1)^{k-j} \binom{j-1}{k-1} \prod_{i=1}^{r-1} (r\alpha_i + j). \tag{3}$$

Where r is positive integer and $\alpha = (\alpha_1, ..., \alpha_{r-1})$ a vector having $(r-1)$ components with $|\alpha| = \alpha_1 + ... + \alpha_{r-1}$.

When $r = 2$, we obtain the classical Bessel operator of the second order

$$B_2 u = u'' + \frac{2\alpha + 1}{x} u' , \tag{4}$$

and for $r = 3$, $\alpha_1 = -2/3$, $\alpha_2 = \nu - 1/3$, we obtain the operator $B_3 u$ studied in [9] and in [5]

$$B_3 u = \frac{d^3}{dx^3} + \frac{3\,\nu}{x} \frac{d^2}{dx^2} - \frac{3\,\nu}{x^2} \frac{d}{dx}, \qquad \nu > 0. \tag{5}$$

For λ being a complex number, let us now consider the system

$$\begin{cases} B_r u(x) &= -\lambda^r u(x), \\ u(0) &= 1, \\ u^k(0) &= 0, \quad k = 1, ..., r-1. \end{cases} \tag{6}$$

The use of the Frobenius method leads us to conclude that (6) has a unique solution which is r-even and given by

$$j_\alpha(\lambda x) = \sum_{m\geq 0} (-1)^m \frac{1}{m!} \prod_{i=1}^{r-1} \frac{\Gamma(\alpha_i+1)}{\Gamma(\alpha_i+m+1)} \left(\frac{\lambda x}{r}\right)^{rm}. \tag{7}$$

In this paper we are concerned with the q-analogue of the j_α higher-order Bessel function (7). This choice is motivated in particular by the context of {[3], [4], [6]}.

The reader will notice that the definition (45) derives from that given in [3] with minor changes. With the help of the q-integral representation we establish the q-integral representation of the Mehler and Sonine types. Moreover, we define the higher-order q-Bessel translation and the higher-order q-Bessel Fourier transform and establish easily some of their properties. Finally, we study the higher-order q-Bessel heat equation.

2. Notation and Preliminary Results

Let q be a fixed real number $0 < q < 1$. Henceforth, we use the following notation:

$$(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b),\ \text{if } n = 0,1,2,...,\infty, \qquad (1+a)_q^t = \frac{(1+a)_q^\infty}{(1+q^t a)_q^\infty},\ \text{if } t\in\mathbb{C}\ . \tag{8}$$

We note for $\lambda \in \mathbb{R}_q$, $n = 0,1,2,...$,

$$(a;\,q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \tag{9}$$

$$(\lambda)_q = \frac{1-q^\lambda}{1-q}, \qquad (\lambda)_n^q = \frac{(q^\lambda;\,q)}{(1-q)^n}, \quad [n]_q! = \frac{(q;\,q)_n}{(1-q)^n}, \tag{10}$$

$$\frac{(\lambda)_n^q}{[n]_q!} = \frac{(q^\lambda;\,q)_n}{(q;\,q)_n}, \quad \frac{(\lambda)_n^q}{(\lambda+n-1)_q} = (\lambda)_{n-1}^q, \quad \frac{(1)_n^q}{(1)_{n-k}^q} = (-1)^k(-n)_k^q q^{nk-\binom{k}{2}}. \tag{11}$$

2.1 The q-Binomial formula

We note a q-Binmial formula by :

$$(ab;\,q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;\,q)_k (b;\,q)_{n-k}, \tag{12}$$

with

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;\, q)_n}{(q;\, q)_{n-k}(q;\, q)_k}, \quad n \in \mathbb{N},\ k = 0, 1, ..., n. \tag{13}$$

2.2 The q-derivative and the q-integral

We denote by D_q the q–derivative of a function

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \ . \tag{14}$$

$$D_q^n f(x) = \frac{q^{-\binom{n}{2}}}{x^n(1-q)^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} f(q^k x),\ n = 0, 1, 2, ... \tag{15}$$

$$D_q^n[f(x)g(x)] = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} f)(q^k x)(D_q^k g)(x),\ n = 0, 1, 2, ... \tag{16}$$

We define the q-schift operators by :

$$(\Lambda_q f)(x) = f(qx) \qquad \textit{and} \qquad (\Lambda_q^{-1} f)(x) = f(q^{-1}x), \tag{17}$$

$$D_q \Lambda_q = q \Lambda_q D_q \qquad \textit{and} \qquad D_q \Lambda_q^{-1} = q^{-1} \Lambda_q^{-1} D_q. \tag{18}$$

and also we note $(\Lambda_{q^\delta}^{-1} f)(x) = f(q^{-\delta} x)$.

The q-Jackson integrals (introduced by Thomae and Jackson [8]) from 0 to a and from aq to ∞ are defined by

$$\int_0^a f(x) d_q x = (1-q) \sum_{j=0}^{\infty} aq^j f(aq^j) \text{ and } \int_{aq}^{\infty} f(t) d_q t = (1-q) \sum_{k=0}^{+\infty} aq^{-k} f(aq^{-k}). \tag{19}$$

Notice that the series on the right hand side are guaranteed to be convergent. See [4].

We define the Jackson integral in a generic interval $[a, b]$ by [8]:

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x \ .$$

This is a special case of the following more general change of variable formula, [18, p 107]. If $u(x) = \alpha x^{\beta}$, then $\int_{u(a)}^{u(b)} f(u)d_q u = \int_a^b f(u(x))D_{q^{1/\beta}}u(x)d_{q^{1/\beta}}x$.

Using the q-Jackson integrals from 0 to 1, we define the q-integral $\int_0^1 ... \int_0^1 f(t_1, ..., t_n)d_q t_1...d_q t_n$ by

$$\int_0^1 ... \int_0^1 f(t_1, ..., t_n)\, d_q t_1...d_q t_n = (1-q)^n \sum_{i_1,...,i_n=0}^{\infty} q^{i_1+...+i_n}\, f(q^{i_1+...+i_n}), \quad (20)$$

provided the sums converge absolutely.

2.3 q-Exponential function

We present two q–analogues exponential function:

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!} = (1+(1-q)x)_q^{\infty} , \quad (21)$$

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}} . \quad (22)$$

Notice that for $q \in (0, 1)$ the series expansion of $e_q(x)$ has radius of convergence $1/(1-q)$. On the contrary, the series expansion of $E_q(x)$ converges for every x. Both product expansions (21) and (22) converge for all x.

2.4 q^{δ}-Basic hypergeometric series

We define the q^{δ}-basic hypergeometric series ${}_r\phi_s^{\delta}$ by

$${}_r\phi_s^{\delta}\left(\begin{matrix} q^{a_1}, ..., q^{a_r} \\ q^{b_1}, ..., q^{b_s} \end{matrix}\middle| q; (q-1)^{1+s-r} z\right) = \sum_{k=0}^{\infty} (q^{\delta})^{\binom{k}{2}} \frac{(a_1;q)_k^q ... (a_r;q)_k^q}{(b_1;q)_k^q ... (b_s;q)_k^q} \frac{z^k}{[k]_q!},$$

$$\lim_{q \uparrow 1} {}_r\phi_s^{\delta}\left(\begin{matrix} q^{a_1}, ..., q^{a_r} \\ q^{b_1}, ..., q^{b_s} \end{matrix}\middle| q; (q-1)^{1+s-r} z\right) = {}_r F_s \left[\begin{matrix} a_1, ...a_r \\ b_1, ...b_s \end{matrix}\middle| z\right]. \quad (23)$$

Where $\delta > 0$ and $r < s + 1$, this expansion converges for all values of z .

For $\delta = 1 + s - r$, we obtain the classic basic hypergeometric series ${}_r\phi_s$, [14, p 11,12].

We note for $\delta > 0$ by $e_q(x, \delta) = \sum_{n=0}^{\infty} q^{\delta\binom{n}{2}} \frac{x^n}{[n]_q!}$, this expansion converges for all values of x .

2.5 q–Gamma and q–Bta functions

The q–gamma function $\Gamma_q(t)$, a q–analogue of Euler's gamma function, was introduced by Thomae and later by Jackson as the infinite product

$$\Gamma_q(t) = \frac{(1-q)_q^{t-1}}{(1-q)^{t-1}} \ , \quad t > 0 \ . \tag{24}$$

The q–Beta function defined by the usual formula

$$\beta_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)} \ , \tag{25}$$

has the q–integral representation, which is a q–analogue of Euler's formula:

$$\beta_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x \ , \quad t,s > 0 \ . \tag{26}$$

The q-duplication formula holds

$$\prod_{i=1}^{r-1} \Gamma_{q^r}(n+\frac{i}{r}) \frac{1}{[rn]_q!} = \prod_{i=1}^{r-1} \Gamma_{q^r}(\frac{i}{r}) \frac{1}{[n]_{q^r}!} \frac{1}{((r)_q)^{rn}}, \tag{27}$$

and

$$((r)_q)^{rn}(1)_n^{q^r} \prod_{i=1}^{r-1} (\frac{i}{r})_n^{q^r} = [rn]_q!. \tag{28}$$

We also denote, $\prod (\alpha_i+1)_n^{q^r} = \prod_{i=1}^{r-1} (\alpha_i+1)_n^{q^r}.$

3. q-Trigonometric Fonction of r-order

The $r - q^{\delta}$-cosinus is defined for $\delta > 0$ by

$$\begin{aligned} \cos_r(x,\ q^r;\delta) = & \ {}_0\phi^{\delta}_{r-1}\left(\left. \begin{matrix} - \\ (q^r)^{1/r}, ..., (q^r)^{(r-1)/r} \end{matrix} \right| q^r; -\frac{(q^r-1)^r x^r}{(1+q+...+q^{r-1})^r} \right) \\ = & \sum_{m\geq 0} (-1)^m b_{rm}(x,\ q^r;\delta) \end{aligned} \tag{29}$$

where

$$b_{rm}(x,\, q^r;\delta) = (q^\delta)^{r\binom{m}{2}} \frac{x^{rm}}{[rm]_q!} = (q^r)^{\delta\binom{m}{2}} \frac{x^{rm}}{\alpha_{rm,q}}. \tag{30}$$

For every $\lambda \in \mathbb{C}$, the function $\cos_r(x,\, q^r;\delta)$ is a unique solution of the system

$$\begin{cases} \Lambda_{q\delta}^{-1} D_q^r u(x) & = & -\lambda^r u(x), \\ \quad u(0) & = & 1, \\ D_q^k u(0) & = & 0, \quad k = 1, ..., r-1. \end{cases} \tag{31}$$

We note $r - q^\delta$-sinus of order (r, l) , $l = 1, ..., r-1$ by

$$\sin_{r,l}(x,\, q^r;\delta) = \sum_{m\geq 0} (-1)^m (q^\delta)^{r\binom{m}{2}} \frac{x^{rm+r-l}}{[rm+r-l]_q!}. \tag{32}$$

Let $\mu = e^{i\pi/r}$ and $w_k = e^{2i\pi(k-1)/r}$, $k = 1, 2, ..., r$. Since

$$\sum_{k=1}^{r} (w_k)^m = \begin{cases} r & \text{for integers m divisible by r,} \\ 0 & \text{for integers m not divisible by r.} \end{cases} \tag{33}$$

and expanding the q-exponential function in series, we obtain

$$\cos_r(x,\, q^r;r\delta) = \frac{1}{r} \sum_{k=1}^{r} e_{q^r}\Big(\frac{\mu w_k x}{q^{(r-1)/2}}\,,\, \delta\Big). \tag{34}$$

When $r = 3$, $\delta = 1$, we obtain the result in [6].

Definition 3.1. *Let $x \in \mathbb{R}$ and $w_k = e^{2i\pi(k-1)/r}$, k=1,2,...,r, a function $f(x)$ is called r even if*

$$f(w_k x) = f(x) \qquad k = 1, ..., r, \tag{35}$$

and r odd of l order if

$$f(x) = w_k^l f(w_k x), \qquad k = 1, ..., r. \tag{36}$$

Proposition 3.2. *The functions* cos_r *and* $sin_{r,l}$ $(l = 1, ..., r-1)$ *are, respectively,r-even and r-odd of order l. From (29) and (32) we obtain the following q-derivative formulas :*

$$D_q^l \cos_r(x,\, q^r;\delta) = -\, q^{-\delta(r-l)} \sin_{r,l}(q^\delta x,\, q^r;\delta), \;\; D_q^r \cos_r(x,\, q^r;\delta) = -\cos_r(q^\delta x,\, q^r;\delta,),$$
$$D_q^{l-m} \sin_{r,m}(x,\, q^r;\delta) = \sin_{r,l}(x,\, q^r;\delta), \;\; D_q^{r-m} \sin_{r,m}(x,\, q^r;\delta) = \cos_r(x,\, q^r;\delta).$$

Proposition 3.3. *The function* $\cos_r(x,\, q^r;1)$ *is* r *even and satisfies,in particular*

$$\cos_r(xt,\, q^r;1) = (-1)^n q^{r(n(n+1)/2)} \frac{1}{x^{rn}} D_{q,t}^{rn}(\cos_r(xtq^{-n},\, q^r;1)). \tag{37}$$

Proposition 3.4. *Let* $x \in \mathbb{R}$ *for* $n \geq 1$*, the function* $b_{rn}(x,\, q^r;1)$ *verify the following properties*

$$b_0(x,\, q^r;1) = 1, \;\; b_{rn}(0,\, q^r;1) = 0 \;\; \textit{and} \;\;\; \Lambda_q^{-1} D_q^r b_{rn}(x,\, q^r;1) = b_{r(n-1)}(x,\, q^r;1). \tag{38}$$

$$|\; b_{rn}(x,\, q^r;\delta) \;|\leq|\; b_{rn}(x,\, q^r;1) \;|\leq \frac{q^{-\binom{r}{2}} x^{rn}}{(rn)!},\;\; \delta \geq 1. \tag{39}$$

Proof. When we put $q = e^{-t}$, $\quad t > 0$. The coefficients $b_{rn}(x,\, q^r;1)$ defined by (29) can be written as

$$b_{rn}(x,\, q^r;1) = \prod_{j=0}^{n-1} \prod_{i=0}^{r-1} \frac{q^j - q^{j+1}}{1 - q^{rj+1+i}} = \prod_{j=0}^{n-1} \prod_{i=0}^{r-1} \frac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-(rj+1+i)t}} \tag{40}$$

Preceding like in [16] ; for $j \neq 0$, the following function $f_i(t) = \dfrac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-(rj+1+i)t}}$ are decreasing in $]0, \infty[$, we obtain to something as limite where t tend to 0 in (40). □

When $|x| \uparrow \infty$, we have $|\cos_r(x,\, q^r;\delta)| \leq q^{-\binom{r}{2}} |\cos_r(x)| \leq q^{-\binom{r}{2}} e^{(r-2)|x|}, \delta \geq 1,$ see [4].

3.1 q^δ-Product formula

We set now the product formula for q^δ-cosinus function. We note by

$$P = \cos_r(x,\, q^r;\delta) \cos_r(y,\, q^r;\delta).$$

Proposition 3.5. Let x and y complex numbers, with $y \neq 0$, we have :

$$P = \sum_{k\geq 0} \frac{(q^{\delta})^{rk^2}}{(1-q)^{rk}[rk]_q!}(-1)^k q^{-\binom{rk}{2}} \big(\frac{x}{y}\big)^{rk} \sum_{s=0}^{rk} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} rk \\ s \end{bmatrix}_q \Lambda_{q^{\delta}}^{-k} \cos_r(yq^{rk-s},\, q^r;\delta).$$

Proof. For $y \neq 0$

$$P = \sum_{k\geq 0} \frac{(q^{\delta})^{rk^2}}{[rk]_q!} \big(\frac{x}{y}\big)^{rk} \sum_{n\geq 0} (-1)^n \frac{(q^{r\delta})^{\binom{n}{2}}}{[r(n-k)]_q!} (q^{\delta})^{-rnk} y^{rn}.$$

Moreover, if we use the previous relation

$$\frac{[rn]_q!}{[r(n-k)]_q!}(1-q)^{rk} = (-1)^{rk} q^{-\binom{rk}{2}+r^2nk} \sum_{s=0}^{rk} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} rk \\ s \end{bmatrix}_q q^{-rns},$$

we obtain that

$$P = \sum_{k\geq 0} \frac{(q^{\delta})^{rk^2}}{(1-q)^{rk}[rk]_q!}(-1)^k q^{-\binom{rk}{2}} \big(\frac{x}{y}\big)^{rk} \sum_{s=0}^{rk} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} rk \\ s \end{bmatrix}_q \cos_r(yq^{k(r-\delta)}q^{-s},\, q^r;\delta).$$

□

4. The q-Bessel Operator of r-order

We suppose now that the components of the vector $\alpha = (\alpha_1, ..., \alpha_{r-1})$ where α_k is a reel number satisfy $\alpha_k \geq -1 + \frac{k}{r}, \quad k = 1, ..., r-1$ and $\delta > 0$.

The q-Bessel operator of r-order is defined by

$$B_{r,\delta} u = \Lambda_{q^{\delta}}^{-1}\Big(\frac{1}{x^{r-1}} \prod_{i=1}^{r-1} \big(q^{r\alpha_i+1} x D_q + (r\alpha_i + 1)_q\,\big) D_q u\Big). \tag{41}$$

Remark 4.1. For $r = 2$, we obtain the q-Bessel operator $B_{2,\delta}$ of the second order studied in [4] for $\delta = 1$

$$B_{2,\delta} u = \Lambda_{q^{\delta}}^{-1}\big(q^{2\alpha+1} D_q^2 u + \frac{(2\alpha+1)_q}{x} D_q u\big). \tag{42}$$

and for $r = 3$, $\alpha_1 = -2/3$, $\alpha_2 = \nu - 1/3$, we obtain the operator $B_{3,\delta}$ studied in [6]

$$B_{3,\delta} u = \Lambda_{q^{\delta}}^{-1}\big(q^{3\nu} D_q^3 u + \frac{1}{q}\frac{(3\,\nu)_q}{x}\, D_q^2 u - \frac{1}{q}\frac{(3\,\nu)_q}{x^2}\, D_q u\big). \tag{43}$$

Proposition 4.2. *For λ being a complex number, the function $j_\alpha(\lambda x, q^r, \delta)$ is a solution of the q-problem*

$$\begin{aligned} B_{r,\delta}u(x) &= -\lambda^r u(x) \\ u(0) = 1,\ D_q^k u(0) &= 0, \quad k = 1, ..., r-1\,. \end{aligned} \tag{44}$$

$$\begin{aligned} j_\alpha(\lambda x, q^r, \delta) &= \sum_{n=0}^{\infty}(-1)^n b_{rn,\alpha}(x, q^r, \delta)\lambda^{rn}, \\ b_{rn,\alpha}(x, q^r, \delta) &= \frac{(q^r)^{\delta\binom{n}{2}} x^{rn}}{((r)_q)^{rn}(1)_n^{q^r}\prod(\alpha_i+1)_n^{q^r}} = \frac{(q^r)^{\delta\binom{n}{2}} x^{rn}}{\alpha_{rn,\alpha,q}}, \\ \alpha_{rn,\alpha,q} &= (1+q+...+q^{r-1})^{rn}\,[n]_{q^r}!\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+n+1)}{\Gamma_{q^r}(\alpha_i+1)}. \end{aligned} \tag{45}$$

$$j_\alpha(x, q^r, \delta) = {}_0\phi_{r-1}^{\delta}\left(\left.\begin{matrix} -, \\ (q^r)^{\alpha_1+1}, ..., (q^r)^{\alpha_{r-1}+1} \end{matrix}\right| q^r; -\frac{(q^r-1)^r x^r}{(1+q+...+q^{r-1})^r}\right). \tag{46}$$

For $\delta = r$, we obtain the q-hypergeometric function ${}_0\phi_{r-1}$.

4.1 Increase of $b_{rn,\alpha}(x, q^r, \delta)$:

Let now $\mid \alpha \mid = \alpha_1 + ... + \alpha_{r-1} = \alpha_0 + ... + \alpha_{r-1}$ with $\alpha_0 = 0$

$$b_{rn,\alpha}(1,\ q^r;\delta) \leq b_{rn,\alpha}(1, q^r, 1) = \frac{(q^r)^{\binom{n}{2}}}{((r)_q)^{rn}(1)_n^{q^r}\prod(\alpha_i+1)_n^{q^r}}, \quad \delta \geq 1 \tag{47}$$

the right term can be written by

$$\frac{\left((q^r)^{-|\alpha|/r}\right)^n}{((r)_q)^{rn}}\prod_{j=0}^{n-1}\prod_{i=0}^{r-1}\frac{(q^r)^{(\alpha_i+j)/r}-(q^r)^{1+(\alpha_i+j)/r}}{1-(q^r)^{1+(\alpha_i+j)}}. \tag{48}$$

Now, by [15], lemma A.1 and [16], proposition A.2, we see that the general terms of product increases to $(j+\alpha_i+1)^{-1}$ if $q \uparrow 1$. Using Stirling's formula, we find that, for some constant C.

$$b_{rn,\alpha}(1,\ q^r;\delta) \leq \frac{\left((q^r)^{-|\alpha|/r}\right)^n}{((r)_q)^{rn}\prod(\alpha_i+1)_n} \leq\ C\ \left((q^r)^{-|\alpha|/r}\right)^n\left(\frac{e}{n(r)_q}\right)^{rn+|\alpha|},$$

this inequality generalizes the inequality in [6].

Use the functional equation of q-Gamma, we obtain the following proposition :

Proposition 4.3. *For $\alpha_i \geq -1 + \frac{i}{r},\ i = 1, ..., r-1$, and n=0,1,2..., we have*

$$D_q j_\alpha(., q^r, \delta)(x) = -\left(\frac{x}{(r)_q}\right)^{r-1} \frac{1}{\prod(\alpha_i + 1)_{q^r}} j_{\alpha+1}(q^\delta x, q^r, \delta), \tag{49}$$

and

$$\{\frac{1}{x^{r-1}} D_q\}^n j_\alpha(x, q^r, \delta) = \left((\frac{1}{(r)_q})^{r-1}\right)^n \frac{(-1)^n (q^\delta)^{\binom{n}{2}}}{\prod(\alpha_i + 1)_n^{q^r}} j_{\alpha+n}(q^{n\delta} x, q^r, \delta). \tag{50}$$

By the q-duplication formula of Γ_q (27), we have, in particular

$$j_{(-1/r, -2/r, ..., -(r-1)/r)}(x, q^r, \delta) = \cos_r(x,\ q^r; \delta). \tag{51}$$

5. q-Integral Representations

In this section, we give two q-integral representations of the $q - j_\alpha$ function (45) involving the q-Jackson integral. We denote by W_α the function

$$W_\alpha(t_1, ..., t_{r-1}; q^r) = \prod_{i=1}^{r-1} \frac{(t_i^r q^r;\ q^r)_\infty}{(t_i^r q^{\alpha_i - \frac{i}{r} + 1};\ q^r)_\infty} t_i^{i-1} = \prod_{i=1}^{r-1} (t_i^r q^r;\ q^r)_{\alpha_i - \frac{i}{r}} t_i^{i-1} \tag{52}$$

$$= \prod_{i=1}^{r-1} (1 - q^r t_i^r)_{q^r}^{\alpha_i - \frac{i}{r}} t_i^{i-1} \tag{53}$$

which tends to $\prod(1 - t_i^r)^{\alpha_i - \frac{i}{r}}\ t_i^{i-1}$ as $q \longrightarrow 1^-$.

5.1 q-Mehler Type

Theorem 5.1. *For $\alpha_i \geq -1 + \frac{i}{r},\ i = 1, ..., r-1$, the function j_α has the following q-integral representation of Mehler type*

$$j_\alpha(z, q^r, \delta) = C_{r,\alpha} \int_0^1 ... \int_0^1 W_\alpha(t_1, ..., t_{r-1}; q^r)\ \cos_r(z t_1, ..., t_{r-1}; q^r, \delta)\, d_q t_1 ... d_q t_{r-1}, \tag{54}$$

where

$$C_{r,\alpha} = ((r)_q)^{r-1}.\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+1)}{\Gamma_{q^r}(\frac{i}{r})\Gamma_{q^r}(\alpha_i-\frac{i}{r}+1)}. \tag{55}$$

Proof. This formula can be proved by expanding $\cos_r(zt, q^r;\delta)$ in a series of power of t and then i grating, there arise q-integrals of the form

$$\int_0^1 t_i^{rm}(1-q^r t_i^r)_{q^r}^{\alpha_i-\frac{i}{r}} t_i^{i-1}\, d_q t_i = \frac{\Gamma_{q^r}(m+\frac{i}{r})\Gamma_{q^r}(\alpha_i-\frac{i}{r}+1)}{(r)_q\Gamma_{q^r}(\alpha_i+m+1)}. \tag{56}$$

Basis of the q-duplication formula for the Γ_q function (27), the formula is proved.

Proposition 5.2. For $\alpha_i \geq -1+\frac{i}{r},\ i=1,...,r-1$, and n=0,1,2..., we have

$$\left|D_q^n\big[j_\alpha(x,q^r,\delta)\big]\right| \leq \prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+1)\Gamma_{q^r}(\frac{n+i}{r})}{\Gamma_{q^r}(\frac{i}{r})\Gamma_{q^r}(\alpha_i+1+\frac{n}{r})}\left|\big[D_{q,x}^n\cos_r(x,\, q^r;\delta)\big]\right|, \tag{57}$$

in particular

$$\big|j_\alpha(x,q^r,\delta)\big| \leq q^{-\binom{r}{2}}\, e^{(r-2)|x|}. \tag{58}$$

5.2 q-Sonine Type

Theorem 5.3. *For $\alpha_i \geq -1+\frac{i}{r},\ i=1,...,r-1$ and $p_i \geq 1$, the function $j_{\alpha+p}$ has the following q-integral representation of Sonine type*

$$j_{\alpha+p}(z,q^r,\delta) = D_{r,\alpha,p}\int_0^1 ... \int_0^1\ V_p(t_1,...,t_{r-1};q^r)\ j_\alpha(zt_1,...,t_{r-1},q^r;\delta)\ d_qt_1...d_qt_{r-1}, \tag{59}$$

where

$$D_{r,\alpha,p} = ((r)_q)^{r-1}.\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+p_i+1)}{\Gamma_{q^r}(p_i)\Gamma_{q^r}(\alpha_i+1)}, \tag{60}$$

$$V_p(t_1,...,t_{r-1};q^r) = \prod_{i=1}^{r-1}(1-q^r t_i^r)_{q^r}^{p_i}\ t_i^{r(\alpha_i-\frac{i}{r}+1)}\ t_i^{i-1}\ . \tag{61}$$

Proof. This formula can be proved by expanding j_α in a series of power of t_i , there arise q-integrals of the form

$$\int_0^1 t_i^{r(\alpha_i-\frac{i}{r}+1+m)}(1-q^r t_i^r)_{q^r}^{p_i-1} t_i^{i-1}\, d_q t_i = \frac{\Gamma_{q^r}(\alpha_i+m+1)\Gamma_{q^r}(p_i)}{(r)_q\Gamma_{q^r}(m+\alpha_i+p_i+1)}.$$

6. q-Fourier Transform

Notation. - Some q-functional spaces will be used to establish our result. We putting

- $\mathbb{R}_q = \{\pm q^k,\ k \in \mathbb{Z}\} \cup \{0\}, \quad \mathbb{R}_q^* = \{\pm q^k,\ k \in \mathbb{Z}\}$,
- $\mathbb{R}_{q,+} = \{q^k,\ k \in \mathbb{Z}\} \cup \{0\}, \quad \mathbb{R}_{q,+}^* = \{q^k,\ k \in \mathbb{Z}\}$.

- We design by $\mathcal{E}_{*,q}(\mathbb{R})$ (resp $\mathcal{E}_{*,q}(\mathbb{R}_q)$) the space of r-even functions defined on $\mathbb{R}$ (resp $\mathbb{R}_q$) infinitely q-derivative, and by $\mathcal{D}_{*,q}(\mathbb{R})$ (resp $\mathcal{D}_{*,q}(\mathbb{R}_q)$) the space of r-even functions defined on $\mathbb{R}$ (resp $\mathbb{R}_q$) infinitely q-derivative with compact support.

In this section we introduce the space $\mathfrak{L}^1_{\alpha,q^\delta}(\mathbb{R}_{q,+},\, d_qx)$ of functions f satisfying

$$\int_0^\infty |f(x) j_\alpha(\lambda x,\, q^r;\delta)|\, d_qx < \infty, \qquad \lambda \in \mathbb{R}_q.$$

Definition 6.1. *The Fourier transform related with $B_{r,\delta}$ of $f \in \mathfrak{L}^1_{\alpha,q^\delta}(\mathbb{R}_{q,+},\, d_qx)$ is the function $\mathcal{F}_{q^\delta}(f)$ defined by*

$$\mathcal{F}_{q^\delta}(f)(\lambda) = \int_0^\infty f(t)\, j_\alpha(\lambda t,\, q^r;\delta)\, d_qt, \quad \lambda \in \mathbb{R}_q. \tag{62}$$

We define also the Fourier transform $\mathcal{F}_{0,q^\delta}$ by

$$\mathcal{F}_{0,q^\delta}(f)(\lambda) = \int_0^\infty f(t) \cos_r(\lambda t,\, q^r;\delta)\, d_qt, \quad \lambda \in \mathbb{R}_q. \tag{63}$$

7. q-Translation and q-Convolution

In this section we study the generalized translation operator associated with the operator $B_{r,\delta}$. We give the following definition related to $\Lambda_{q\delta}^{-1} D_q^r$.

Definition 7.1. *The translation operator τ_{x,q^δ}, $x \in \mathbb{R}$ (resp $\mathbb{R}_q$) associated with the r-order derivative operator $\Lambda_{q\delta}^{-1} D_q^r$ is defined for f in $\mathcal{E}_{*,q}(\mathbb{R})$ (resp $\mathcal{E}_{*,q}(\mathbb{R}_q)$) and $y \in \mathbb{R}$ (resp $\mathbb{R}_q$) by*

$$\tau_{x,\,q^\delta}(f)(y)=\sum_{n=0}^{\infty} b_{rn}(y,\,q^r;\delta)\,(\Lambda_{q\delta}^{-1}D_q^r)^{(n)}\,f(x), \tag{64}$$

the functions $b_{rn}(y,\,q^r;\delta)$ *are given by (29).*

We have the product formula

$$\cos_r(\lambda x,\,q^r;\delta).\cos_r(\lambda y,\,q^r;\delta)=\tau_{x,\,q^\delta}\cos_r(\lambda y,\,q^r;\delta)=\tau_{y,\,q^\delta}\cos_r(\lambda x,\,q^r;\delta).$$

Proposition 7.2. The operators $\tau_{x,\,q^\delta}$ satisfy :

1. For $x\in\mathbb{R}$, $\tau_{x,\,q^\delta}$ belong in $\mathcal{L}(\mathcal{E}_{*,q}(\mathbb{R}),\,\mathcal{E}_{*,q}(\mathbb{R}))$.
2. The map $x\longrightarrow\tau_{x,\,q^\delta}$ is infinitely q-derivative, r-even.

Lemma 7.3. *For* $f\in D_{*,q}(\mathbb{R})$, $n\in\mathbb{N}$, *we have*

$$\left(\Lambda_{q^\delta}^{-1}D_q^r\right)^n f(x)=\frac{q^{-\binom{rn}{2}}}{(1-q)^{rn}(q^{-\delta n})^{rn}b_{rn}(x,\,q^r;\delta)}\sum_{k=0}^{rn}\frac{(-1)^k q^{\binom{rn-k}{2}}}{[rn-k]_q![k]_q!}\Lambda_{q^\delta}^{-n}f(q^kx).$$

Proof. For $\delta>0$, by [17] and (15)

$$D_{q,x}^{rn}f(x)=\frac{q^{-\binom{rn}{2}}}{(1-q)^{rn}x^{rn}}\sum_{k=0}^{rn}(-1)^k\begin{bmatrix}rn\\k\end{bmatrix}_q q^{\binom{rn-k}{2}}f(q^kx),$$

$$\left(\Lambda_{q^\delta}^{-1}D_q^r\right)^n=\left((q^\delta)^r\right)^{-\binom{n}{2}}\Lambda_{q^\delta}^{-n}D_q^{rn},$$

$$\left(\Lambda_{q^\delta}^{-1}D_q^r\right)^n f(x)=\frac{q^{-\binom{rn}{2}}\left((q^\delta)^r\right)^{-\binom{n}{2}}}{(1-q)^{rn}(q^{-\delta n}x)^{rn}}\sum_{k=0}^{rn}(-1)^k\begin{bmatrix}rn\\k\end{bmatrix}_q q^{\binom{rn-k}{2}}\Lambda_{q^\delta}^{-n}f(q^kx).$$

□

We obtain for $\delta>0$

$$\tau_{y,q^\delta}f(x)=\sum_{n=0}^{\infty}\frac{b_{rn}(1,\,q^r;\delta)}{(1-q)^{rn}(q^{-\delta n})^{rn}}\left(\frac{y}{x}\right)^{rn}q^{-\binom{rn}{2}}\sum_{k=0}^{rn}(-1)^k\begin{bmatrix}rn\\k\end{bmatrix}_q q^{\binom{rn-k}{2}}\Lambda_{q^\delta}^{-n}f(q^kx).$$

Proposition 7.4. For $f \in \mathcal{D}_{*,\,q}(\mathbb{R}_q)$ we have :

$$\mathcal{F}_{0,\,q^\delta}({}^t\tau_{x,\,q^\delta} f)(\lambda) = \cos_r(\lambda x,\, q^3; \delta)\mathcal{F}_{0,\,q^\delta}(f)(\lambda).$$

The convolution product of two functions f and g of $\mathcal{D}_{*,\,q}(\mathbb{R}_q)$ is defined by :

$$f \star_{q^\delta} g(x) = \int_0^\infty {}^t\tau_{x,\,q^\delta} f(y)g(y)\, d_q y = \int_0^\infty f(y)\tau_{x,\,q^\delta} g(y)\, d_q y.$$

in $\mathcal{D}_{*,\,q}(\mathbb{R}_q)$ and we have : $\mathcal{F}_{0,\,q^\delta}(f \star_{q^\delta} g)(\lambda) = \mathcal{F}_{0,\,q^\delta}(f)(\lambda).\,\mathcal{F}_{0,\,q^\delta}(g)(\lambda)$.

These previous properties can be extended for the operator $B_{r,\delta}$ and suggest the following definition.

Definition 7.5. *We call generalized translation operators associated with* $B_{r,\delta}$ *the operators* $\mathrm{T}^\alpha_{x,\,q^\delta}$, $x \in \mathbb{R}$ *(resp* $\mathbb{R}_q$*), defined on* $\mathcal{E}_{*,\,q}(\mathbb{R})$ *(resp* $\mathcal{E}_{*,\,q}(\mathbb{R}_q)$*) by :*

$$\mathrm{T}^\alpha_{x,\,q^\delta}(f)(y) = \sum_{n=0}^{\infty} b_{rn,\,\alpha}(y,\, q^r; \delta)\, B^n_{r,\delta}(f)(y), \qquad y \in \mathbb{R}\ (resp\ \mathbb{R}_q), \tag{65}$$

where the functions $b_{rn,\alpha}(y, q^r, \delta)$ *is given by (45).*

We summarize the properties of $\mathrm{T}^\alpha_{x,\,q^\delta}$ in this proposition.

Proposition 7.6. The operators $\mathrm{T}^\alpha_{x,\,q^\delta}$ satisfy :

1. For $x \in \mathbb{R}$, $\mathrm{T}^\alpha_{x,\,q^\delta}$ in $\mathcal{L}(\mathcal{E}_{*,q}(\mathbb{R}),\, \mathcal{E}_{*,q}(\mathbb{R}))$
2. The map $x \longrightarrow \mathrm{T}^\alpha_{x,\,q^\delta}$ are infinitely q-derivative and r-even.
3. For all functions f in $\mathcal{E}_{*,q}(\mathbb{R})$:
   - $\mathrm{T}^\alpha_{x,\,q^\delta} f(y) = \mathrm{T}^\alpha_{y,\,q^\delta} f(x)$;
   - $\mathrm{T}^\alpha_{0,\,q^\delta} f(y) = f(y)$.
4. For given f in $\mathcal{E}_{*,q}(\mathbb{R})$, we put : $u(x,\, y) = \mathrm{T}^\alpha_{x,\,q^\delta}\, f(y)$.

Then the function u is solution of the Cauchy problem :

$$(\mathbb{I}) \qquad \begin{cases} B_{x,r,\delta} u(x,\, y) = B_{y,r,\delta} u(x,\, y), \\ u(x,\, 0) = f(x); D_{q,\,y} u(x,\, 0) = 0; \\ D^k_{q,y} u(x,\, 0) = 0\; k = 0.1.2...r-1. \end{cases}$$

$$\mathrm{T}^{\alpha}_{x,\,q^\delta} j_\alpha(\lambda y, q^r, \delta) = j_\alpha(\lambda x, q^r, \delta) j_\alpha(\lambda y, q^r, \delta) = \mathrm{T}^{\alpha}_{y,\,q^\delta} j_\alpha(\lambda x, q^r, \delta). \qquad (66)$$

Now we are able to define the convolution product related to the operator $B_{r,\delta}$.

Definition 7.7. *The convolution product associated with $B_{r,\delta}$ of two functions f and g in $\mathcal{D}_{*,\,q}(\mathbb{R}_q)$ is the function $f \star_{\alpha,\,q^\delta} g$ defined by :*

$$f \star_{\alpha,\,q^\delta} g(y) = \int_0^\infty f(x) \mathrm{T}^{\alpha}_{y,\,q^\delta} g(x)\, d_q x = \int_0^\infty {}^t\mathrm{T}^{\alpha}_{y,\,q^\delta} f(x)\, g(x)\, d_q x. \qquad (67)$$

8. Higher-order q-Bessel Heat Polynomials

We recall that the function $e_{q^r}(-z^r t) j_\alpha(xz;\, q^r;\, \delta)$ is analytic in z^r. We thus have, for $t \in \mathbb{R}$ and $\delta \geq 1$,

$$e_{q^r}(-z^r t) j_\alpha(xz;\, q^r;\, \delta) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{rn}}{\alpha_{rn,\alpha,q}} p_n^\alpha(x,\, t,\, q^r;\, \delta), \qquad (68)$$

$$p_n^\alpha(x,\, t,\, q^r;\, \delta) = \sum_{k=0}^{n} (q^r)^{\delta\binom{n-k}{2}} \frac{(x^r)^{n-k} t^k}{[k]_{q^r}!} \frac{\alpha_{rn,\alpha,q}}{\alpha_{r(n-k),\alpha,q}} \qquad (69)$$

$$= \frac{\prod(\alpha_i+1)_n^{q^r}}{(1+q+\ldots+q^{r-1})^{-rn}} t^n \sum_{k=0}^{\infty} \frac{(-1)^k (-n)_k^{q^r} (q^r)^{(\delta-1)\binom{k}{2}} (q^r)^{nk} (x^r)^k t^{-k}}{\prod(\alpha_i+1)_k^{q^r} (1+q+\ldots+q^{r-1})^{rk} [k]_{q^r}!} \qquad (70)$$

$$= \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} t^n\, {}_1\phi_{r-1}^{\delta-1} \left(\left. \begin{matrix} (q^r)^{-n}, \\ (q^r)^{\alpha_1+1}, \ldots, (q^r)^{\alpha_{r-1}+1} \end{matrix} \right| q^r; \frac{(q^r-1)^{r-1}(-x^r (q^r)^n)}{(1+q+\ldots+q^{r-1})^r t} \right). \qquad (71)$$

9. q-Heat Equation

We give an applications of the Fourier transform related with $B_{r,\delta}$. We begin by recalling that

$$\int_0^\infty e_{q^r}(-c x^r)\, (c^n x^{rn}) x^{r\alpha_k + (r-1)}\, d_q x = \frac{(q^r)^{-n(\alpha_k+1)-\binom{n}{2}} (\alpha_k+1)_n^{q^r} \mathbf{I}(\alpha_k+1, q^r)}{c^{\alpha_k+1}(1+q+\ldots+q^{r-1})}, \qquad (72)$$

$$\mathbf{I}(\alpha_k+1;\, q^r) = \int_0^\infty e_{q^r}(-x)\, x^{\alpha_k}\, d_{q^r} x \quad \text{and} \quad \mathbf{H}_{q^r}(\alpha_k+1) = \frac{\mathbf{I}(\alpha_k+1;\, q^r)}{\Gamma_{q^r}(\alpha_k+1)}. \qquad (73)$$

We note by $d\eta_{q,\alpha_k}(y) = \frac{y^{r\alpha_k+(r-1)}}{(1+q+...+q^{r-1})^{\alpha_k}\Gamma_{q^r}(\alpha_k+1)}\, d_qy$ and we define for $\delta > 1$ the fundamental solution $\mathcal{K}_{\alpha_k}(x, t,\, q^r; \delta)$ by

$$\mathcal{K}_{\alpha_k}(x, t,\, q^r; \delta) = \int_0^\infty e_{q^r}(-ty^r) j_\alpha(xy,\, q^r; \delta)\, d\eta_{q,\alpha_k}(y),$$

$$= \frac{\mathbf{H}_{q^r}(\alpha_k+1)}{(t(1+q+...+q^{r-1}))^{\alpha_k+1}} \times \sum_{n=0}^{\infty} \frac{(-1)^n (q^\delta)^{r\binom{n}{2}} \times (q^r)^{-(\alpha_k+1)n-\binom{n}{2}} x^{rn} t^{-n}}{(1+q+...+q^{r-1})^{rn}[n]_{q^r}! \prod_{i\neq k}(\alpha_i+1)_n^{q^r}},$$

$$= \frac{\mathbf{H}_{q^r}(\alpha_k+1)}{(t(1+q+...+q^{r-1}))^{\alpha_k+1}} \times$$

$${}_0\phi_{r-2}^{\delta-1}\left(\begin{matrix} -, \\ (q^r)^{\alpha_1+1}, .., (q^r)^{\alpha_{k-1}+1}, (q^r)^{\alpha_{k+1}+1}, ..., (q^r)^{\alpha_{r-1}+1} \end{matrix} \middle| q^r; (\frac{-x^r (q^r)^{-(\alpha_k+1)} (q^r-1)^{r-1}}{(1+q+...+q^{r-1})^r t}) \right).$$

For $\delta = r$, we obtain the basic hypergeometric series .

We consider the q-problem for $t, x \geq 0$

$$(\,\mathrm{II}\,) \begin{cases} B_{r,\delta} u(x,\, t) = D_{q^r,\, t} u(x,\, t) \\ D_q^k u(0,\, t) = 0, \quad k = 1, ..., r-1 \\ u(w_k\, x,\, t) = u(x,\, t), \quad k = 1, ..., r-1 \\ u(x, 0) = f(x). \end{cases} \tag{74}$$

Theorem 9.1. *Let $f \in \mathfrak{L}^1_{\alpha,\, q^\delta}(\mathbb{R}_{q,+},\, d_qx)$, the function*

$$u(x,\, t) = \int_0^\infty \mathrm{T}^\alpha_{y,\, q^\delta} \mathcal{K}_{\alpha_k}(x, t,\, q^r; \delta)\, f(y)\, d_qy = \big(f \star_{\alpha, q^\delta} \mathcal{K}_{\alpha_k}(.\,,\, t,\, q^r; \delta)\big)(x), \tag{75}$$

is a solution of the equation (II) for $\alpha_k \geq -1 + \frac{k}{r}, k = 1, ..., r-1,\ t, x \in \mathbb{R}_{q,+}$.

10. Analytic Cauchy Problem Related to The r-order q-Bessel operator $B_{r,\delta}$

We say that a function $u(x, t)$ in $\mathcal{H}_\alpha([0, a] \times [0, \sigma])$ if

$$B_{r,\delta}\, u(x, t) = D_{q^r, t}\, u(x, t) \tag{76}$$

$x \in [0, a]$ and $t \in [0, \sigma]$. The diffusion polynomials $p_n^\alpha(x, t)$ satisfy the q-equation(76). Hence we expect to obtain infinite series expansions $u(x, t) = \sum_{m=0}^\infty a_m p_m^\alpha(x,\, t,\, q^r;\, \delta))$ with possible convergence in a strip $|t| < \sigma$.

Let $\delta \geq 1$, we note

$$\mathcal{R}^{\delta}_{\alpha,q}(x) = \sum_{n=0}^{\infty} \frac{(q^r)^{(\delta-1)\binom{n}{2}} x^{rn}}{(1+q+...+q^{r-1})^{rn} \prod(\alpha_i+1)^{q^r}_n}$$
$$= {}_1\varphi^{\delta-1}_{r-1}\left(\begin{matrix} (q^r)^1 \\ (q^r)^{\alpha_1+1}, ..., (q^r)^{\alpha_{r-1}+1} \end{matrix} \middle| q^r; \frac{(q^r-1)^{r-1}x^r}{(1+q+..+q^{r-1})^r} \right).$$

Lemma 10.1. *Let $s > 0$ and $\delta \geq 1$*

$$\frac{p^{\alpha}_n(|x|,|t|,q^r,\delta)}{\alpha_{rn,\alpha,q}} \leq \frac{s^n}{[n]_{q^r}!}(1+\frac{|t|}{s})^n \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}).$$

Proof. We have

$$\frac{p^{\alpha}_n(|x|,|t|,q^r,\delta)}{\alpha_{rn,\alpha,q}} \leq \frac{s^n}{[n]_{q^r}!} \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q^r} \left(\frac{|t|}{s}\right)^{n-k} \frac{(q^r)^{\delta\binom{k}{2}} \frac{|x|^{rk}}{s^k}}{((r)_q)^{rk} \prod(\alpha_i+1)^{q^r}_n}$$
$$\leq \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}) \frac{s^n}{[n]_{q^r}!} \sum_{k=0}^{\infty} (q^r)^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^r} \left(\frac{|t|}{s}\right)^{n-k}$$
$$= \frac{s^n}{[n]_{q^r}!}\left(1+\frac{|t|}{s}\right)^n_{q^r} \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}) \leq \frac{s^n}{[n]_{q^r}!}\left(1+\frac{|t|}{s}\right)^n \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}),$$

since, $$\frac{(q^r)^{\delta-1\binom{k}{2}} \frac{|x|^{rk}}{s^k}}{((r)_q)^{rk} \prod(\alpha_i+1)^{q^r}_n} < \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}).$$ □

Lemma 10.2.

$$p^{\alpha}_n(x,t,q^r,\delta) \geq \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} t^n, \quad \textit{for } t,x > 0,\ \delta > 0.$$

Proof. Since the coefficients of p^{α}_n are positive, it follows that

$$p^{\alpha}_n(x,t,q^r,\delta) \geq p^{\alpha}_n(0,t,q^r,\delta) = \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} t^n.$$

□

Theorem 10.3. *If the series $\sum_{n=0}^{\infty} a_n p^{\alpha}_n(x_0,t_0,q^r,\delta)$ converges for $t_0 > 0$ and $x_0 > 0$, then the series $\sum_{n=0}^{\infty} a_n p^{\alpha}_n(x_0,t_0,q^r,\delta)$ and $\sum_{n=0}^{\infty} d_{rn,\alpha,q} a_n p^{\alpha}_{n-1}(x,t,q^r,\delta)$ converge absolutely and locally uniformly in the strip $|t| < t_0$ and $\sum_{n=0}^{\infty} a_n p^{\alpha}_n(x_0,t_0,q^r,\delta)$ is in $\mathcal{H}_{\alpha}(R_+)$ for $|t| < t_0$.*

Proof. We note by $d_{rn,\alpha,q} = \alpha_{rn,\alpha,q}/\alpha_{r(n-1),\alpha,q}$. Since the general term of a convergent series must go to zero, $\lim_{n\longrightarrow\infty} a_n p_n^\alpha(x,t,q^r,\delta) = 0$. By lemma 10.2, it therefore follows that $a_n = O\big(\frac{[n]_{q^r}!}{\alpha_{rn,\alpha,q}t_0^n}\big)$. Using Lemma 10.1, we get for $s > 0$ and $\delta \geq 1$

$$\sum_{n=0}^{\infty} a_n d_{rn,\alpha,q} p_{n-1}^\alpha(x,t,q^r,\delta) \leq M \sum_{n=1}^{\infty} \frac{[n]_{q^r}!}{\alpha_{rn,\alpha,q}t_0^n} \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} (s+|t|)^n \mathcal{R}_{\alpha,q}^\delta\Big(\frac{|x|}{s^{1/r}}\Big)$$
$$\leq M\mathcal{R}_{\alpha,q}^\delta\Big(\frac{|x|}{s^{1/r}}\Big) \sum_{n=0}^{\infty} \Big(\frac{s+|t|}{t_0}\Big)^n,$$

which converges for $s + |t| < t_0$. Since $s > 0$ is arbitrary it converges for $(s + |t|) < t_0$, and as before for $|t| < t_0$. □

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be an entire function of order ρ, $\rho > 0$, and of type $0 < \sigma < \infty$. The type is determined by $\limsup_{n\to\infty} \frac{rn}{e\rho} |a_n|^{\frac{\rho}{rn}} = \sigma$. Therefore,

$$|a_n| \leq M \left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho}. \tag{77}$$

Theorem 10.4. *If $f(z)$ is an entire function of order ρ with $0 < \rho < r/r-1$ and of type σ, $0 < \sigma < \infty$, then*

$$u(x,t) = \sum_{n=0}^{\infty} a_n p_n^\alpha(x,t,q^r,\delta) \tag{78}$$

is in $\mathcal{H}_\alpha(R)$ in the strip $|t| < 1/(\sigma\rho)^{r/\rho}$ and $u(x,0) = f(x)$.

Proof. Using (77) and lemma 10.1, for $s > 0$ we obtain

$$\sum_{n=0}^{\infty} a_n p_n^\alpha(x,t,q^r,\delta) \leq M \sum_{n=0}^{\infty} \left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho} \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} (s+|t|)^n {\mathcal{R}^\delta}_{\alpha,q}\Big(\frac{|x|}{s^{1/r}}\Big). \tag{79}$$

Since, $\left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho} \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} \leq \left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho} r^{rn} \prod_{i=1}^{r-1} \frac{\Gamma_{q^r}(\alpha_i+n+1)}{\Gamma_{q^r}(\alpha_i+1)}$, or for $n \uparrow \infty$, we have $\prod_{i=1}^{r-1} \frac{\Gamma_{q^r}(\alpha_i+n+1)}{\Gamma_{q^r}(\alpha_i+1)} \sim \prod_{i=1}^{r-1} \Gamma_{q^r}(\alpha_i+n+1)$, by [15, p. 53] , for $n \uparrow \infty$ $\prod_{i=1}^{r-1} \Gamma_{q^r}(\alpha_i+n+1) \leq \prod_{i=1}^{r-1} \Gamma(\alpha_i+n+1)$. Using Stirling's formula, we get

$$\left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho} r^{rn} \prod_{i=1}^{r-1} \Gamma(\alpha_i+n+1) \sim \left[\frac{e^{1-\frac{r-1}{r}\rho}\, r^{\rho-1}}{n^{1-\frac{r-1}{r}\rho+(\sum \alpha_i+\frac{r-1}{2})\rho/rn}}\right]^{rn/\rho} (2\pi)^{\frac{r-1}{2}} (\sigma\rho)^{rn/\rho}$$

for $0 < \rho < \frac{r}{r-1}$. Thus the series in (79) is dominated by

$$M_{t,q} \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}) \sum_{n=0}^{\infty} \{(\sigma\rho)^{r/\rho}(s+|t|)\}^n ,$$

which converges for $(\sigma\rho)^{3/\rho}(s+|t|) < 1$. Since $s > 0$ is arbitrary, we get absolute and local uniform convergence for $|t| < \frac{1}{(\sigma\rho)^{3/\rho}}$. Since the order and type of entire function is not changed by taking derivatives, a similar type argument shows that the derived series $\sum_{n=1}^{\infty} a_n d_{rn,\alpha,q} p^{\alpha}_{n-1}(x,t,q^r,\delta)$, also converges absolutely and locally uniformly for $|t| < \frac{1}{(\sigma\rho)^{r/\rho}}$. It follows that $u(x,t)$ given by (78) is in $\mathcal{H}_\alpha$ in the stated strip. □

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