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Optimal Review Period and Backorder Rate in a Periodic Review Inventory Model with Controllable Lead Time

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Abstract

In this study, we consider a periodic review inventory model involving variable lead time with partial backorders. The objective is simultaneously to optimize the length of review period, the backorder rate, and the lead time in order to minimize the total expected annual cost. We first assume that the protection interval (i.e., review period plus lead time) demand follows a normal distribution, and then relaxes this assumption to consider the distribution free case where only the mean and the standard deviation of the protection interval demand are known. Two algorithm procedures of finding the optimal solution are developed. Also two numerical examples are given to illustrate the results.

Keywords and Phrases: *Inventory, Periodic review, Protection interval, Backorder rate, Minimax distribution free procedure.*

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1. Introduction

In classical economic order quantity (EOQ) inventory model dealing with the problem of shortages, it was often assumed that during the stock-out period, shortages are either completely backordered or completely lost. However, in many market situations, we can often observe that some customers may prefer their demands to be backordered while shortages occur, and some may refuse the backorder case. When a shortage occurs, many factors may affect customers' willingness of accepting backorders. It is obvious that for well-famed products or fashionable goods such as certain brand gum shoes, hi-fi equipment, and clothes, customers may prefer to wait in order to satisfy their demands. Besides the products themselves, there is a potential factor that may motivate the customers' desire for backorders. The factor is in some extent a price discount offered by the supplier. In general, provided that the supplier could offer a price discount on the stock-out item by negotiation to secure more backorders, it may make the customers more willing to wait for the desired items. In other words, the higher the price discounts of a supplier, the higher the advantage of the customers, and hence, higher backorder rate may result. This phenomenon reveals that, as unsatisfied demands occur during the stock-out period, how to find an optimal backorder rate through controlling a price discount from supplier to minimize the relevant inventory total cost is a decision-making problem worth discussing.

In 2001, Pan and Hsiao [10] presented an EOQ inventory model with back-order discount and variable lead time. Later, Ouyang et al. [7] considered a periodic review inventory model with review period and backorder discounts viewed as decision variables, but the lead time is treated as a fixed constant. However, as pointed out by Silver [11], if the quantitative models are to be more useful as aids for managerial decision making, they must permit some of the usual parameters to be treated as decision variables. In many practical situations, lead time can be reduced at an added crashing cost; in other words, it is controllable. By shortening the lead time, we can lower the safety stock, reduce the stock-out loss, and improve the customer service level so as to gain the competitive advantages in business. Recently, the Japanese successful experience of using Just-In-Time (JIT) production also evidenced that substantial advantages and benefits can be attained by controlling lead time.

In the literature of inventory theory, the lead time reduction in the continuous review inventory models have been continually modified (see, e.g., Liao and Shyu [5], Ben-Daya and Raouf [1], Ouyang *et al*. [9], Moon and Choi [6], Hariga and Ben-Daya [4], Pan and Hsiao [10], and Chuang et al. [2]) so as to accommodate more practical features of the real production/inventory systems. It is noted that the reduction of lead time in the periodic review inventory model is quite sparse.

The purpose of this paper is to examine the effects of the reduction of lead time associated with the controllable backorder rate in the periodic review inventory model. That is, the study proposes a general model which allows review period, T, backorder rate, β, (or price discount, π_x), and lead time, L, as decision variables to accommodate a more realistic inventory situation. In this paper, we start with a protection interval demand that follows a normal distribution, and try to determine the optimal ordering policy. We next relax this assumption by only assuming that the first and second moments of the probability distribution of the protection interval demand to be known and finite, and then solve this inventory model by using the minimax distribution free approach.

This paper is organized as follows. In the next section, we first review and extend the Ouyang et al.'s [7] model. The model in which the protection interval demand has perfect information is formulated in Section 3, and the model in which there is only partial information for the protection interval demand is asserted in Section 4. Two numerical examples are provided to illustrate the proposed models in Section 5, and Section 6 is a summary of the work done in this article.

2. Review and Extension of the Ouyang et al.'s Model

Ouyang et al. [7] considered a periodic review inventory model with backorder rate (or price discount) and asserted the total expected annual cost as follows:

$$EAC(T,\pi_x) = \frac{A}{T} + h\left[R - DL - \frac{DT}{2} + \left(1 - \frac{\beta_0\pi_x}{\pi_0}\right)E(X-R)^+\right] + \frac{1}{T}\left(\frac{\beta_0\pi_x^{\ 2}}{\pi_0} + \pi_0 - \beta_0\pi_x\right)E(X-R)^+, \qquad (1)$$

where notation used are:

D = average demand per year

A = fixed ordering cost per order

h = inventory holding cost per item per year

R = target level

β = backorder rate, $0 < \beta < 1$

β_0 = upper bound of the backorder rate

π_x = backorder price discount offered by the supplier per unit

π_0 = marginal profit (i.e. cost of lost demand) per unit

T = length of a review period

L = length of lead time

X = the protection interval, $T+L$, demand which has a probability density function (*p.d.f.*) f_X with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$

$E(\cdot)$ = mathematical expectation

x^+ = maximum value of x and 0, i.e., $x^+ = Max\{x,0\}$

$E(X-R)^+$ = the expected demand short at the end of cycle,

and assumptions are:

1. The inventory level is reviewed every T units of time. A sufficient quantity is ordered up to the target level R, and the ordering quantity is received after L units of time.

2. The length of the lead time L does not exceed an inventory cycle time T so that there is never more than a single order outstanding in any cycle.

3. The target level R = expected demand during the protection interval + safety stock (SS), and SS = $k\times$(standard deviation of protection interval demand), i.e., $R = D(T+L) + k\sigma\sqrt{T+L}$, where k is the safety factor and satisfies $P(X>R)=q$, q is given to represent the allowable stock-out probability during the protection interval.

4. During the stock-out period, the backorder rate, β, is variable and is in proportion to the price discount offered by the supplier per unit π_x. The backorder rate is defined as $\beta = \beta_0\pi_x/\pi_0$, where $0 \le \beta_0 < 1$ and $0 \le \pi_x \le \pi_0$.

It is noted that the lead time, L, in model (1) is viewed as a fixed constant. However, as mentioned previously, in many practical situations, lead time can be reduced at an added crashing cost; in other words, it is controllable. In this study, we will consider the lead time as a decision variable and assume that:

5. The lead time L consists of n mutually independent components. The i-th component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Further, for convenience, we rearrange c_i such that $c_1 \le c_2 \le \cdots \le c_n$. Then, it is clear that the reduction of lead time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.

6. We let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components $1, 2, \cdots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j)$, $i = 1, 2, \cdots, n$; and the lead time crashing cost $C(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$ is given by $C(L) = c_i (L_{i-1} - L) + \sum_{j=1}^{i-1} c_j (b_j - a_j)$.

Therefore, the objective of our problem is to minimize the following total expected annual cost:

$$
\begin{aligned}
EAC(T, \pi_x, L) = &\text{ ordering cost + holding cost + stock-out cost} \\
&+ \text{lead time crashing cost} \\
= &\frac{A}{T} + h\left[R - DL - \frac{DT}{2} + \left(1 - \frac{\beta_0 \pi_x}{\pi_0}\right) E(X - R)^+ \right] \\
&+ \frac{1}{T}\left(\frac{\beta_0 {\pi_x}^2}{\pi_0} + \pi_0 - \beta_0 \pi_x \right) E(X - R)^+ + \frac{C(L)}{T}. \qquad (2)
\end{aligned}
$$

3. Basic Model

We first assume that the protection interval demand X follows a normal distribution with mean $D(T + L)$ and standard deviation $\sigma \sqrt{T + L}$.

Given that $R = D(T+L) + k\sigma\sqrt{T+L}$, the expected shortage quantity $E(X-R)^+$ at the end of the cycle can be expressed as

$$E(X-R)^+ = \int_R^\infty (x-R) f_X(x)dx = \sigma\sqrt{T+L}\;\psi(k) > 0,$$

where $\psi(k) \equiv \phi(k) - k[1-\Phi(k)]$, $\phi(k)$ and $\Phi(k)$ denote the standard normal *p.d.f.* and distribution function (*d.f.*), respectively.

Therefore, the total expected annual cost Eq (2) becomes

$$EAC(T,\pi_x,L) = \frac{A+C(L)}{T} + h\left(\frac{DT}{2} + k\sigma\sqrt{T+L}\right)$$
$$+\left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right) + \frac{1}{T}\left(\frac{\beta_0{\pi_x}^2}{\pi_0} + \pi_0 - \beta_0\pi_x\right)\right]\sigma\sqrt{T+L}\;\psi(k). \quad (3)$$

The problem is to find the optimal values of T, π_x and L, such that $EAC(T,\pi_x,L)$ in Eq (3) is minimized. Taking the first partial derivatives of $EAC(T,\pi_x,L)$ with respect to T, π_x and $L \in [L_i, L_{i-1}]$, respectively, we obtain

$$\frac{\partial EAC(T,\pi_x,L)}{\partial T} = -\frac{A+C(L)}{T^2} + h\left(\frac{D}{2} + \frac{k\sigma}{2\sqrt{T+L}}\right) - \frac{G(\pi_x)\,\sigma\sqrt{T+L}\;\psi(k)}{T^2}$$
$$+\frac{\left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right) + \frac{G(\pi_x)}{T}\right]\sigma\,\psi(k)}{2\sqrt{T+L}}, \quad (4)$$

$$\frac{\partial EAC(T,\pi_x,L)}{\partial \pi_x} = \left(\frac{\frac{2\beta_0\pi_x}{\pi_0} - \beta_0}{T} - \frac{h\beta_0}{\pi_0}\right)\sigma\sqrt{T+L}\;\psi(k), \quad (5)$$

and

$$\frac{\partial EAC(T,\pi_x,L)}{\partial L} = \frac{hk\sigma}{2\sqrt{T+L}} + \frac{\left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right) + \frac{G(\pi_x)}{T}\right]\sigma\,\psi(k)}{2\sqrt{T+L}} - \frac{c_i}{T}, \quad (6)$$

where $G(\pi_x) = \pi_0 - \beta_0 \pi_x + \dfrac{\beta_0 \pi_x{}^2}{\pi_0} > 0$ (because $\dfrac{\pi_0}{\pi_x} > \beta_0 \left(1 - \dfrac{\pi_x}{\pi_0}\right) > 0$).

By examining the second order sufficient conditions, it can be easily verified that $EAC(T, \pi_x, L)$ is not a convex function of (T, π_x, L). However, for fixed T and π_x, $EAC(T, \pi_x, L)$ is concave in $L \in (L_i, L_{i-1})$, since

$$\frac{\partial^2 EAC(T, \pi_x, L)}{\partial L^2} = -\frac{hk\sigma}{4(T+L)^{3/2}} - \frac{\left[h\left(1 - \dfrac{\beta_0 \pi_x}{\pi_0}\right) + \dfrac{G(\pi_x)}{T}\right]\sigma\, \psi(k)}{4(T+L)^{3/2}} < 0 .$$

Therefore, for fixed T and π_x, the minimum total expected annual cost will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, for a given value of $L \in [L_i, L_{i-1}]$, $EAC(T, \pi_x, L)$ is a convex function of (T, π_x) (the proof is the same as that in Ouyang et al. [7], we omit it here). Thus, for fixed $L \in [L_i, L_{i-1}]$, the minimum value of $EAC(T, \pi_x, L)$ will occur at the point (T, π_x), which satisfies $\dfrac{\partial EAC(T, \pi_x, L)}{\partial T} = 0$ and $\dfrac{\partial EAC(T, \pi_x, L)}{\partial \pi_x} = 0$ simultaneously.

Hence, setting Eq (4) and Eq (5) to zero and solving it, respectively, produces

$$\frac{A + C(L)}{T^2} = \frac{hD}{2} + \frac{h\sigma}{2\sqrt{T+L}}\left[k + \left(1 - \frac{\beta_0 \pi_x}{\pi_0}\right)\psi(k)\right] - \frac{G(\pi_x)\sigma\sqrt{T+L}\,\psi(k)}{T^2} + \frac{G(\pi_x)\sigma\psi(k)}{2T\sqrt{T+L}}, \tag{7}$$

and

$$\pi_x = \frac{Th + \pi_0}{2}. \tag{8}$$

Substituting Eq (8) into Eq (7) leads to

$$\frac{A + C(L)}{T^2} = \frac{hD}{2} + \frac{h\sigma}{2\sqrt{T+L}}\left\{k + \left[1 - \frac{\beta_0}{\pi_0}\left(\frac{Th + \pi_0}{2}\right)\right]\psi(k)\right\}$$

$$-\frac{H(T)\sigma\sqrt{T+L}\ \psi(k)}{T^2}+\frac{H(T)\sigma\psi(k)}{2T\sqrt{T+L}}, \tag{9}$$

where $H(T)\equiv G\left(\frac{Th+\pi_0}{2}\right)=\pi_0-\beta_0\left(\frac{Th+\pi_0}{2}\right)+\frac{\beta_0}{\pi_0}\left(\frac{Th+\pi_0}{2}\right)^2$.

Thus, we can establish the following algorithm to find the optimal T, π_x and L.

Algorithm 1

Step1. For each L_i, $i=0,1,2,\cdots,n,$ and a given q (and hence, the value of k can be found directly from the normal distribution table), use a numerical search technique to obtain T_i which satisfies Eq (9) and compute π_{x_i} from Eq (8). And compare π_{x_i} and π_0.

(i) If $\pi_{x_i}\le\pi_0$, π_{x_i} is feasible, then go to Step2.

(ii) If $\pi_{x_i}>\pi_0$, π_{x_i} is not feasible. Set $\pi_{x_i}=\pi_0$ and calculate the corresponding value of T_i from Eq (7), then go to Step2.

Step2. For each (T_i,π_{x_i},L_i), compute the corresponding total expected annual cost $EAC(T_i,\pi_{x_i},L_i)$, $i=0,1,2\cdots,n.$

Step3. Find $\underset{i=0,1,2,\ldots,n}{Min}\ EAC(T_i,\pi_{x_i},L_i)$. If $EAC(T^*,\pi_x{}^*,L^*)=\underset{i=0,1,2,\ldots,n}{Min}\ EAC(T_i,\pi_{x_i},L_i)$, then $(T^*,\pi_x{}^*,L^*)$ is the optimal solution.

Once, the optimal solution $(T^*,\pi_x{}^*,L^*)$ obtain, then the optimal target level is $R^*=D(T^*+L^*)+k\sigma\sqrt{T^*+L^*}$. And the optimal backorder rate is $\beta^*=\beta_0\pi_x^*/\pi_0$.

4. Distribution Free Model

In many practical situations, the distributional information of the protection interval demand is often quite limited. Hence, in this section, we relax the assumption about the normal distribution of the protection interval demand and only assume that the protection interval demand X has given finite first two moments; i.e., the $p.d.f.\ f_X$ belongs to the class Ω of *p.d.f.*'s with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$. Since the probability distribution of X is unknown, we can not find the exact value of $E(X-R)^+$. We propose to apply the minimax distribution free procedure for our problem. The minimax distribution free approach for this problem is to find the "most unfavorable" $p.d.f.\ f_X$ in Ω for each (T,π_x,L) and then minimize the total expected annual cost over (T,π_x,L); more exactly, we need to solve

$$\underset{(T,\pi_x,L)}{Min}\ \underset{f_X\in\Omega}{Max}\, EAC(T,\pi_x,L)\,. \tag{10}$$

For this purpose, we need the following proposition which can be verified by the similar method as in Gallego and Moon [3].

Proposition 1

For any $f_X \in \Omega$,

$$E(X-R)^+ \le \frac{1}{2}\left\{\sqrt{\sigma^2(T+L)+[R-D(T+L)]^2}-[R-D(T+L)]\right\}. \tag{11}$$

Moreover, the upper bound of Eq (11) is tight.

Given that $R = D(T+L)+k\sigma\sqrt{T+L}$, and for any probability distribution of the protection interval demand X, the above inequality always holds. Then, using model (2) and inequality (11), the problem (10) is reduced to minimize

$$EAC^W(T,\pi_x,L)$$
$$=\frac{A+C(L)}{T}+h\left[\frac{DT}{2}+k\sigma\sqrt{T+L}+\frac{1}{2}\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\sigma\sqrt{T+L}\left(\sqrt{1+k^2}-k\right)\right]$$
$$+\frac{G(\pi_x)\sigma\sqrt{T+L}}{2T}\left(\sqrt{1+k^2}-k\right). \tag{12}$$

Once again the approach employed in the previous section is utilized to solve problem (12). For fixed $L\in[L_i,L_{i-1}]$, it can be shown that $EAC^W(T,\pi_x,L)$ is convex at the point (T,π_x) which satisfies $\frac{\partial EAC^W(T,\pi_x,L)}{\partial T}=0$ and $\frac{\partial EAC^W(T,\pi_x,L)}{\partial \pi_x}=0$,i.e.,

$$\frac{A+C(L)}{T^2}=\frac{hD}{2}+\frac{h\sigma}{2\sqrt{T+L}}\left[k+\frac{1}{2}\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\left(\sqrt{1+k^2}-k\right)\right]$$
$$+\frac{G(\pi_x)\sigma}{4T\sqrt{T+L}}\left(\sqrt{1+k^2}-k\right)-\frac{G(\pi_x)\sigma\sqrt{T+L}}{2T^2}\left(\sqrt{1+k^2}-k\right), \qquad (13)$$

and

$$\pi_x=\frac{Th+\pi_0}{2}. \qquad (14)$$

Substituting Eq (14) into Eq (13), we get

$$\frac{A+C(L)}{T^2}=\frac{hD}{2}+\frac{h\sigma}{2\sqrt{T+L}}\left[k+\frac{1}{2}\left(1-\frac{\beta_0(Th+\pi_0)}{2\pi_0}\right)\left(\sqrt{1+k^2}-k\right)\right]$$
$$+\frac{H(T)\sigma}{4T\sqrt{T+L}}\left(\sqrt{1+k^2}-k\right)-\frac{H(T)\sigma\sqrt{T+L}}{2T^2}\left(\sqrt{1+k^2}-k\right), \qquad (15)$$

where $H(T)$ be defined as above.

Next, for fixed T and π_x, $EAC^W(T,\pi_x,L)$ is concave in $L\in(L_i,L_{i-1})$, since

$$\frac{\partial^2 EAC^W(T,\pi_x,L)}{\partial L^2}=-\frac{h\sigma}{4(T+L)^{3/2}}\left[k+\frac{1}{2}\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\left(\sqrt{1+k^2}-k\right)\right]$$
$$-\frac{G(\pi_x)\sigma}{8T(T+L)^{3/2}}\left(\sqrt{1+k^2}-k\right)<0.$$

Therefore, for fixed T and π_x, the minimum total expected annual cost will occur at the end points of the interval $[L_i,L_{i-1}]$.

Theoretically, for given $A, D, h, \pi_0, \beta_0, \sigma, k$ (which depends on the allowable stock-out probability q and the *p.d.f.* $f_X(x)$) and each $L_i\ (i=0,1,2,\cdots,n)$, from Eq

(14) and Eq (15), we can obtain the optimal value (T_i, π_{x_i}), and the corresponding total expected annual cost $EAC^W(T_i, \pi_{x_i}, L_i)$ for $i = 0, 1, 2, \cdots, n$. Thus, the minimum total expected annual cost can be obtained. However, in practice, since the *p.d.f.* $f_X(x)$ is unknown, even if the value of q is given, we can not get the exact value of k. Therefore, in order to find the value of k, we need the following proposition.

Proposition 2

Let X *represent the protection interval demand which has a p.d.f.* $f_X(x)$ *with finite mean* $D(T+L)$ *and standard deviation* $\sigma\sqrt{T+L}$, *then for any real number* $c > D(T+L)$,

$$P(X > c) \le \frac{\sigma^2(T+L)}{\sigma^2(T+L) + [c - D(T+L)]^2}. \tag{16}$$

The proof is similar to that of Ouyang and Wu [8], and hence we omit it here.

Because the target level $R = D(T+L) + k\sigma\sqrt{T+L}$ as mentioned earlier, if we take R instead of c in inequality (16), we get

$$P(X > R) \le \frac{1}{1+k^2}. \tag{17}$$

Further it is assumed that the allowable stock-out probability q during the protection interval is given, that is, $q = P(X > R)$, then from Eq (17) we can obtain $0 \le k \le \sqrt{\frac{1}{q} - 1}$.

It is easy to verify that $EAC^W(T, \pi_x, L)$ has a smooth curve for $k \in [0, \sqrt{\frac{1}{q} - 1}]$. Thus, we can establish the following algorithm to obtain the suitable k, and hence the optimal T, π_x and L.

Algorithm 2

Step1. For a given q, we divide the interval $[0, \sqrt{\frac{1}{q} - 1}]$ into N equal subintervals, where N is large enough. And we let $k_0 = 0$, $k_N = \sqrt{\frac{1}{q} - 1}$ and $k_j = k_{j-1} + \dfrac{k_N - k_0}{N}$, $j = 1, 2, \cdots, N-1$.

Step2. For each L_i, i =0, $1, 2, \cdots, n$, perform (i) to (iv).

(i) For given $k_j \in \{k_0, k_1, \cdots, k_N\}$, $j = 1, 2, \cdots, N$, we can use a numerical search technique to obtain T_{i,k_j} which satisfies Eq (15).

(ii) Substitute T_{i,k_j} into Eq (14) and compute $\pi_{x_{i,k_j}}$. And compare $\pi_{x_{i,k_j}}$ and π_0.

(a)If $\pi_{x_{i,k_j}} \le \pi_0$, $\pi_{x_{i,k_j}}$ is feasible, then go to (iii).

(b)If $\pi_{x_{i,k_j}} > \pi_0$, $\pi_{x_{i,k_j}}$ is not feasible. Set $\pi_{x_{i,k_j}} = \pi_0$ and calculate the corresponding value of T_{i,k_j} from Eq (13), then go to (iii).

(iii) Compute the corresponding total expected annual cost

$$EAC^W(T_{i,k_j}, \pi_{x_{i,k_j}}, L_i)$$
$$= \frac{A + C(L_i)}{T_{i,k_j}}$$
$$+h\left[\frac{DT_{i,k_j}}{2} + k_j\,\sigma\sqrt{T_{i,k_j} + L_i} + \frac{1}{2}\left(1 - \frac{\beta_0 \pi_{x_{i,k_j}}}{\pi_0}\right)\sigma\sqrt{T_{i,k_j} + L_i}\left(\sqrt{1 + {k_j}^2} - k_j\right)\right]$$
$$+\frac{G(\pi_{x_{i,k_j}})\,\sigma\sqrt{T_{i,k_j} + L_i}}{2T_{i,k_j}}\left(\sqrt{1 + {k_j}^2} - k_j\right).$$

(iv) Find $\underset{k_j \in \{k_o, k_1, \ldots, k_N\}}{Min} EAC^W(T_{i,k_j}, \pi_{x_{i,k_j}}, L_i)$, and let

$$EAC^W(T_{i,k_{s(i)}}, \pi_{x_{i,k_{s(i)}}}, L_i) = \underset{k_j \in \{k_o, k_1, \ldots, k_N\}}{Min} EAC^W(T_{i,k_j}, \pi_{x_{i,k_j}}, L_i).$$

Step3. Find $\underset{i=0,1,2,\ldots,n}{Min} EAC^W(T_{i,k_{s(i)}}, \pi_{x_{i,k_{s(i)}}}, L_i)$.

If $EAC^W(T^{**}, {\pi_x}^{**}, L^{**}) = \underset{i=0,1,2,\ldots,n}{Min} EAC^W(T_{i,k_{s(i)}}, \pi_{x_{i,k_{s(i)}}}, L_i)$, then $(T^{**}, {\pi_x}^{**}, L^{**})$ is the optimal solution; the value of $k_{s(i)}$ such that

$EAC^W(T^{**}, \pi_x^{**}, L^{**})$ exists is the optimal safety factor and we denote it by k^{**}.

Once, the optimal solution $(T^{**}, \pi_x^{**}, L^{**})$ obtain, then the optimal target level is $R^{**} = D(T^{**} + L^{**}) + k^{**}\sigma\sqrt{T^{**} + L^{**}}$. And the optimal backorder rate is $\beta^{**} = \beta_0 \pi_x^{**} / \pi_0$.

5. Numerical Examples

In order to illustrate the above solution procedure, let us consider an inventory system with the following data used in [7] : D = 600 units per year, A = \$200 per order, h = \$20 per unit per year, $\pi_0 = \$150$ per unit short, $\sigma = 7$ units per week, Besides, we assume that the lead time has three components with data shown in Table 1.

Lead time Component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 1 Lead time data

Example 1. Suppose that the protection interval demand follows a normal distribution. We want to solve the cases when the upper bounds of the backorder rate $\beta_0 =$ 0.2, 0.35, 0.5, 0.65, 0.8 and 0.95, and $q =$ 0.2 (in this situation, the value of safety factor k can be found directly from the standard normal table, and is given by 0.845). Applying the Algorithm 1 procedure yields the results as tabulated in Table 2. From this table, the optimal inventory policy can easily be found by comparing $EAC(T_i, \pi_{x_i}, L_i)$, for $i = 0, 1, 2, 3,$ and thus we summarize these in Table 3.

β_0	i	L_i	$C(L_i)$	T_i	π_{x_i}	R_i	$EAC(T_i, \pi_{x_i}, L_i)$	
0.2	0	8	0	14.98	77.88	293.54	$4898.58	
	1	6	5.6	14.56	77.80	264.05	4806.41	
	2	4	22.4	14.24	77.74	235.74	4746.27	*
	3	3	57.4	14.47	77.78	226.31	4809.95	
0.35	0	8	0	14.79	77.84	291.15	4819.88	
	1	6	5.6	14.38	77.76	261.80	4729.99	
	2	4	22.4	14.08	77.71	233.73	4672.85	*
	3	3	57.4	14.32	77.75	224.44	4739.17	
0.5	0	8	0	14.59	77.81	288.73	4740.54	
	1	6	5.6	14.19	77.73	259.54	4653.01	
	2	4	22.4	13.91	77.67	231.67	4598.94	*
	3	3	57.4	14.16	77.72	222.54	4668.00	
0.65	0	8	0	14.39	77.77	286.29	4660.55	
	1	6	5.6	14.00	77.69	257.25	4575.44	
	2	4	22.4	13.74	77.64	229.59	4524.55	*
	3	3	57.4	14.01	77.69	220.62	4596.42	
0.8	0	8	0	14.18	77.73	283.82	4579.87	
	1	6	5.6	13.81	77.66	254.94	4497.27	
	2	4	22.4	13.57	77.61	227.49	4449.66	*
	3	3	57.4	13.85	77.66	218.69	4524.43	
0.95	0	8	0	13.98	77.69	281.33	4498.48	
	1	6	5.6	13.62	77.62	252.61	4418.46	
	2	4	22.4	13.39	77.58	225.36	4374.24	*
	3	3	57.4	13.69	77.63	216.75	4452.00	

Table 2 Solution procedures of Example 1 $(T_i, L_i \text{ in week})$

β_0	L^*	T^*	π_x^*	R^*	$EAC(T^*, \pi_x^*, L^*)$
0.2	4	14.24	77.74	235.74	$4746.27
0.35	4	14.08	77.71	233.73	4672.85
0.5	4	13.91	77.67	231.67	4598.94
0.65	4	13.74	77.64	229.59	4524.55
0.8	4	13.57	77.61	227.49	4449.66
0.95	4	13.39	77.58	225.36	4374.24

Table 3 Summary of the optimal solutions of Example 1 $(T^*, L^* \text{ in week})$

From Table 3, it is interesting to observe that increasing the value of upper bound of the backorder rate β_0 will results in a decreases in the total expected annual cost, the review period, the price discount, and target level.

Example 2. The data are the same as in Example 1, except that the probability distribution of the protection interval demand is unknown. We solve the cases when the upper bounds of the backorder rate $\beta_0 =$ 0.2, 0.35, 0.5, 0.65, 0.8, and 0.95, and $q = 0.2$ (in this situation, we have $k_0 = 0$, and $k_N = 2$), let $k_j = k_{j-1} + \frac{k_N - k_0}{N}$, $j = 1, 2, \cdots, N-1$, and take N=500. Applying the Algorithm 2, the summarized optimal values are presented in Table 4.

β_0	L^{**}	T^{**}	π_x^{**}	R^{**}	$EAC^W(T^{**}, \pi_x^{**}, L^{**})$
0.2	4	11.87	77.28	258.45	$5454.74
0.35	4	11.85	77.28	256.54	5388.63
0.5	4	11.83	77.27	254.60	5321.05
0.65	4	11.82	77.27	252.60	5251.89
0.8	4	11.80	77.26	250.56	5158.04
0.95	4	11.78	77.26	248.48	5108.37

Table 4 Summary of the optimal solutions of Example 2 $(T^{**}, L^{**}\ in\ week)$

Furthermore, we examine the performance of distribution free approach against the normal distribution. The optimal solutions of distribution free and normal distribution cases are $(T^{**}, \pi_x^{**}, L^{**})$ and $(T^{*}, \pi_x^{*}, L^{*})$, respectively. Substituting them into Eq (3), the added cost by utilizing the minimax distribution free procedure instead of the normal distribution is $EAC(T^{**}, \pi_x^{**}, L^{**}) - EAC(T^{*}, \pi_x^{*}, L^{*})$. This is the largest amount that we would be willing to pay for the knowledge of *p.d.f.* f_X. This quantity can be regarded as the expected value of additional information (EVAI), a summary is presented in Table 5.

β_0	$EAC(T^{**},\pi_x^{**},L^{**})$	$EAC(T^{*},\pi_x^{*},L^{*})$	$EVAI$
0.2	\$4877.07	\$4746.27	\$138.80
0.35	4799.77	4672.85	126.92
0.5	4722.35	4598.94	123.41
0.65	4644.31	4524.55	119.76
0.8	4566.65	4449.66	116.99
0.95	4488.86	4374.24	114.62

Table 5 Calculation of EVAI for (T,π_x,L) model

Table5 reveals that the amount of EVAI decrease as the value of upper bound of the backorder rate β_0 increase.

6. Concluding Remarks

In this study, we seek to extend Ouyang et al. [7] model, and propose more general model that allow the lead time as a control variable, rather than as a fixed parameter. The objective is to minimize the total expected annual cost, which is the sum of the ordering cost, holding cost, stock-out cost, and lead time crashing cost. We first assume that the protection interval demand follows a normal distribution, and then relax the assumption about probability distributional form of the protection interval demand and apply the minimax distribution free procedure to solve the problem.

For further consideration on this problem, it would be interested to deal with a mixed stochastic inventory model that the stock-out cost term in the objective function is replaced by a service level constraint.

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