Tamsui Oxford Journal of Mathematical Sciences **25(4)** (2009) 415-423
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D-Graceful Labeling of a Path*

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Received December 31, 2008, Accepted May 19, 2009.

Abstract

Let G be a undirected graph with a vertex set V and an edge set E. Given a nonnegative integer set D. A *D-graceful labeling* f of G is an injection $f : V \to D$ such that

$$\{|f(x) - f(y)| \mid xy \in E\} = \{1, 2, 3, \ldots, |E|\}.$$

A graph is called *D-graceful* if it has a D-graceful labeling. We call a graph G *graceful* if G is $\{0, 1, \ldots, |E|\}$-graceful. Let $Z(n; a, b)$ denote the set $\{0, 1, \cdots, a-1, b+1, \cdots, n+b-a\}$. In this paper, we showed that P_n is D-graceful for some D. And we conjecture that P_n is $Z(n; t, t)$-graceful except $n = 2t = 2$ and $n = 3, 4$.

Keywords and Phrases: *Graceful labeling, D-Graceful, Labeling, Path.*

1. Introduction

In 1964, Ringel [9] conjectured that K_{2n+1}, the complete graph on $2n + 1$ vertices, can be decomposed into $2n + 1$ isomorphic copies of a given tree

*2000 *Mathematics Subject Classification.* Primary 05C78.
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with n vertices. In 1967, Rosa [10] introduced β-labelings as a tool to attack Ringels's conjecture. This labeling was called *graceful* by Golomb.

Let G be a undirected graph with a vertex set V and an edge set E. Given a nonnegative integer set D, a *D-graceful labeling* f of G is an injection $f : V \to D$ such that

$$\{|f(x) - f(y)| \mid xy \in E\} = \{1, 2, 3, \ldots, |E|\}.$$

A graph is called *D-graceful* if it has a D-graceful labeling, where the set D is called a *graceful set* of G. In addition, we use a *graceful labeling* to represent a $\{0, 1, \ldots, |E|\}$-graceful labeling. A graph is called *graceful* if it has a graceful labeling. The gracefulness will get influenced by various properties of set theory. The following lemma shows the result due to some set extension.

Lemma 1. *Suppose $A \subseteq D$ and G is A-graceful. Then G is D-graceful.*

A graceful set D of G is exactly if D has no proper subset A such that G is A-graceful. It is trivial that $|D| = |V(G)|$ if D is an exactly graceful set of G. A *complete labeling*, introduced by Barrientos [1] in 2005, is a D-graceful labeling for some exactly graceful set D. Chang and Yan [2] show that the gracefulness of $C_m \cup P_n$ by D-gracefulness.

Lemma 2. *If D is an exactly graceful set of G, then*

$$\max D - \min D \leq \frac{|E(G)|(|E(G)| - 1)}{2}.$$

Theorem 3. *[7]K_n is graceful if and only if $n \leq 4$.*

Lemma 4. *An integer set D is an exactly graceful set of a complete graph K_n if and only if one of following conditions holds.*
(1) $n = 1$ and $D = \{t\}$ for any integer t.
(2) $n = 2$ and $D = \{t, t + 1\}$ for any integer t.
(3) $n = 3$ and $D = \{t, t + 1, t + 3\}$ or $\{t, t + 2, t + 3\}$ for any integer t.
(4) $n = 4$ and $D = \{t, t+1, t+4, t+6\}$ or $\{t, t+2, t+5, t+6\}$ for any integer t.

It is trivial that G is $(D - t)$-graceful for each integer t if G is D-graceful, where $D - t = \{d - t|d \in D\}$. So we only consider the exactly graceful set D which $\min D = 0$. Given two positive integers $a \leq b$, let $Z(n; a, b)$ denote the set $\{0, 1, \cdots, a-1, b+1, \cdots, n+b-a\}$. It is trivial that P_n is $Z(n; a, b)$-graceful if $a \geq n$. Hence, the assumption of $a < n$ is made for our discussion later in this paper. In addition, "t-graceful" is used to denote $Z(n; t, t)$-graceful for notation simplification. We showed that P_n is D-graceful for some D. And we conjecture that P_n is t-graceful except $n = 2t = 2$ or 4.

2. D-Graceful labeling of P_n

A $[k, l]$-D-graceful labeling of P_n is a D-graceful labeling of P_n in which the end vertices of P_n are labeled by k and l. P_n is $[k, l]$-D-graceful if it has a $[k, l]$-D-graceful labeling.

Lemma 5. *Let $1 \leq t \leq n-2$. If P_n is $[0, l]$-$Z(n; 1, t)$-graceful, then P_{n+2t+1} is $[0, n+3t-l]$-$Z(n+2t+1; 1, t)$-graceful.*

Proof. Let g be a $[0, l]$-$Z(n; 1, t)$-graceful labeling of $P_n : v_1, v_2, \cdots, v_n$. We define a labeling f of $P_{n+2t+1} : x_1, x_2, \cdots, x_{n+2t+1}$ such that

$$f(x_i) = \begin{cases} 0, & i = 1; \\ 2t + 1 - k, & i = 2k+1,\ 1 \leq k \leq t; \\ n + 2t - 1 + k, & i = 2k,\ 1 \leq k \leq t; \\ n + 3t - g(v_{i-2t-1}), & 2t + 2 \leq i \leq n + 2t + 1. \end{cases}$$

Then it is easy to check that f is a bijection from the vertex set of P_{n+2t+1} to the set $Z(n + 2t + 1; 1, t)$. And we have

$$|f(x_i) - f(x_{i+1})| = \begin{cases} n + 2t, & i = 1; \\ n + i - 2, & 2 \leq i \leq 2t + 1; \\ |g(v_{i-2t-1}) - g(v_{i-2t})|, & 2t + 2 \leq i \leq n + 2t. \end{cases}$$

Thus, f is a$[0, n + 3t - l]$-$Z(n + 2t + 1; 1, t)$-graceful labeling of P_{n+2t+1}. □

Theorem 6. *P_n is $Z(n; 1, 1)$-graceful if and only if $n \neq 2$.*

Proof. By Lemma 5, we have P_{n+3} is $[0, n + 3 - l]$-$Z(n + 3; 1, 1)$-graceful when P_n is $[0, l]$-$Z(n; 1, 1)$-graceful. Therefore, the theorem holds under the following labelings.

$P_3 : 0, 2, 3,$
$P_4 : 0, 3, 4, 2,$
$P_5 : 0, 4, 3, 5, 2,$ □

Theorem 7. *P_n is $Z(n;1,2)$-graceful if and only if $n \neq 2,3$.*

Proof. It is trivial that P_n is not $Z(n;1,2)$-graceful for $n=2,3$. By Lemma 5, we have P_{n+5} is $[0,n+6-l]$-$Z(n+5;1,2)$-graceful when P_n is $[0,l]$-$Z(n;1,2)$-graceful. Therefore, the theorem holds under the following labelings.

$P_4: 0,3,5,4,$
$P_5: 0,4,5,3,6,$
$P_6: 0,5,6,4,7,3,$
$P_7: 0,6,4,5,8,2,7,$
$P_8: 0,7,4,5,9,3,8,6.$ □

Lemma 8. *Let a,b be two integers and $a \le b \le 2a-2$. If P_n is $[0,l]$-$Z(n;a,b)$-graceful and $n>a$, then P_{n+2a-1} is $[0,n-a+2b+1-l]$-$Z(n+2a-1;a,b)$-graceful.*

Proof. Let g be a $[0,l]$-$Z(n;a,b)$-graceful labeling of $P_n: v_1,v_2,\cdots,v_n$. We define a labeling f of $P_{n+2a-1}: x_1,x_2,\cdots,x_{n+2a-1}$ such that

$$f(x_i)=\begin{cases} k, & i=2k+1,\ 0\le k\le a-1;\\ n+2(b-a)+2-k, & i=2k,\ 1\le k\le b-a+1;\\ n+2b+1-k, & i=2k,\ b-a+2\le k\le a-1;\\ n-a+2b+1-g(v_{i-2a+1}), & 2a\le i\le n+2a-1.\end{cases}$$

Then f is a bijection from the vertex set of P_{n+2a-1} to the set $Z(n+2a-1;a,b)$. Noted that, $f(a_{2a-2})=n+2b-a+2<n-a+2b+1=f(x_{2a})$ if $b<2a-2$ and $f(a_{2a-2})=n+b-a+1<n-a+2b+1=f(x_{2a})$ if $b=2a-2$. Since

$$|f(x_i)-f(x_{i+1})|=\begin{cases} n+2(b-a)+2-i, & 1\le i\le 2(b-a)+2;\\ n+2b+1-i, & 2(b-a)+3\le i\le 2a-1;\\ |g(v_{i-2a+1})-g(v_{i-2a+2})|, & 2a\le i\le n+2a-2,\end{cases}$$

we have $0\le |f(x_i)-f(x_{i+1})|\le n+2a-2$ and $|f(x_i)-f(x_{i+1})|\neq |f(x_j)-f(x_{j+1})|$ if $i\neq j$. Thus, f is a $[0,n-a+2b+1-l]$-$Z(n+2a-1;a,b)$-graceful labeling of P_{n+2a-1}. □

Corollary 9. *Let $2\le t\le n-1$. If P_n is $[0,l]$-$Z(n;t,t)$-graceful, then P_{n+2t-1} is $[0,n+t+1-l]$-$Z(n+2t-1;t,t)$-graceful.*

Corollary 10. *Let $3\le t\le n-1$. If P_n is $[0,l]$-$Z(n;t,t+1)$-graceful, then P_{n+2t-1} is $[0,n+t+3-l]$-$Z(n+2t-1;t,t+1)$-graceful.*

Theorem 11. *P_n is $Z(n;2,2)$-graceful if and only if $n \neq 4$. Moreover, P_n is $[0,\frac{n}{2}]$-$Z(n;2,2)$-graceful if n is even and $n \neq 4$. And P_n is $[0,\frac{n+3}{2}]$-$Z(n;2,2)$-graceful if n is odd and $n \geq 2$.*

Proof. By Corollary 9, we have P_{n+3}, $n \geq 3$, is $[0, n+3-l]$-$Z(n+3;2,2)$-graceful when P_n is $[0,l]$-$Z(n;2,2)$-graceful. Consider the followings labelings of P_2, P_3, P_5, and P_7, we have P_n is $[0,\frac{n}{2}]$-$Z(n;2,2)$-graceful if n is even and $n \neq 4$ and P_n is $[0,\frac{n+3}{2}]$-$Z(n;2,2)$-graceful if n is odd and $n \geq 2$.

$P_2 : 0, 1$
$P_3 : 0, 1, 3$
$P_5 : 0, 3, 1, 5, 4$
$P_7 : 0, 3, 7, 1, 6, 4, 5.$

It could be checked that P_4 is not $Z(n;2,2)$-graceful and P_1 is $Z(n;2,2)$-graceful. □

Theorem 12. *P_n is $Z(n;3,3)$-graceful.*

Proof. By Corollary 9, we have P_{n+5} is $[0, n+5-l]$-$Z(n+5;3,3)$-graceful when P_n is $[0,l]$-$Z(n;3,3)$-graceful. Therefore, the theorem holds under the following labelings.

$P_1 : 0;$
$P_2 : 0, 1;$
$P_3 : 0, 2, 1;$
$P_4 : 0, 2, 1, 4;$
$P_5 : 0, 1, 5, 2, 4;$
$P_6 : 0, 5, 6, 2, 4, 1;$
$P_7 : 0, 6, 2, 7, 5, 4, 1;$
$P_8 : 0, 7, 2, 8, 6, 5, 1, 4.$ □

Lemma 13. *If $n \geq 3$ and there is a $Z(n;2,3)$-graceful labeling g in $P_n : x_1, x_2, x_3, \ldots, x_n$ with $g(x_1) = 0$, then there is a $Z(n+9;2,3)$-graceful labeling f in $P_{n+9} : v_1, v_2, v_3, \ldots, v_{n+9}$ with $f(v_1) = 0$.*

Proof. Let $f(v_1) = 0$, $f(v_2) = n+5$, $f(v_3) = 5$, $f(v_4) = n+6$, $f(v_5) = 4$, $f(v_6) = n+10$, $f(v_7) = 6$, $f(v_8) = n+9$, $f(v_9) = 1$, and $f(v_k) = n+8-g(x_{k-9})$ for $k \geq 10$. Then we have $\{|f(x_i) - f(x_{i+1})| \mid 1 \leq i \leq 9\} = \{n, n+1, \ldots, n+8\}$ and $\{|f(x_i) - f(x_{i+1})| \mid 10 \leq i \leq n+8\} = \{|g(x_{j+1}) - g(x_j)| | 1 \leq j \leq n-1\} = \{1, 2, \ldots, n-1\}$. Hence, f is $Z(n+9;2,3)$-graceful labeling. □

Theorem 14. *P_n is $Z(n;2,3)$-graceful except n= 3, 4, 5.*

Proof. It could be checked that P_n is not $Z(n;2,3)$-graceful if $n=3,4,5$. By Lemma 13, theorem holds under the following labelings.

$P_1 : 0,$
$P_2 : 0,1,$
$P_6 : 0,5,1,4,6,7,$
$P_7 : 0,6,1,4,8,7,5,$
$P_8 : 0,7,1,4,9,5,6,8,$
$P_9 : 0,8,1,4,10,5,9,7,6,$
$P_{12} : 0,11,1,10,4,5,13,6,9,7,12,8,$
$P_{13} : 0,12,1,7,9,4,14,5,13,6,10,11,8,$
$P_{14} : 0,13,1,8,9,4,15,5,14,6,12,10,7,11.$ □

Theorem 15. *P_n is $Z(n;t,t)$-graceful if $4t-1 \le n \le 4t+3$.*

Proof. Let $P_n : x_1, x_2, \cdots, x_n$. For the case of $n = 4t-1$, the labeling of P_{4t-1} is as following.

$$f(x_i) = \begin{cases} k, & i = 2k+1,\ 0 \le k \le t-1, \\ k+1, & i = 2k+1,\ t \le k \le 2t-2, \\ n-k, & i = 2k,\ 1 \le k \le 2t-1, \\ n, & i = 4t-1. \end{cases}$$

For the case of $n = 4t$, the labeling of P_{4t} is as following.

$$f(x_i) = \begin{cases} k, & i = 2k+1,\ 0 \le k \le t-1, \\ k+1, & i = 2k+1,\ t \le k \le 2t-1, \\ n-k, & i = 2k,\ 1 \le k \le 2t-1, \\ n, & i = 4t. \end{cases}$$

For the case of $n = 4t+1$, the labeling of P_{4t+1} is as following.

$$f(x_i) = \begin{cases} k, & i = 2k+1,\ 0 \le k \le t-1, \\ n-k, & i = 2k,\ 1 \le k \le t, \\ n-k-1, & i = 2k+1,\ t \le k \le 2t, \\ k, & i = 2k,\ t+1 \le k \le 2t-1, \\ n, & i = 4t. \end{cases}$$

For the case of $n = 4t+2$, the labeling of P_{4t+2} is as following.

$$f(x_i) = \begin{cases} k-1, & i = 2k,\ 1 \le k \le t, \\ k, & i = 2k,\ t+1 \le k \le 2t, \\ n-k-1, & i = 2k+1,\ 0 \le k \le 2t, \\ n, & i = 4t+2. \end{cases}$$

For the case of $n = 4t + 3$, the labeling of P_{4t+3} is as following.

$$f(x_i) = \begin{cases} k-1, & i = 2k,\ 1 \le k \le t, \\ k, & i = 2k,\ t+1 \le k \le 2t+1, \\ n-k-1, & i = 2k+1,\ 0 \le k \le 2t, \\ n, & i = 4t+3. \end{cases}$$

We can check that these labelings are $Z(n; t, t)$-graceful. So, P_n is $Z(n; t, t)$-graceful if $4t - 1 \le n \le 4t + 3$. □

By Corollary 9 and Theorem 15, we have the following corollary.

Corollary 16. *P_n is $Z(n; t, t)$-graceful if $2mt - m + 1 \le n \le 2mt - m + 5$ for $k \ge 2$.*

After summarizing the results discussed in this section, we have the following two conjectures.

Conjecture 17. *For each positive integer t, P_n is $Z(n; t, t)$-graceful except $n = 2t \ne 2, 4$.*

Conjecture 18. *For given integer $1 \le a \le b$, there exist integer N such that P_n is $Z(n; a, b)$-graceful if $n \ge N$.*

3. Gacefulness of $P_{2,n} \cup P_m$

In this section, we showed some graphs are graceful with the support of “D-gracefulness”.

Let u and v be two vertices. We connect u and v by means of ”b” internally disjoint paths of length ”a” each. The resulting graph is denoted by $P_{a,b}$. Kathiresan[5] has shown that $P_{a,b}$ is graceful for a is even and b is odd and he conjectured $P_{a,b}$ is graceful except when $a = 2r + 1$ and $b = 4s + 2$.

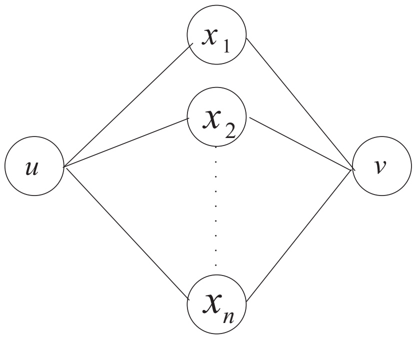

Figure 1: $P_{2,n}$

Lemma 19. *$P_{2,n}$ is graceful.*

Proof. $P_{2,n}$ is shown in Figure 1. Define the labeling f in $P_{2,n}$ with $f(u) = 0$, $f(v) = n$, and $f(x_i) = 2n - i + 1$. Then we have f is a graceful labeling. □

Theorem 20. *$P_{2,n} \cup P_m$ is graceful except $m = n = 2$ or $n = 1$.*

Proof. Define the labeling f in $P_{2,n} \cup P_m$ as follow.
1. For the vertices in the part of $P_{2,n}$, let $f(u) = 0$, $f(v) = n$ and $f(x_i) = m + 2n - i$, $1 \leq i \leq n$.
2. For the vertices in the part of P_m, if $m = 2$, then $n \geq 3$. We label the two vertices of P_2 by 1 and 2. Assume $m \neq 2$. By Theorem 6, we have a $Z(m; 1, 1)$-graceful g of P_m. Let $f(w) = g(w) + n - 1$ for each vertex w in the part of P_m.

Then f is a graceful labeling of $P_{2,n} \cup P_m$. □

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