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A New General Integral Operator *

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Abstract

In this paper we define a new general integral operator for certain holomorphic functions in the unit disc $\mathcal{U}$ and give some properties for this integral operator on some classes of univalent functions

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1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C}, |z| < 1\}$ be the open unit disc of the complex plane. Denote by $\mathcal{A}$ the class of holomorphic functions in $\mathcal{U}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Recently, D. Breaz and N. Breaz in [3] introduced and studied the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\mu_1} \cdot ... \cdot \left(\frac{f_n(t)}{t}\right)^{\mu_n} dt, \tag{1}$$

where $\mu_1, \ldots, \mu_n$ are real numbers, $n \in \mathbb{N} = \{1, 2, 3, \ldots, \}$.

Sălăgean [2] has introduced the following operator $D^k : \mathcal{A} \to \mathcal{A}$ $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^k f(z) = D(D^{k-1} f(z)) = z + \sum_{n=2}^{\infty} n^k a_n z^n.$$

The purpose of this paper is to define a new integral operator which derive from the above mentioned integral operators with the help of the operator D^k.

Definition 1. We define the general integral operator $I_{k,n,\lambda,\mu} : \mathcal{A}^n \to \mathcal{A}$ by

$$I_{k,n,\lambda,\mu}(f_1, \ldots, f_n) = F, \quad D^k F(z) = \int_0^z \left(\frac{D^{\lambda_1} f_1(t)}{t}\right)^{\mu_1} \cdots \left(\frac{D^{\lambda_n} f_n(t)}{t}\right)^{\mu_n} dt \tag{2}$$

where $f_1, \ldots, f_n \in \mathcal{A}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}_0^n$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Remark 1. If we take $k = 0$ or $k = 1$ and $\lambda_1 = ... = \lambda_n = 0$ in Definition, we obtain the definitions given by the first author and N.Breaz in [3].

In order to prove our results we need the following lemma:

Lemma. *Let* $F = I_{k,n,\lambda,\mu}(f_1, \ldots, f_n)$, *where* $f_1, \ldots, f_n \in \mathcal{A}$, *and* k, n, λ, μ *are as in Definition 1; then*

$$\frac{D^{k+2}F(z)}{D^{k+1}F(z)} = \sum_{i=1}^{n} \mu_i \frac{D^{\lambda_i+1} f_i(z)}{D^{\lambda_i} f_i(z)} + 1 - \sum_{i=1}^{n} \mu_i. \tag{3}$$

Proof. By (2) we have

$$\frac{D^{k+1}F(z)}{z} = \left(\frac{D^{\lambda_1}f_1(t)}{t}\right)^{\mu_1} \cdots \left(\frac{D^{\lambda_n}f_n(t)}{t}\right)^{\mu_n}.$$

Also we obtain

$$\frac{D^{k+2}F(z) - D^{k+1}F(z)}{z^2}$$
$$= \sum_{i=1}^{n} \mu_i \left(\frac{D^{\lambda_i}f_i(z)}{z}\right)^{\mu_i} \cdot \left(\frac{D^{\lambda_i+1}f_i(z) - D^{\lambda_i}f_i(z)}{zD^{\lambda_i}f_i(z)}\right) \prod_{j=1, j\neq i}^{n} \left(\frac{D^{\lambda_j}f_j(z)}{z}\right)^{\mu_j}$$

Hence,

$$\frac{\dfrac{D^{k+2}F(z) - D^{k+1}F(z)}{z^2}}{\dfrac{D^{k+1}F(z)}{z}} = \frac{\sum_{i=1}^{n} \mu_i \left(\frac{D^{\lambda_i}f_i(z)}{z}\right)^{\mu_i} \cdot \left(\frac{D^{\lambda_i+1}f_i(z) - D^{\lambda_i}f_i(z)}{zD^{\lambda_i}f_i(z)}\right) \prod_{j=1, j\neq i}^{n} \left(\frac{D^{\lambda_j}f_j(z)}{z}\right)^{\mu_j}}{\left(\frac{D^{\lambda_1}f_1(z)}{z}\right)^{\mu_1} \cdots \left(\frac{D^{\lambda_n}f_n(z)}{z}\right)^{\mu_n}},$$

or, after simplifications,

$$\frac{D^{k+2}F(z)}{zD^{k+1}F(z)} - \frac{1}{z} = \sum_{i=1}^{n} \mu_i \left(\frac{D^{\lambda_i+1}f_i(z) - D^{\lambda_i}f_i(z)}{zD^{\lambda_i}f_i(z)}\right)$$

and finally we obtain

$$\frac{D^{k+2}F(z)}{D^{k+1}F(z)} - 1 = \sum_{i=1}^{n} \mu_i \left(\frac{D^{\lambda_i+1}f_i(z) - D^{\lambda_i}f_i(z)}{D^{\lambda_i}f_i(z)}\right) = \sum_{i=1}^{n} \mu_i \frac{D^{\lambda_i+1}f_i(z)}{D^{\lambda_i}f_i(z)} - \sum_{i=1}^{n} \mu_i.$$

Definition 2. ([5]). The class of *k-starlike functions of order* δ is defined by

$$\mathcal{S}_k(\delta) = \left\{ f \in \mathcal{A} : \mathbf{Re}\frac{D^{k+1}f(z)}{D^k f(z)} > \delta,\ z \in \mathcal{U} \right\},$$

where $\delta \in [0, 1)$ and $k \in \mathbb{N}_0$.

Remark 2. Particularly, $\mathcal{S}_1(0)$ is the class of convex functions and $\mathcal{S}_0(0)$ is the class of starlike functions ([5]).

Definition 3. ([2], [4]). We define the class of *k-uniform starlike function of order* δ *and type* α by

$$\mathcal{US}_k(\alpha,\delta)=\left\{f\in\mathcal{A}:\mathbf{Re}\frac{D^{k+1}f(z)}{D^kf(z)}\geq\alpha\left|\frac{D^{k+1}f(z)}{D^kf(z)}-1\right|+\delta;\ \ z\in\mathcal{U}\right\}$$

where $\alpha\geq 0,\delta\in[-1,1),\alpha+\delta\geq 0,k\in\mathbb{N}$.

Remark 3. ([2], [4]). The geometric interpretation is that if $f\in\mathcal{US}_k(\alpha,\delta)$, then, for all $z\in\mathcal{U}$, the values of $D^{k+1}f(z)/D^kf(z)$ belongs to the convex domain $\Delta_{\alpha,\delta}$ included in the right half plane $\{w:\mathbf{Re}\,w>\frac{\alpha+\delta}{\alpha+1}\}$, where $\Delta_{\alpha,\delta}$ is the half plane $\{w:\mathbf{Re}\,w>\delta\}$ for $\alpha=0$, a hyperbolic region for $0<\alpha<1$, a parabolic region for $\alpha=1$, or an elliptic region for $\alpha>1$. From this we deduce that $\mathcal{US}_k(\alpha,\delta)\subset\mathcal{S}_k\left(\frac{\alpha+\delta}{\alpha+1}\right)$ and $\mathcal{US}_k\left(0,\delta\right)=\mathcal{S}_k\left(\delta\right)$.

Definition 4. ([1]). We define the class $\mathcal{SH}_k(\alpha)$ by

$$\begin{aligned}&\mathcal{SH}_k(\alpha)\\ =&\left\{f\in\mathcal{A}:\left|\frac{D^{k+1}f(z)}{D^kf(z)}-2\alpha\left(\sqrt{2}-1\right)\right|<\mathbf{Re}\left\{\sqrt{2}\frac{D^{k+1}f(z)}{D^kf(z)}\right\}+2\alpha\left(\sqrt{2}-1\right),\ z\in U\right\},\end{aligned}\tag{4}$$

where $\alpha>0$ and $n\in\mathbb{N}_0$.

2. Main results

Theorem 1. *Let* k, n, λ, μ *be as in Definition 1, let* $\delta_i\in[-1,1)$, $\alpha_i\geq 0$, $\alpha_i+\delta_i\geq 0$ *for* $i=\{1,...,n\}$ *and suppose that* $\sum_{i=1}^n\mu_i\frac{1-\delta_i}{1+\alpha_i}\leq 1$. *If* $f_i\in\mathcal{US}_{\lambda_i}(\alpha_i,\delta_i)$ *for* $i=\{1,...,n\}$, *then* $F=I_{k,n,\lambda,\mu}$ *belongs to* $\mathcal{S}_{k+1}(0)$..

Proof. Since $f_i\in\mathcal{US}_{\lambda_i}(\alpha_i,\delta_i)$, from Remark 3 and Definition 2 we have

$$\mathbf{Re}\frac{D^{\lambda_i+1}f_i(z)}{D^{\lambda_i}f_i(z)}>\frac{\alpha_i+\delta_i}{\alpha_i+1}\tag{5}$$

From (3) and (5) we have

$$\mathbf{Re}\frac{D^{k+2}F(z)}{D^{k+1}F(z)}=\sum_{i=1}^n\mu_i\,\mathbf{Re}\frac{D^{\lambda_i+1}f_i(z)}{D^{\lambda_i}f_i(z)}-\sum_{i=1}^n\mu_i+1>$$

$$\sum_{i=1}^{n}\mu_i\frac{\alpha_i+\delta_i}{\alpha_i+1}-\sum_{i=1}^{n}\mu_i+1=\sum_{i=1}^{n}\mu_i\frac{\delta_i-1}{\alpha_i+1}+1\geq 0,\quad z\in\mathcal{U}$$

and by Definition 2 it follows that $F\in\mathcal{S}_{k+1}(0)$

Corollary 1. *Let* k, n, λ, μ *be as in Definition 1, let* $\delta_1,\ldots,\delta_n\in[0,1)$ *and suppose that* $\sum_{i=1}^{n}\mu_i(1-\delta_i)\leq 1$. *If* $f_i\in\mathcal{S}_{\lambda_i}(\delta_i)$ *for* $i=\{1,...,n\}$, *then* $F=I_{k,n,\lambda,\mu}\in\mathcal{S}_{k+1}(0)$.

Proof. In Theorem 1 we put $\alpha_i=0$ for $i=\{1,...,n\}$.

Theorem 2. *Let* k, n, λ, μ *be as in Definition 1, let* $\delta_i\in[-1,1)$, $\alpha_i\geq 0, \alpha_i+\delta_i\geq 0$ *for* $i\in\{1,...,n\}$ *and suppose that* $\sum_{i=1}^{n}\mu_i\leq 1$. *If* $f_i\in\mathcal{US}_{\lambda_i}(\alpha_i,\delta_i)$ *for* $i=\{1,...,n\}$, *then* $F=I_{k,n,\lambda,\mu}$ *belongs to* $\mathcal{S}_{k+1}(\delta^*)$, *where* $\delta^*=\sum_{i=1}^{n}\mu_i\frac{\alpha_i+\delta_i}{\alpha_i+1}$.

The proof is similar to those of Theorem 1 and is omitted.

Corollary 2. *Let* k, n, λ, μ *be as in Definition 1, let* $\delta_1,\ldots,\delta_n\in[0,1)$ *and suppose that* $\sum_{i=1}^{n}\mu_i\leq 1$. *If* $f_i\in\mathcal{S}_{\lambda_i}(\delta_i)$ *for* $i\in\{1,...,n\}$, *then* $F=I_{k,n,\lambda,\mu}\in\mathcal{S}_{k+1}(\delta^*)$, where $\delta^*=\sum_{i=1}^{n}\mu_i\delta_i$.

Proof. In Theorem 2 we put $\alpha_i=0$ for $i=\{1,...,n\}$.

Remark 4. In the particular case $n=\mu_1=1$, $k=\delta_1=\lambda_1=0$ and $f_1=f$ we reobtain the well-known result: if f is a starlike function and $f(z)=zF'(z)$, then F is a convex function.

Theorem 3. *Let* k, n, λ, μ *be as in Definition 1, let* $\alpha\geq 0$, $\delta\in[-1,1)$, $\alpha+\delta\geq 0$ *and suppose that* $\sum_{i=1}^{n}\mu_i\leq 1$. *If* $f_i\in\mathcal{US}_k(\alpha,\delta)$ *for* $i=\{1,...,n\}$, *then* $F=I_{k,n,\lambda,\mu}$ *belongs to* $\mathcal{US}_{k+1}(\alpha,\delta)$.

Proof. By using (3) and Definition 3 we deduce

$$\begin{aligned}
&\mathbf{Re}\frac{D^{k+2}F(z)}{D^{k+1}F(z)}-\alpha\left|\frac{D^{k+2}F(z)}{D^{k+1}F(z)}-1\right|-\delta\\
=\;&\sum_{i=1}^{n}\mu_i\mathbf{Re}\frac{D^{k_i+1}f_i(z)}{D^{k_i}f_i(z)}-\sum_{i=1}^{n}\mu_i+1-\alpha\left|\sum_{i=1}^{n}\mu_i\left(\frac{D^{k_i+1}f_i(z)}{D^{k_i}f_i(z)}-1\right)\right|-\delta\\
\geq\;&\sum_{i=1}^{n}\mu_i\left[\mathbf{Re}\frac{D^{k_i+1}f_i(z)}{D^{k_i}f_i(z)}-\alpha\left|\sum_{i=1}^{n}\frac{D^{k_i+1}f_i(z)}{D^{k_i}f_i(z)}-1\right|-\delta\right]+\sum_{i=1}^{n}\mu_i\delta-\sum_{i=1}^{n}\mu_i+1-\delta\\
\geq\;&(1-\delta)\left(1-\sum_{i=1}^{n}\mu_i\right)\geq 0,\quad z\in\mathcal{U}
\end{aligned}$$

Theorem 4. *Let k, n, λ, μ be as in Definition 1, let $\alpha \geq 0$ and suppose that $\sum_{i=1}^{n} \mu_i \leq 1$. If $f_i \in \mathcal{SH}_{\lambda_i}(\alpha)$ for $i = \{1, ..., n\}$, then $F = I_{k,n,\lambda,\mu}$ belongs to $\mathcal{SH}_{k+1}(\alpha)$.*

Proof. Since $f_i \in \mathcal{SH}_{\lambda_i(\alpha)}$ for $i = \{1, ..., n\}$, from Definition 4 we have

$$\mathbf{Re}\left\{\sqrt{2}\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}\right\}+2\alpha\left(\sqrt{2}-1\right)-\left|\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}-2\alpha\left(\sqrt{2}-1\right)\right|>0,\ z\in U. \tag{6}$$

By (3) and (6) we have

$$\begin{aligned}
&\mathbf{Re}\left\{\sqrt{2}\frac{D^{k+2}F(z)}{D^{k+1}F(z)}\right\}-\left|\frac{D^{k+2}F(z)}{D^{k+1}F(z)}-2\alpha\left(\sqrt{2}-1\right)\right|+2\alpha\left(\sqrt{2}-1\right)\\
=\ &\sum_{i=1}^{n}\mu_i\mathbf{Re}\left\{\sqrt{2}\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}\right\}+\sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\mu_i\\
&-\left|\sum_{i=1}^{n}\mu_i\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}+1-\sum_{i=1}^{n}\mu_i-2\alpha\left(\sqrt{2}-1\right)\right|+2\alpha\left(\sqrt{2}-1\right)\\
=\ &\sum_{i=1}^{n}\mu_i\left[\mathbf{Re}\left\{\sqrt{2}\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}\right\}+2\alpha(\sqrt{2}-1)\right]-\sum_{i=1}^{n}\mu_i\left[2\alpha(\sqrt{2}-1)+\sqrt{2}\right]+\sqrt{2}\\
&-\left|\sum_{i=1}^{n}\mu_i\left[\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}-2\alpha\left(\sqrt{2}-1\right)\right]+2\alpha\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\mu_i+1-\sum_{i=1}^{n}\mu_i-2\alpha\left(\sqrt{2}-1\right)\right|\\
&+2\alpha\left(\sqrt{2}-1\right)\\
\geq\ &\sum_{i=1}^{n}\mu_i\left[\mathbf{Re}\left\{\sqrt{2}\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}\right\}+2\alpha(\sqrt{2}-1)-\left|\frac{D^{\lambda_i+1}f(z)}{D^{\lambda_i}f(z)}-2\alpha\left(\sqrt{2}-1\right)\right|\right]\\
>\ &\left[2\alpha(\sqrt{2}-1)+\sqrt{2}\right]\left(1-\sum_{i=1}^{n}\mu_i\right)-\left|1-2\alpha(\sqrt{2}-1)\right|\left(1-\sum_{i=1}^{n}\mu_i\right)\\
>\ &\left(1-\sum_{i=1}^{n}\mu_i\right)\mathbf{min}\left\{(\sqrt{2}-1)(1+4\alpha),\sqrt{2}+1\right\}\geq 0,\quad z\in U.
\end{aligned}$$

Now by Definition 4 we obtain that $F \in \mathcal{SH}_{k+1}(\alpha)$.

Theorem 5. *Let k, n, λ, μ be as in Definition 1, let $\alpha \geq 0$ and let $f_i \in \mathcal{SH}_{\lambda_i}(\alpha)$ for $i = \{1, ..., n\}$. If*

$$1-\sum_{i=1}^{n}\mu_i\left[2\alpha\left(\sqrt{2}-1\right)+1\right]>0$$

then $F_n \in \mathcal{S}_{k+1}(0)$.

Proof.

Multiplying the equality (3) by $\sqrt{2}$ and using (6) we obtain

$$\begin{aligned}
& \mathbf{Re}\sqrt{2}\frac{D^{k+2}F_n(z)}{D^{k+1}F_n(z)} \\
= \;& \mathbf{Re}\left[\sum_{i=1}^{n}\sqrt{2}\alpha_i\frac{D^{k+1}f_i(z)}{D^k f_i(z)}\right] + \sum_{i=1}^{n}2\alpha_i\alpha\left(\sqrt{2}-1\right) \\
& -\sum_{i=1}^{n}2\alpha_i\alpha\left(\sqrt{2}-1\right) - \sum_{i=1}^{n}\sqrt{2}\alpha_i + \sqrt{2} \\
> \;& \sum_{i=1}^{n}\alpha_i\left|\frac{D^{k+1}f_i(z)}{D^k f_i(z)} - 2\alpha(\sqrt{2}-1)\right| \\
& -\sum_{i=1}^{n}\alpha_i\left[2\alpha(\sqrt{2}-1)+\sqrt{2}\right] + \sqrt{2} \\
> \;& \sqrt{2}\left(1-\sum_{i=1}^{n}\alpha_i\left[\sqrt{2}\alpha(\sqrt{2}-1)+1\right]\right) > 0
\end{aligned}$$

So, by Definition 2, $F \in \mathcal{S}_{k+1}(0)$.

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