Tamsui Oxford Journal of Mathematical Sciences **25(4)** (2009) 393-405
Aletheia University

Some Inclusion properties for Certain Subclasses of Strongly Starlike and Strongly Convex Functions associated with the Dziok-Srivastava operator *

J. K. Prajapat†

Department of Mathematics, Sobhasaria Engineering College
NH-11 Gokulpura, Sikar 332001, Rajasthan, India

Received May 27, 2008, Accepted July 1, 2008.

Abstract

The purpose of the present article is to investigate several new subclasses of analytic functions defined by using Dziok-Srivastava operator and investigate linebreak various inclusion relationships for these subclasses. Some interesting corollaries and consequences of the results presented here are also discussed.

Keywords and Phrases: *Analytic functions; Strongly starlike functions; Strongly convex functions; Gauss hypergeometric function; Dziok-Srivastava operator; Inclusion properties.*

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

*2000 *Mathematics Subject Classification.* 30C45, 30C75.

†E-mail: jkp_0007@rediffmail.com

which are *analytic* in the *open unit* disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\,\beta \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1), \tag{1.2}$$

then the function $f(z)$ is said to be strongly starlike of order β and type α in $\mathbb{U}$. We denote this subclass of $\mathcal{A}$ by $\mathcal{S}_s^*(\alpha,\beta)$. If, on the other hand, $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\,\beta \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1), \tag{1.3}$$

then the function $f(z)$ is said to be strongly convex of order β and type α in $\mathbb{U}$. We denote this subclass of $\mathcal{A}$ by $\mathcal{K}_c(\alpha,\beta)$. It is obvious that

$$f(z) \in \mathcal{K}_c(\alpha,\beta) \iff zf'(z) \in \mathcal{S}_s^*(\alpha,\beta).$$

If $f \in \mathcal{A}$ and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.5}$$

Making use of the Hadamard product (or convolution) given by (1.5) Dziok and Srivastava [4](see also [5]) introduced a linear operator

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) \ : \ \mathcal{A} \longrightarrow \mathcal{A},$$

which is defined by

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = z\ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z) * f(z) \qquad (z \in \mathbb{U};\ f \in \mathcal{A}), \tag{1.6}$$

where ${}_qF_s$ denote the familiar generalized hypergeometric function given by

$${}_qF_s(z) \ \equiv \ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z)$$

$$=: \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \, ... \, (\alpha_q)_n}{(\beta_1)_n \, ... \, (\beta_s)_n} . \, \frac{z^n}{n!}$$

$(z \in \mathbb{U};\ \alpha_j \in \mathbb{C}(j = 1, ..., q),\ \ \beta_j \in \mathbb{C} \backslash \{0, -1, -2, ...\}(j = 1, ..., s),\ q \leq s{+}1;\ q, s \in \mathbb{N}_0)$, where $(x)_k$ is the Pochhammer symbol, defined by

$$(x)_0 = 1, \qquad (x)_k = x(x+1)...(x+k-1);\ k \in \mathbb{N}.$$

Note the Dziok-Srivastava linear operator defined by (1.6) above, contains such well known operators as the Hohlov linear operator [7], the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [15], the Bernardi-Libera-Livingston operator [1], and the Srivastava-Owa fractional derivative operator [16].

Very recently Kwon and Cho [9] introduced the following family of linear operator $H_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ analogous to $H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$:

$$H_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) \ : \ \mathcal{A} \longrightarrow \mathcal{A},$$

defined by

$$H_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = \mathcal{F}_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z) * f(z) \qquad (1.7)$$

$(\alpha_i, \beta_j \in \mathbb{C} \backslash \{0, -1, -2, ...\}(i = 1, ..., q;\ j = 1, ..., s);\ \lambda > -1;\ z \in \mathbb{U};\ f \in \mathcal{A})$, where $\mathcal{F}_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z)$ is the function defined by

$$z\ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z) * \mathcal{F}_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z) = \frac{z}{(1-z)^\lambda} \qquad (\lambda > -1). \qquad (1.8)$$

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1) = H_\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s),$$

Note that

$$H_{0,1,0}(1) f(z) = f(z), \quad H_{1,1,0}(1) f(z) = z f'(z) \quad \text{and} \quad H_{2,1,0}(1) f(z) = \frac{1}{2} z^2 f''(z) + z f'(z).$$

It is easily verified from the above definition of the operator $H_{\lambda,q,s}(\alpha_1) f$ that

$$z(H_{\lambda,q,s}(\alpha_1+1) f(z))' = \alpha_1 \ H_{\lambda,q,s}(\alpha_1) f(z) - (\alpha_1 - 1) H_{\lambda,q,s}(\alpha_1+1) f(z), \ (1.9)$$

and

$$z(H_{\lambda,q,s}(\alpha_1)f(z))' = \lambda\, H_{\lambda+1,q,s}(\alpha_1)f(z) - (\lambda-1)H_{\lambda,q,s}(\alpha_1)f(z). \tag{1.10}$$

The definition of the linear operator $H_{\lambda,q,s}(\alpha_1)$ is motivated essentially by the Choi-Saigo-Srivastava operator [3] for analytic functions, which includes a simpler integral operator studied earlier by Noor [11] (see also [12]).

Using the linear operator $H_{\lambda,q,s}(\alpha_1)$, we now introduce the following subclasses of $\mathcal{A}$:

$$\mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta)$$
$$:= \left\{ f : f(z) \in \mathcal{A},\ H_{\lambda,q,s}(\alpha_1)f(z) \in \mathcal{S}^*_s(\alpha,\beta) \text{ and } \frac{z\big(H_{\lambda,q,s}(\alpha_1)f(z)\big)'}{H_{\lambda,q,s}(\alpha_1)f(z)} \neq \alpha \quad (z \in \mathbb{U}) \right\} \tag{1.11}$$

and

$$\mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta)$$
$$:= \left\{ f : f(z) \in \mathcal{A},\ H_{\lambda,q,s}(\alpha_1)f(z) \in \mathcal{K}_c(\alpha,\beta) \text{ and } \frac{\Big(z\big(H_{\lambda,q,s}(\alpha_1)f(z)\big)'\Big)'}{\big(H_{\lambda,q,s}(\alpha_1)f(z)\big)'} \neq \alpha \quad (z \in \mathbb{U}) \right\}. \tag{1.12}$$

It is obvious from the definitions (1.11) and (1.12) that

$$f(z) \in \mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta) \iff zf'(z) \in \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta).$$

In this paper we establish some inclusion relationships for the above-mentioned function classes involving the linear operator $H_{\lambda,q,s}(\alpha_1)$ defined by (1.7) above. Some corollaries and consequences of our main inclusion relationships are also mentioned.

2. Main Inclusion Relationships

In order to derive our main inclusion relationships, we recall here the following lemma.

Lemma 1. (see [13]). *Let a function $p(z)$ be analytic in $\mathbb{U}$ with*

$$p(0) = 1 \qquad \textit{and} \qquad p(z) \neq 0 \quad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ *such that*

$$|\arg\big(p(z)\big)| < \frac{\pi}{2}\,\beta \quad (|z| < |z_0|) \quad \textit{and} \quad |\arg\big(p(z_0)\big)| = \frac{\pi}{2}\,\beta \quad (0 < \beta \leqq 1), \tag{2.1}$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta, \tag{2.2}$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \qquad \arg\big(p(z_0)\big) = \frac{\pi}{2}\,\beta, \tag{2.3}$$

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \qquad \arg\big(p(z_0)\big) = -\frac{\pi}{2}\,\beta, \tag{2.4}$$

and

$$\big(p(z_0)\big)^{\frac{1}{\beta}} = \pm\, ia \qquad (a > 0).$$

Theorem 1 below gives our first main inclusion relationship.

Theorem 1. *Let* $f \in \mathcal{A}$. *Suppose also that*

$$\alpha + \lambda > 1, \quad \alpha + \alpha_1 > 1, \quad 0 \leqq \alpha < 1 \quad \textit{and} \quad 0 < \beta \leqq 1.$$

Then

$$\mathcal{S}^*_{\lambda+1,\alpha_1}(q,s,\alpha,\beta) \subset \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta) \subset \mathcal{S}^*_{\lambda,\alpha_1+1}(q,s,\alpha,\beta).$$

Proof. First of all we show that

$$\mathcal{S}^*_{\lambda+1,\alpha_1}(q,s,\alpha,\beta) \subset \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta).$$

Let $f \in \mathcal{S}^*_{\lambda+1,\alpha_1}(q,s,\alpha,\beta)$ and set

$$p(z) = \frac{1}{1-\alpha}\left(\frac{z\big(H_{\lambda,q,s}(\alpha_1)f(z)\big)'}{H_{\lambda,q,s}(\alpha_1)f(z)} - \alpha\right) \qquad (z \in \mathbb{U}), \tag{2.5}$$

where the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1$ and $p(z) \neq 0$ for $z \in \mathbb{U}$. Now using the identity (1.10) in (2.5), and differentiating with respect to z, we get

$$\frac{z\big(H_{\lambda+1,q,s}(\alpha_1)f(z)\big)'}{H_{\lambda+1,q,s}(\alpha_1)f(z)} - \alpha = (1-\alpha)p(z) + \frac{(1-\alpha)zp'(z)}{\alpha+\lambda-1+(1-\alpha)p(z)}.$$

Suppose now that there exists a point $z_0 \in \mathbb{U}$ such that the conditions (2.1) to (2.4) of Lemma 1 are satisfied. Thus, if

$$\arg\big(p(z_0)\big) = \frac{\pi}{2}\,\beta \qquad (z_0 \in \mathbb{U}),$$

then

$$\frac{z_0\big(H_{\lambda+1,q,s}(\alpha_1)f(z_0)\big)'}{H_{\lambda+1,q,s}(\alpha_1)f(z_0)} - \alpha = (1-\alpha)p(z_0)\left(1+\frac{\left(\frac{z_0p'(z_0)}{p(z_0)}\right)}{\alpha+\lambda-1+(1-\alpha)p(z_0)}\right)$$

$$= (1-\alpha)\,a^{\beta}e^{\frac{i\pi\beta}{2}}\left(1+\frac{ik\beta}{\alpha+\lambda-1+(1-\alpha)a^{\beta}e^{\frac{i\pi\beta}{2}}}\right).$$

This implies that

$$\arg\left(\frac{z_0\big(H_{\lambda+1,q,s}(\alpha_1)f(z_0)\big)'}{H_{\lambda+1,q,s}(\alpha_1)f(z_0)} - \alpha\right) = \frac{\pi\beta}{2}+\arg\left(1+\frac{ik\beta}{\alpha+\lambda-1+(1-\alpha)a^{\beta}e^{\frac{i\pi\beta}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{k\beta\left(\alpha+\lambda-1+(1-\alpha)a^{\beta}\cos\frac{\pi\beta}{2}\right)}{(\alpha+\lambda-1)^2+(1-\alpha)^2a^{2\beta}+2(\alpha+\lambda-1)(1-\alpha)a^{\beta}\cos\frac{\pi\beta}{2}+k\beta(1-\alpha)a^{\beta}\sin\frac{\pi\beta}{2}}\right)$$

$$\geqq 0, \qquad \left(\text{since}\quad k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \geqq 1, \quad \text{and} \quad z_0 \in \mathbb{U}.\right)$$

Thus, in view of (1.2) and (1.11), this last inequality would contradict our assumption. On the other hand, if we set

$$\arg\big(p(z_0)\big) = -\frac{\pi}{2}\,\beta,$$

then it can *similarly* be shown that

$$\arg\left(\frac{z\big(H_{\lambda+1,q,s}(\alpha_1)f(z)\big)'}{H_{\lambda+1,q,s}(\alpha_1)f(z)} - \alpha\right) \leq -\frac{\pi}{2}\beta \quad \left(\text{since} k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \leqq -1 \quad \text{and} \quad z_0 \in \mathbb{U}\right),$$

which again contradicts the assumption. Hence the function $p(z)$ defined by (2.5) satisfies the following inequality:

$$|\arg\big(p(z)\big)| < \frac{\pi}{2}\,\beta \qquad (z \in \mathbb{U}),$$

which implies that $f \in \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta)$.

To prove second part, let $f \in \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta)$ and put

$$q(z) = \frac{1}{1-\alpha}\left(\frac{z\big(H_{\lambda,q,s}(\alpha_1+1)f(z)\big)'}{H_{\lambda,q,s}(\alpha_1+1)f(z)} - \alpha\right) \qquad (z \in \mathbb{U}),$$

where $q(z)$ is analytic function in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for $z \in \mathbb{U}$. Then by using the argument similar to those detailed above with (1.9) it follows that

$$|\arg\big(q(z)\big)| < \frac{\pi}{2}\,\beta \qquad (z \in \mathbb{U}),$$

which implies that $f \in \mathcal{S}^*_{\lambda,\alpha_1+1}(q,s,\alpha,\beta)$. This completes the proof of Theorem 1.

We next prove the following inclusion relationships.

Theorem 2. *Let $f \in \mathcal{A}$. Then under the parametric constraints stated with Theorem 1, we have*

$$\mathcal{K}_{\lambda+1,\alpha_1}(q,s,\alpha,\beta) \subset \mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta) \subset \mathcal{K}_{\lambda,\alpha_1+1}(q,s,\alpha,\beta).$$

Proof. We observe from Theorem 1 that

$$\begin{aligned} f(z) \in \mathcal{K}_{\lambda+1,\alpha_1}(q,s,\alpha,\beta) &\iff zf'(z) \in \mathcal{S}^*_{\lambda+1,\alpha_1}(q,s,\alpha,\beta) \\ &\implies zf'(z) \in \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta) \\ &\iff f(z) \in \mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta), \\ f(z) \in \mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta) &\iff zf'(z) \in \mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta) \\ &\iff zf'(z) \in \mathcal{S}^*_{\lambda,\alpha_1+1}(q,s,\alpha,\beta) \\ &\iff f(z) \in \mathcal{K}_{\lambda,\alpha_1+1}(q,s,\alpha,\beta). \end{aligned}$$

which establishes Theorem 2.

Theorem 3. *Let $f \in \mathcal{A}$ and $\mathcal{I}_c$ be the integral operator defined by*

$$\mathcal{I}_c f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \qquad (c > -1;\ f \in \mathcal{A}). \tag{2.6}$$

Suppose also

$$c > -\alpha, \quad 0 \leqq \alpha < 1, \quad 0 < \beta \leqq 1 \quad \text{and} \quad \frac{z\big(H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_c f(z)\big)'}{H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_c f(z)} \neq \alpha \qquad (z \in \mathbb{U}).$$

Then

$$f(z)\in\mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta)\implies\mathcal{I}_cf(z)\in\mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta). \tag{2.7}$$

Proof. We begin by assuming that $f(z)\in\mathcal{S}^*_{\lambda,\alpha_1}(q,s,\alpha,\beta)$ and defining a function $r(z)$ by

$$r(z)=\frac{1}{1-\alpha}\left(\frac{z\big(H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_cf(z)\big)'}{H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_cf(z)}-\alpha\right)\qquad(z\in\mathbb{U}), \tag{2.8}$$

where $r(z)$ is analytic in $\mathbb{U}$, with $r(0)=1$ and $r(z)\neq 0$ for $z\in\mathbb{U}$. It can easily be verified from (1.9) and (2.6) that

$$z\big(H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_cf(z)\big)'=(c+1)H_{\lambda,q,s}(\alpha_1)f(z)-c\,H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_cf(z). \tag{2.9}$$

Thus, by using (2.9) in (2.8), we find that

$$\frac{z\big(H_{\lambda,q,s}(\alpha_1)f(z)\big)'}{H_{\lambda,q,s}(\alpha_1)f(z)}-\alpha=(1-\alpha)r(z)+\frac{(1-\alpha)zr'(z)}{c+\alpha+(1-\alpha)r(z)}.$$

The remaining part of the proof of the Theorem 3 is much akin to that of Theorem 1. Therefore, we choose to omit the analogous details involved.

From Theorem 3, we easily see the following result.

Theorem 4. *Under the parametric constraints stated with Theorem 3, let*

$$f(z)\in\mathcal{A}\qquad\textit{and}\qquad\frac{\Big(z\big(H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_cf(z)\big)'\Big)'}{\big(H_{\lambda,q,s}(\alpha_1)\,\mathcal{I}_cf(z)\big)'}\neq\alpha\qquad(z\in\mathbb{U}).$$

Then

$$f(z)\in\mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta)\implies\mathcal{I}_cf(z)\in\mathcal{K}_{\lambda,\alpha_1}(q,s,\alpha,\beta). \tag{2.10}$$

Remark. On setting

$$q=2,\ \ s=1,\ \ \lambda=1,\ \ \alpha_1=2-\nu,\ \ \alpha_2=2+\mu+\eta\ \text{ and }\ \beta_1=2-\nu+\eta \tag{2.11}$$

in (1.7), and restricting the parameters as

$$\mu>0\quad\text{and}\quad\min\{\mu+\eta,-\nu+\eta,-\nu\}>-2,$$

the linear operator $H_{\lambda,q,s}(\alpha_1)$ reduces to the following fractional integral operator $J_{0,z}^{\mu,\nu,\eta}$ due to Srivastava *et al.* [16](see also [14]):

$$J_{0,z}^{\mu,\nu,\eta} f(z) = z+\sum_{n=2}^{\infty} \frac{(2)_{n-1}\,(2-\nu+\eta)_{n-1}}{(2-\nu)_{n-1}\,(2+\mu+\eta)_{n-1}}\; a_n\, z^n.$$

$$= \frac{\Gamma(2-\nu)\,\Gamma(2+\mu+\eta)\,z^{-\mu}}{\Gamma(2-\nu+\eta)\,\Gamma(\mu)} \int_0^z (z-t)^{\mu-1}\,{}_2F_1\left(\mu+\nu,-\eta;\;\mu;\;1-\frac{t}{z}\right) f(t)\,dt$$

$$(z \in \mathbb{U};\; f \in \mathcal{A}). \qquad (2.12)$$

Therefore for the parametric substitution given by (2.11), Theorems 1 to 4 would yield the corresponding known results due to Prajapat *et al.* [14].

On the other hand if we set

$$q=2,\quad s=1,\quad \lambda=0,\quad \alpha_1=1,\quad \alpha_2=\nu+\mu+1 \;\text{ and }\; \beta_1=\nu+1 \qquad (2.13)$$

in (1.7), and restricting the parameters as $\mu > 0$ and $\nu > -1$, we obtain the multiplier transformation operator Ω_ν^μ, which was introduced and studied by Jung *et al.* [8], as follows

$$\Omega_\nu^\mu = z+\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\sum_{n=2}^{\infty}\frac{\Gamma(n+\nu)}{\Gamma(n+\mu+\nu)}\; a_n z^n$$

$$= \binom{\mu+\nu}{\nu}\frac{\mu}{z^\nu}\int_0^z t^{\nu-1}\left(1-\frac{t}{z}\right)^{\mu-1} f(t)\,dt \qquad (\mu>0;\;\nu>-1;\; f\in\mathcal{A}). \qquad (2.14)$$

Therefore setting the parametric substitution given by (2.13) in Theorems 1 to 4, we get results obtained by Liu [10]. Furthermore taking parametric substitution given by (2.13) alongwith $\beta = 1$ in first part of Theorem 1 and 2, we get results given by Gao et al. [6, p. 1790, Theorem 1; p. 1791, Theorem 2].

3. Corollaries and Consequences

In this concluding section, we consider some corollaries and consequences of our main results (Theorems 1 to 4) established in Section 2.

First of all, on setting

$$q = 1, \quad s = 0, \quad \alpha_1 = 1 \quad \text{and} \quad \lambda = 0,$$

Theorem 1 would yield the following result.

Corollary 1. *If*

$$f(z) \in \mathcal{A} \quad \textit{and} \quad f(z) \neq \alpha\, \mathcal{I}_0[f](z) \qquad (z \in \mathbb{U};\ 0 \leq \alpha < 1),$$

then

$$\{f:\ f(z) \in \mathcal{K}_c(\alpha, \beta)\} \ \subset\ \{f:\ f(z) \in \mathcal{S}_s^*(\alpha, \beta)\} \ \subset\ \{f:\ \mathcal{I}_0[f](z) \in \mathcal{S}_s^*(\alpha, \beta)\}. \tag{3.1}$$

Next if we set

$$q = 1, \quad s = 0, \quad \alpha_1 = 1 \quad \text{and} \quad \lambda = 1$$

in Theorem 1 we get Corollary 2 below.

Corollary 2. *If*

$$f(z) \in \mathcal{A}, \quad (1-\alpha)zf'(z) + z^2 f''(z) \neq 0 \qquad (z \in \mathbb{U};\ 0 \leq \alpha < 1),$$

and

$$(1-\alpha)zf'(z) + \left(2 - \frac{\alpha}{2}\right) z^2 f''(z) + \frac{1}{2} z^3 f'''(z) \neq 0 \qquad (z \in \mathbb{U};\ 0 \leq \alpha < 1),$$

then

$$\left\{f:\ \frac{1}{2}\left(z^2 f'(z)\right)' \in \mathcal{S}_s^*(\alpha, \beta)\right\} \ \subset\ \{f:\ f(z) \in \mathcal{K}_c(\alpha, \beta)\} \ \subset\ \{f:\ f(z) \in \mathcal{S}_s^*(\alpha, \beta)\}. \tag{3.2}$$

By, setting

$$q = 1, \quad s = 0, \quad \alpha_1 = 1 \quad \text{and} \quad \lambda = 0,$$

in Theorem 2, we arrive at Corollary 3 below.

Corollary 3. *If*

$$f(z) \in \mathcal{A}, \quad f(z) \neq \alpha\, \mathcal{I}_0[f](z) \qquad (z \in \mathbb{U};\ 0 \leq \alpha < 1),$$

and

$$(1-\alpha)f'(z)+(3-\alpha)\,zf''(z)+z^2f'''(z)\neq 0 \qquad (z\in\mathbb{U};\ 0\leq\alpha<1),$$

then

$$\{f:\ zf'(z)\in\mathcal{K}_c(\alpha,\beta)\}\ \subset\ \{f:\ f(z)\in\mathcal{K}_c(\alpha,\beta)\}\ \subset\ \{f:\ \mathcal{I}_0[f](z)\in\mathcal{K}_c(\alpha,\beta)\}\,. \tag{3.3}$$

Upon setting

$$q=1,\quad s=0,\quad \alpha_1=2\quad\text{and}\quad \lambda=0,$$

Theorem 3 would yield the following result.

Corollary 4. *If*

$$f(z)\in\mathcal{A}\qquad \textit{and}\qquad z\Big(\mathcal{I}_0\big(\mathcal{I}_0[f](z)\big)\Big)'\neq\alpha\,\mathcal{I}_0\big(\mathcal{I}_0[f](z)\big)\qquad (z\in\mathbb{U}), \tag{3.4}$$

then

$$\mathcal{I}_0[f](z)\in\mathcal{S}_s^*(\alpha,\beta)\implies \mathcal{I}_0\big(\mathcal{I}_0[f](z)\big)\in\mathcal{S}_s^*(\alpha,\beta). \tag{3.5}$$

Numerous other applications and consequences of our main results (Theorem 1 to 4) can indeed be derived similarly.

References

[1] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135**(1969), 429-446.

[2] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* **15(4)**(1984), 737-745.

[3] J. H. Choi, M. Saigo, and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, *J. Math. Anal. Appl.* **276(1)**(2002), 432-445.

[4] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric functions, *Appl. Math. Comput.* **103**(1999), 1-13.

[5] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric functions, *Integral Transform Spec. Funct.* **14**(2003), 7-18.

[6] C.-Y. Gao, S.-M. Yuan, and H. M. Srivastava, Some functional inequalities and inclusion relationships associated with certain families of integral operators, *Comput. Math. Appl.* **49**(2005), 1787–1795.

[7] Yu. E. Hohlov, Operators and operations in the class of univalent functions, *Izv. Vyssh. Uchebn. Zaved. Mat.* **10**(1978), 83-89.

[8] I. B. Jung, Y. C. Kim, and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.* **176**(1993), 138–147.

[9] Oh S. Kwon and N. E. Cho, Inclusion properties for certain subclasses of analytic functions associated with the Dziok-Srivastava operator, *J. Inequal. Appl.* **Volume 2007**, Article ID 51079, 10.

[10] J.-L. Liu, Certain integral operator and strongly starlike functions, *Int. J. Math. Math. Sci.* **30**(2002), 569–574.

[11] K. I. Noor, On new classes of integral operators, *J. Natur. Geom.* **16(1-2)** (1999), 71-80.

[12] K. I. Noor and M. A. Noor, On integral operators, *J. Math. Anal. Appl.* **238(2)** (1999), 341-352.

[13] M. Nunokawa, On properties of non-Carathéodory functions, *Proc. Japan Acad. Ser. A Math. Sci.* **68**(1992), 152–153.

[14] J. K. Prajapat, R. K. Raina, and H. M. Srivastava, Some inclusion properties for certain subclasses of strongly starlike and strongly convex functions involving a family of fractional integral operators, *Integral Transform Spec. Funct.* **18(9)**(2007), 639-651.

[15] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49**(1975), 109-115.

[16] H. M. Srivastava, M. Saigo, and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.* **131** (1988), 412-420.