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Coefficient Bounds for Certain Subclasses of Functions Analytic with Respect to Symmetric Conjugate Points *

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Abstract

We introduce several new subclasses of functions analytic with respect to symmetric conjugate points, which are defined by using subordination for analytic functions ϕ and ψ on the unit disk. The object of this paper is to investigate some structural formulas and various interesting properties of Hadamard product in these subclasses. Furthermore, the upper bounds on the coefficient functional $|a_3 - \mu a_2^2|$ for functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in these subclasses are also obtained.

Keywords and Phrases: *Starlike functions, Close-to-convex functions, Subordination, Hadamard product (or convolution), Linear operators.*

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathcal{S}$, $\mathcal{C}$, $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions which are *univalent*, *close-to-convex*, *starlike of order* γ, and *convex of order* γ in $\mathbb{U}$ (see, e.g., [9]). In particular, the classes

$$\mathcal{S}^*(0) = \mathcal{S}^* \qquad \text{and} \qquad \mathcal{K}(0) = \mathcal{K}$$

are the familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

Given two functions f and g, which are analytic in $\mathbb{U}$ with $f(0) = g(0)$, the function f is said to be *subordinate* to g in $\mathbb{U}$ if there exists a function w, analytic in $\mathbb{U}$ such that

$$w(0) = 0, \quad |w(z)| < 1 \ (z \in \mathbb{U}) \quad \text{and} \quad f(z) = g(w(z)) \ (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \quad \text{in } \mathbb{U}.$$

We also observe that

$$f(z) \prec g(z) \quad \text{in } \mathbb{U}$$

if and only if

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U})$$

whenever g is univalent in $\mathbb{U}$.

Let $\mathcal{M}$ be the class of analytic functions $\phi(z)$ in $\mathbb{U}$ normalized by $\phi(0) = 1$, and let $\mathcal{N}$ be the subclass of $\mathcal{M}$ consisting of those functions ϕ which are univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex and $\text{Re}\ \phi(z) > 0$ $(z \in \mathbb{U})$.

Making use of the principle of subordination between analytic functions, Ma and Minda [6] and Kim *et al.* [5] investigate the subclasses $\mathfrak{S}^*(\phi)$, $\mathfrak{K}(\phi)$, and $\mathfrak{C}(\phi,\psi)$ of the class $\mathcal{A}$ for ϕ, $\psi\in\mathcal{N}$, which are defined by

$$\mathfrak{S}^*(\phi) := \left\{ f\in\mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \quad \text{in } \mathbb{U} \right\},$$

$$\mathfrak{K}(\phi) := \left\{ f\in\mathcal{A} : 1+\frac{zf''(z)}{f'(z)} \prec \phi(z) \quad \text{in } \mathbb{U} \right\},$$

and

$$\mathfrak{C}(\phi,\psi) := \left\{ f\in\mathcal{A} : \exists\, g\in\mathfrak{K}(\phi) \text{ s.t. } \frac{f'(z)}{g'(z)} \prec \psi(z) \quad \text{in } \mathbb{U} \right\}.$$

Obviously, for special choices for the functions ϕ and ψ involved in these definitions, we have the following relationships:

$$\mathfrak{S}^*\left(\frac{1+z}{1-z}\right) = \mathcal{S}^*, \qquad \mathfrak{K}\left(\frac{1+z}{1-z}\right) = \mathcal{K},$$

$$\mathfrak{C}\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\right) \subset \mathcal{C}, \tag{1.2}$$

and

$$\mathfrak{S}^*\left(\frac{1+(1-2\gamma)z}{1-z}\right) = \mathcal{S}^*(\gamma) \qquad (0 \leqq \gamma < 1).$$

Since

$$f(z)\in\mathfrak{K}(\phi) \Longleftrightarrow zf'\in\mathfrak{S}^*(\phi),$$

we also have

$$f\in\mathfrak{C}(\phi,\psi) \Longleftrightarrow \exists\, h\in\mathfrak{S}^*(\phi) \text{ s.t } \frac{zf'(z)}{h(z)} \prec \psi(z) \quad \text{in } \mathbb{U}. \tag{1.3}$$

For the functions f and g given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined, as usual, by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = f(z) * g(z).$$

We now introduce two operators $\mathcal{D}$ and $\mathcal{T}$ as follows [2] (see also [10]):

(1) *The Operator* $\mathcal{T}$. For $f \in \mathcal{A}$, let

$$\mathcal{T}f(z) = \frac{1}{2}\left\{f(z) - \overline{f(-\overline{z})}\right\} = z + \sum_{n=2}^{\infty} \frac{1}{2}\left[a_n - (-1)^n \overline{a_n}\right] z^n. \tag{1.4}$$

(2) *The operator* $\mathcal{D}$. For $f \in \mathcal{A}$ and $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$, let

$$\mathcal{D}^0 f(z) = f(z), \qquad \mathcal{D}f(z) = zf'(z),$$

$$\mathcal{D}^{n+1} f(z) = \mathcal{D}(\mathcal{D}^n f(z)) \qquad (n \in \mathbb{N}).$$

It is easily verified that the operators $\mathcal{T}$ and $\mathcal{D}$ are well defined on $\mathcal{A}$ and have the following properties:

(I) $\mathcal{T}$ and $\mathcal{D}$ are linear operators on $\mathcal{A}$.
(II) $\mathcal{D}\mathcal{T} = \mathcal{T}\mathcal{D}$.
(III) $\mathcal{T}\mathcal{T} = \mathcal{T}$.

Next, by using the operators $\mathcal{T}$ and $\mathcal{D}$, we introduce the following new classes of analytic functions for $\alpha \geqq 0$ and $\phi, \psi \in \mathcal{N}$:

$$\mathfrak{S}^*_{sc}(\alpha, \phi) := \left\{f \in \mathcal{A} : \frac{\mathcal{D}\mathcal{D}_\alpha f(z)}{\mathcal{D}_\alpha \mathcal{T} f(z)} \prec \phi(z) \quad \text{in } \mathbb{U}\right\}, \tag{1.5}$$

and

$$\mathfrak{C}_{sc}(\alpha, \phi, \psi) = \left\{f \in \mathcal{A} : \exists\, h \in \mathfrak{S}^*_{sc}(\alpha, \phi) \text{ s.t. } \frac{\mathcal{D}\mathcal{D}_\alpha f(z)}{\mathcal{D}_\alpha \mathcal{T} h(z)} \prec \psi(z) \quad \text{in } \mathbb{U}\right\}, \tag{1.6}$$

where $\mathcal{D}_\alpha = \alpha\mathcal{D} + (1-\alpha)\mathcal{D}^0$.

We note that $\mathcal{D}_\alpha f = (1-\alpha)f + \alpha z f' \in \mathcal{A}$ for $f \in \mathcal{A}$ and $f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ $(\mathfrak{C}_{sc}(\alpha, \phi, \psi))$ is equivalent to $\mathcal{D}_\alpha f \in \mathfrak{S}^*_{sc}(0, \phi) \equiv \mathfrak{S}^*_{sc}(\phi)$ $(\mathfrak{C}_{sc}(0, \phi, \psi) \equiv \mathfrak{C}_{sc}(\phi, \psi))$. In particular, by taking

$$\alpha \geqq 0 \quad \text{and} \quad \phi(z) = \psi(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U})$$

in (1.5) and (1.6), we have the relationships

$$\mathfrak{S}^*_{sc}\left(\alpha,\ \frac{1+z}{1-z}\right) = \mathcal{S}^*_{sc}(\alpha) \tag{1.7}$$

and

$$\mathfrak{C}_{sc}\left(\alpha,\ \frac{1+z}{1-z},\ \frac{1+z}{1-z}\right) = \mathcal{C}_{sc}(\alpha),$$

which were studied by Chen *et al.* [2]. Furthermore, putting $\alpha = 0$ in (1.7), then $\mathfrak{S}^*_{sc}((1+z)/(1-z)) = \mathcal{S}^*_{sc}(0)$, which is the class of functions starlike with respect to symmetric conjugate points due to El-Ashwah and Thomas [4] (see also [2]).

In this paper, we investigate various convolution properties and some structural representations of functions in the classes $\mathfrak{S}^*_{sc}(\alpha,\phi)$ and $\mathfrak{C}_{sc}(\alpha,\phi,\psi)$. Moreover, the coefficient bounds of these classes are also considered.

2. Some results of the classes $\mathfrak{S}^*_{sc}(\alpha,\phi)$ and $\mathfrak{C}_{sc}(\alpha,\phi,\psi)$

We begin by recalling the following results.

Lemma 1. *Let $p_i(z) \prec \phi(z)$ $(\phi \in \mathcal{N}; i = 1,2)$ in $\mathbb{U}$ and α,β any positive real numbers. Then*

$$\frac{1}{\alpha+\beta}\left\{\alpha p_1(z) + \beta p_2(z)\right\} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Proof. This follows easily from the definition of $\mathcal{N}$.

Lemma 2. (Ruscheweyh and Sheil-Small [8])**.** *Suppose either $g \in \mathcal{K}$, $h \in \mathcal{S}^*$ or else $g, h \in \mathcal{S}^*(1/2)$. Then for any analytic function G in $\mathbb{U}$, we have*

$$\frac{(g * hG)(z)}{(g * h)(z)} \in \overline{co}G(\mathbb{U}) \qquad (z \in \mathbb{U}),$$

where $\overline{co}G(\mathbb{U})$ is the closed convex hull of $G(\mathbb{U})$.

Lemma 3. (Ma and Minda [6])**.** *Let $\phi \in \mathcal{N}$. For $g \in \mathcal{K}$ and $h \in \mathfrak{S}^*(\phi)$, we have $g * h \in \mathfrak{S}^*(\phi)$.*

Lemma 4. *If* $f \in \mathfrak{S}^*_{sc}(\phi)$ *for* $\phi \in \mathcal{N}$, *then* $\mathcal{T}f \in \mathfrak{S}^*(\phi)$.

Proof. Let $f \in \mathfrak{S}^*_{sc}(\phi)$ for $\phi \in \mathcal{N}$. If we set

$$p(z) = \frac{\mathcal{D}f(z)}{\mathcal{T}f(z)} = \frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}},$$

then $p(z) \prec \phi(z)$ in $\mathbb{U}$ and

$$\overline{p(-\overline{z})} = \frac{\mathcal{D}\overline{(-f(-\overline{z}))}}{\mathcal{T}f(z)}.$$

Since $\phi \in \mathcal{N}$, we also have $\overline{p(-\overline{z})} \prec \phi(z)$ in $\mathbb{U}$. Applying the method of proof of the aforementioned result of Chen *et al.* [2, Lemma 2] with Lemma 1, we obtain

$$\frac{z\,(\mathcal{T}f(z))'}{\mathcal{T}f(z)} = \frac{1}{2}\left(p(z) + \overline{p(-\overline{z})}\right) \prec \phi(z) \qquad (z \in \mathbb{U}),$$

and hence the proof is complete.

By virtue of (1.3) and Lemma 4, we observe that if $f \in \mathfrak{S}^*_{sc}(\phi)$ for $\phi \in \mathcal{N}$, then $f \in \mathfrak{C}(\phi, \phi)$. Furthermore, it follows from (1.2) that $\mathfrak{S}^*_{sc}((1+z)/(1-z)) \subset \mathcal{C}$.

First, by using Lemma 4, we prove

Theorem 1. *Let* $\alpha \geqq 0$ *and* $\phi \in \mathcal{N}$. *If* $f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$, *then* $\mathcal{D}_\alpha \mathcal{T}f \in \mathfrak{S}^*(\phi)$ *and* $\mathcal{T}f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$.

Proof. Since $f(z) \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ if and only if $\mathcal{D}_\alpha f(z) \in \mathfrak{S}^*_{sc}(\phi)$, by applying $\mathcal{T}\mathcal{D} = \mathcal{D}\mathcal{T}$ and Lemma 4, we have

$$\mathcal{D}_\alpha \mathcal{T}f(z) = \mathcal{T}\mathcal{D}_\alpha f(z) \in \mathfrak{S}^*(\phi) \qquad (\phi \in \mathcal{N}; z \in \mathbb{U}). \tag{2.1}$$

Furthermore, from $\mathcal{T}\mathcal{T} = \mathcal{T}$ and (2.1) we obtain

$$\frac{\mathcal{D}(\mathcal{D}_\alpha(\mathcal{T}f(z)))}{\mathcal{D}_\alpha(\mathcal{T}(\mathcal{T}f(z)))} = \frac{\mathcal{D}(\mathcal{D}_\alpha(\mathcal{T}f(z)))}{\mathcal{D}_\alpha(\mathcal{T}f(z))} \prec \phi(z) \qquad (\phi \in \mathcal{N}; z \in \mathbb{U}).$$

Hence $\mathcal{T}f(z) \in \mathfrak{S}^*_{sc}(\alpha, \phi)$, which completes the proof of Theorem 1. □

Remark 1. In its special case when

$$\phi(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}),$$

Theorem 1 would reduce immediately to a known result due to Chen *et al.* [2, Theorem 1].

Theorem 2. *Let* $\alpha \geqq 0$.

(a) *Let* $\phi \in \mathcal{N}$. *If* $f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ *and* $g \in \mathcal{K}$ *with real coefficients, then* $g * f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$.

(b) *Let* $\phi \in \mathcal{N}$ *and* $\mathrm{Re}\ \phi(z) > 1/2$ $(z \in \mathbb{U})$. *If* $f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ *and* $g \in \mathcal{S}^*(1/2)$ *with real coefficients, then* $g * f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$.

Proof. First, we prove (a). Let $f(z) \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ and set

$$G(z) = \frac{\mathcal{D}\mathcal{D}_\alpha f(z)}{\mathcal{D}_\alpha \mathcal{T} f(z)} \qquad (\alpha \geqq 0; z \in \mathbb{U}). \tag{2.2}$$

Then $G(z) \prec \phi(z)$ in $\mathbb{U}$. Since $g(z) \in \mathcal{K}$ with real coefficients, we can easily verify that

$$\begin{aligned} \mathcal{D}_\alpha \mathcal{T}(g * f)(z) &= g(z) * \mathcal{D}_\alpha \mathcal{T} f(z) \\ \mathcal{D}\mathcal{D}_\alpha (g * f)(z) &= g(z) * \mathcal{D}\mathcal{D}_\alpha f(z). \end{aligned} \tag{2.3}$$

Making use of (2.2) and (2.3), we get

$$\frac{\mathcal{D}\mathcal{D}_\alpha (g * f)(z)}{\mathcal{D}_\alpha \mathcal{T}(g * f)(z)} = \frac{g(z) * \mathcal{D}\mathcal{D}_\alpha f(z)}{g(z) * \mathcal{D}_\alpha \mathcal{T} f(z)} = \frac{g(z) * G(z)(\mathcal{D}_\alpha \mathcal{T} f(z))}{g(z) * \mathcal{D}_\alpha \mathcal{T} f(z)}. \tag{2.4}$$

By using $\phi(z) \in \mathcal{N}$ and Theorem 1, we observe that $\phi(\mathbb{U})$ is convex with $\mathrm{Re}\ \phi(z) > 0$ $(z \in \mathbb{U})$ and $\mathcal{D}_\alpha \mathcal{T} f(z) \in \mathcal{S}^*$. Therefore, using (2.4) and Lemma 2, we obtain

$$\frac{\mathcal{D}\mathcal{D}_\alpha (g * f)(z)}{\mathcal{D}_\alpha \mathcal{T}(g * f)(z)} \in \overline{co}G(\mathbb{U}) \subset \overline{\phi(\mathbb{U})}.$$

Since $\mathcal{D}\mathcal{D}_\alpha (g * f)(z)/\mathcal{D}_\alpha \mathcal{T}(g * f)(z)$ is analytic in $\mathbb{U}$, we have

$$\frac{\mathcal{D}\mathcal{D}_\alpha (g * f)(z)}{\mathcal{D}_\alpha \mathcal{T}(g * f)(z)} \in \phi(\mathbb{U}),$$

which yields that

$$\frac{\mathcal{D}\mathcal{D}_\alpha (g * f)(z)}{\mathcal{D}_\alpha \mathcal{T}(g * f)(z)} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Hence, $(g * f)(z) \in \mathfrak{S}^*_{sc}(\alpha, \phi)$. Assertion (b) can be shown similarly. □

Corollary 1. *Let $0 < \alpha \leqq 1$ and $\phi \in \mathcal{N}$. Then*

$$\mathfrak{S}^*_{sc}(\alpha, \phi) \subset \mathfrak{S}^*_{sc}(0, \phi) = \mathfrak{S}^*_{sc}(\phi).$$

Proof. Let $f(z) \in \mathfrak{S}^*_{sc}(\alpha, \phi)$, and let

$$K_\alpha(z) = z + \sum_{n=2}^{\infty} \frac{z^n}{1+(n-1)\alpha} = \frac{\gamma+1}{z^\gamma} \int_0^z \frac{t^\gamma}{1-t} dt \qquad (z \in \mathbb{U}), \tag{2.5}$$

where $\gamma = 1/\alpha - 1 \geqq 0$. Then, by virtue of $\phi(z) \in \mathcal{N}$ and Theorem 1, we have $\mathcal{D}_\alpha \mathcal{T} f(z) \in \mathcal{S}^*$, and it is well known that $K_\alpha(z) \in \mathcal{K}$ with real coefficients (see [1]). Since $\mathcal{D}_\alpha(K_\alpha * f)(z) = f(z)$ and

$$\frac{\mathcal{D}f(z)}{\mathcal{T}f(z)} = \frac{K_\alpha(z) * (\mathcal{D}\mathcal{D}_\alpha f(z)/\mathcal{D}_\alpha \mathcal{T} f(z))(\mathcal{D}_\alpha \mathcal{T} f(z))}{K_\alpha(z) * \mathcal{D}_\alpha \mathcal{T} f(z)},$$

by applying Lemma 2, we obtain

$$\frac{\mathcal{D}f(z)}{\mathcal{T}f(z)} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Hence $f(z) \in \mathfrak{S}^*_{sc}(\phi)$, which evidently proves Corollary 1. □

Theorem 3. *Let $\alpha \geqq 0$ and $\phi \in \mathcal{N}$. A function $f \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ if and only if there exist a function $p(z) \prec \phi(z)$ in $\mathbb{U}$ and a function $F \in \mathfrak{S}^*(\phi)$ with real coefficients satisfying*

$$\frac{zF'(z)}{F(z)} = \frac{1}{2}\left(p(iz) + \overline{p(i\overline{z})}\right) \qquad (z \in \mathbb{U})$$

such that

$$f'(z) = \begin{cases} i\dfrac{p(z)F(-iz)}{z} & (\alpha = 0) \\[2ex] i\dfrac{\gamma+1}{z^{\gamma+1}} \displaystyle\int_0^z p(t)F(-it)t^{\gamma-1} dt & (\alpha > 0), \end{cases}$$

where $\gamma = 1/\alpha - 1 > -1$.

Proof. By using Lemmas 3, 4 and Theorem 1, the proof of Theorem 3 is similar to the corresponding results obtained by Chen *et al.* [2]. The details may be omitted. □

Next, by using arguments similar to those above with (1.6), we can prove the following results for the class $\mathfrak{C}_{sc}(\alpha,\phi,\psi)$.

Theorem 4. *Let $\phi,\psi\in\mathcal{N}$.*

(a) *If $f\in\mathfrak{C}_{sc}(\phi,\psi)$, then $\mathcal{T}f\in\mathfrak{C}(\phi,\psi)$.*

(b) *If $f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$ for $\alpha\geqq 0$, then $\mathcal{D}_\alpha\mathcal{T}f\in\mathfrak{C}(\phi,\psi)$ and $\mathcal{T}f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$.*

Remark 2. If we take

$$\phi(z)=\psi(z)=\frac{1+z}{1-z}\qquad(z\in\mathbb{U})$$

in Theorem 4 (b), we infer the result due to Chen *et al.* [2, Theorem 7].

Theorem 5. *Let $\alpha\geqq 0$.*

(a) *Let $\phi,\psi\in\mathcal{N}$. If $f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$ and $g\in\mathcal{K}$ with real coefficients, then $g*f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$.*

(b) *Let $\phi,\psi\in\mathcal{N}$ and* Re $\phi(z)>1/2$ $(z\in\mathbb{U})$. *If $f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$ and $g\in\mathcal{S}^*(1/2)$ with real coefficients, then $g*f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$.*

Corollary 2. *Let $0<\alpha\leqq 1$ and $\phi,\psi\in\mathcal{N}$. Then $\mathfrak{C}_{sc}(\alpha,\phi,\psi)\subset\mathfrak{C}_{sc}(\phi,\psi)$.*

Theorem 6. *Let $\alpha\geqq 0$ and $\phi,\psi\in\mathcal{N}$. A function $f\in\mathfrak{C}_{sc}(\alpha,\phi,\psi)$ if and only if there exist a function $p(z)\prec\psi(z)$ in $\mathbb{U}$ and a function $F\in\mathfrak{S}^*(\phi)$ with real coefficients such that*

$$f'(z)=\begin{cases} i\dfrac{p(z)F(-iz)}{z} & (\alpha=0)\\[2ex] i\dfrac{\gamma+1}{z^{\gamma+1}}\displaystyle\int_0^z p(t)F(-it)t^{\gamma-1}dt & (\alpha>0),\end{cases}$$

where $\gamma=1/\alpha-1>-1$.

3. Coefficient bounds for the classes $\mathfrak{S}^*_{sc}(\alpha,\phi)$ and $\mathfrak{C}_{sc}(\alpha,\phi,\psi)$

In order to prove the coefficient bounds for the classes $\mathfrak{S}^*_{sc}(\alpha,\phi)$ and $\mathfrak{C}_{sc}(\alpha,\phi,\psi)$, we now recall the following lemma due to Kim *et al.* [5].

Lemma 5. *Assume that* $\eta(z) = e_1 + e_2 z + \cdots$ *is analytic in* $\mathbb{U}$ *with* $|\eta(z)| \leqq 1$. *Then* $|e_1|^2 + |e_2| \leq 1$.

Lemma 6. *Let* $\lambda \in \mathbb{R}$ *and* $\phi(z) = 1 + A_1 z + A_2 z^2 + \cdots \in \mathcal{N}$ *with* $\phi'(0) > 0$. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathfrak{S}^*(\phi)$, *then* $|a_3 - \lambda a_2^2| \leqq K(\lambda, A_1, A_2)$, *where*

$$K(\lambda, A_1, A_2) = \begin{cases} (A_2 - 2\lambda A_1^2 + A_1^2)/2 & if\ 2\lambda A_1^2 \leqq A_2 + A_1^2 - A_1 \\ A_1/2 & if\ A_2 + A_1^2 - A_1 \leqq 2\lambda A_1^2 \leqq A_2 + A_1^2 + A_1 \\ (2\lambda A_1^2 - A_1^2 - A_2)/2 & if\ A_2 + A_1^2 + A_1 \leqq 2\lambda A_1^2. \end{cases}$$

Proof. Set

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}).$$

Then, by using same argument of [6, Theorem 3], we can easily verify Lemma 6, and so we omit it. □

By applying Lemmas 5 and 6, we derive

Theorem 7. *Let* $\alpha \geqq 0$ *and* $\mu \in \mathbb{R}$. *Suppose that* $\phi(z) = 1 + A_1 z + A_2 z^2 + \cdots \in \mathcal{N}$ *and* $\phi'(0) > 0$. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathfrak{S}^*_{sc}(\alpha, \phi)$, *then*

$$|a_3 - \mu a_2^2| \leqq M(\alpha, \mu, A_1, A_2) + N(\alpha, \mu, A_1, A_2),$$

where

$$M(\alpha, \mu, A_1, A_2) = \begin{cases} \dfrac{1}{6(1+2\alpha)}\left(A_2 - \dfrac{3(1+2\alpha)}{2(1+\alpha)^2}\mu A_1^2 + A_1^2\right) \\ \qquad if\ 3(1+2\alpha)\mu A_1^2 \leqq 2(1+\alpha)^2(A_2 + A_1^2 - A_1) \\ \dfrac{A_1}{6(1+2\alpha)} \\ \qquad if\ A_2 + A_1^2 - A_1 \leqq \dfrac{3(1+2\alpha)}{2(1+\alpha)^2}\mu A_1^2 \leqq A_2 + A_1^2 + A_1 \\ \dfrac{1}{6(1+2\alpha)}\left(\dfrac{3(1+2\alpha)}{2(1+\alpha)^2}\mu A_1^2 - A_2 - A_1^2\right) \\ \qquad if\ 2(1+\alpha)^2(A_2 + A_1^2 + A_1) \leqq 3(1+2\alpha)\mu A_1^2 \end{cases} \tag{3.1}$$

and

$$N(\alpha,\mu,A_1,A_2)$$

$$= \begin{cases} \dfrac{1}{3(1+2\alpha)}\left\{\left|A_2-\dfrac{3(1+2\alpha)\mu}{4(1+\alpha)^2}A_1^2\right|+A_1^2\left|1-\dfrac{3(1+2\alpha)\mu}{2(1+\alpha)^2}\right|\right\} \\ if\ A_1^2\left|1-\dfrac{3(1+2\alpha)\mu}{2(1+\alpha)^2}\right| \geqq 2\left(A_1-\left|A_2-\dfrac{3(1+2\alpha)\mu}{4(1+\alpha)^2}A_1^2\right|\right) \\ \dfrac{A_1}{3(1+2\alpha)}\left\{1+\dfrac{A_1^3\left(1-\frac{3(1+2\alpha)\mu}{2(1+\alpha)}\right)^2}{4\left(A_1-\left|A_2-\frac{3(1+2\alpha)\mu}{4(1+\alpha)^2}A_1^2\right|\right)}\right\} \qquad otherwise. \end{cases} \tag{3.2}$$

Proof. Let $f(z) \in \mathfrak{S}^*_{sc}(\alpha,\phi)$, and set

$$p(z) = \frac{\mathcal{D}\mathcal{D}_\alpha f(z)}{\mathcal{D}_\alpha \mathcal{T} f(z)} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}). \tag{3.3}$$

Then, from (1.5) we see that

$$p(z) = \phi(w(z)) \qquad (z \in \mathbb{U}), \tag{3.4}$$

where w is an analytic in $\mathbb{U}$ such that $w(0) = 0$ and $|w(z)| \leqq |z|$ for $z \in \mathbb{U}$. Since $\mathcal{D}_\alpha f(z) = z + \sum_{n=2}^{\infty}(1+(n-1)\alpha)a_n z^n$, by virtue of (1.4) and (3.3), $\mathcal{D}\mathcal{D}_\alpha f(z) = p(z)\mathcal{D}_\alpha \mathcal{T} f(z)$ and simple calculations show that

$$a_2 = \frac{c_1}{2(1+\alpha)} + \frac{1}{4}(a_2 - \overline{a_2})$$

and

$$a_3 = \frac{c_2}{3(1+2\alpha)} + \frac{c_1(1+\alpha)(a_2-\overline{a_2})}{6(1+2\alpha)} + \frac{1}{6}(a_3+\overline{a_3}).$$

Therefore, we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{6}(a_3+\overline{a_3}) - \frac{\mu}{16}(a_2-\overline{a_2})^2 + \frac{1}{3(1+2\alpha)}\left(c_2 - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2}c_1^2\right) \\ &\quad + \left(\frac{1+\alpha}{6(1+2\alpha)} - \frac{\mu}{4(1+\alpha)}\right)c_1(a_2-\overline{a_2}). \end{aligned} \tag{3.5}$$

By applying Theorem 1 and Lemma 6 with $\lambda = 3(1+2\alpha)\mu/4(1+\alpha)^2$, it is easily observed that $\mathcal{D}_\alpha \mathcal{T} f(z) \in \mathfrak{S}^*(\phi)$, which yields

$$\left|\frac{1}{6}(a_3+\overline{a_3}) - \frac{\mu}{16}(a_2-\overline{a_2})^2\right| \leqq M(\alpha,\mu,A_1,A_2). \tag{3.6}$$

In view of (3.3) and (3.4), if we write $w(z) = e_1 z + e_2 z^2 + \cdots$, then $c_1 = A_1 e_1$ and $c_2 = A_1 e_2 + A_2 e_1^2$. Since

$$\frac{\mathcal{D}\mathcal{D}_\alpha \mathcal{T} f(z)}{\mathcal{D}_\alpha \mathcal{T} f(z)} \prec \phi(z) \qquad (z \in \mathbb{U})$$

by Theorem 1, Rogosinski's result [7] (see also [3] (p.192)) implies $|a_2 - \overline{a_2}| \leqq 2A_1/(1+\alpha)$. Thus we have

$$\begin{aligned}
&\left|\frac{1}{3(1+2\alpha)}\left(c_2 - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2}c_1^2\right) + \left(\frac{1+\alpha}{6(1+2\alpha)} - \frac{\mu}{4(1+\alpha)}\right)c_1(a_2-\overline{a_2})\right| \\
\leqq\ & \frac{1}{3(1+2\alpha)}\left\{A_1|e_2| + \left|A_2 - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2}A_1^2\right| |e_1|^2\right\} \\
&+ \left|\frac{1+\alpha}{6(1+2\alpha)} - \frac{\mu}{4(1+\alpha)}\right| |A_1 e_1||a_2 - \overline{a_2}| \\
\leqq\ & \frac{1}{3(1+2\alpha)}\left\{A_1|e_2| + \left|A_2 - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2}A_1^2\right| |e_1|^2\right\} \\
&+ \left|\frac{1}{3(1+2\alpha)} - \frac{\mu}{2(1+\alpha)^2}\right| A_1^2\ |e_1|.
\end{aligned}$$

Then, the same techniques as in the proof of [4, Theorem 3.1] show that

$$\begin{aligned}
&\left|\frac{1}{3(1+2\alpha)}\left(c_2 - \frac{3(1+2\alpha)\mu}{4(1+\alpha)^2}c_1^2\right) + \left(\frac{1+\alpha}{6(1+2\alpha)} - \frac{\mu}{4(1+\alpha)}\right)c_1(a_2-\overline{a_2})\right| \\
\leqq\ & N(\alpha,\mu,A_1,A_2).
\end{aligned} \tag{3.7}$$

Hence, making use of (3.6) and (3.7) in equality (3.5), we obtain

$$|a_3 - \mu a_2^2| \leqq M(\alpha,\mu,A_1,A_2) + N(\alpha,\mu,A_1,A_2),$$

which proves Theorem 7. □

Corollary 3. *Let* $\alpha \geqq 0$, $0 \leqq \gamma < 1/2$ *and* $\mu \in \mathbb{R}$. *If*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathfrak{S}^*_{sc}\left(\alpha, \frac{1+(1-2\gamma)z}{1-z}\right) \quad (z \in \mathbb{U}),$$

then

$$|a_3 - \mu a_2^2| \leqq \begin{cases} (1-\gamma)\left(\dfrac{3-2\gamma}{1+2\alpha} - \dfrac{4(1-\gamma)\mu}{(1+\alpha)^2}\right) \qquad if \quad \mu \leqq \dfrac{(1+\alpha)^2}{3(1+2\alpha)} \\[2ex] \dfrac{(1-\gamma)^2}{3(1+2\alpha)}\left(\dfrac{4(1+\alpha)^2(1-\gamma)}{3(1+2\alpha)\mu} + \dfrac{1+2\gamma}{1-\gamma} - 4\alpha + \dfrac{3\alpha(1+2\alpha)(2+\alpha)\mu}{(1+\alpha)^2}\right) \\[2ex] \qquad if \quad \dfrac{(1+\alpha)^2}{3(1+2\alpha)} \leqq \mu \leqq \dfrac{2(1+\alpha)^2}{3(1+2\alpha)} \\[2ex] \dfrac{(1-\gamma)^2}{3(1+2\alpha)}\left(\dfrac{4\gamma-1}{1-\gamma} - 4\alpha + \dfrac{4(1+\alpha)^2}{3(1+2\alpha)\mu} + 3(1+2\alpha)\mu\right) \\[2ex] \qquad if \quad \dfrac{2(1+\alpha)^2}{3(1+2\alpha)} \leqq \mu \leqq \dfrac{2(1+\alpha)^2}{3(1+2\alpha)(1-\gamma)} \\[2ex] \dfrac{1-\gamma}{1+2\alpha}\left(1 + \dfrac{(1-\gamma)^2\ [2(1+\alpha) - 3(1+2\alpha)\mu]^2}{12(1+\alpha)^2 - 9(1+2\alpha)(1-\gamma)\mu}\right) \\[2ex] \qquad if \quad \dfrac{2(1+\alpha)^2}{3(1+2\alpha)(1-\gamma)} \leqq \mu \leqq \dfrac{(1+\alpha)^2(3-2\gamma)}{3(1+2\alpha)(1-\gamma)} \\[2ex] \dfrac{1-\gamma}{3(1+2\alpha)}\left(4\gamma - 5 + \dfrac{9(1+2\alpha)(1-\gamma)\mu}{(1+\alpha)^2}\right) \\[2ex] \qquad if \quad \dfrac{(1+\alpha)^2(3-2\gamma)}{3(1+2\alpha)(1-\gamma)} \leqq \mu \leqq \dfrac{2(1+\alpha)^2(2-\gamma)}{3(1+2\alpha)(1-\gamma)} \\[2ex] \dfrac{1-\gamma}{1+2\alpha}\left(2\gamma - 3 + \dfrac{4(1+2\alpha)(1-\gamma)\mu}{(1+\alpha)^2}\right) \qquad if \quad \dfrac{2(1+\alpha)^2(2-\gamma)}{3(1+2\alpha)(1-\gamma)} \leqq \mu. \end{cases}$$

Proof. Setting

$$\phi(z) = \frac{1+(1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)\left(z + z^2 + \cdots\right) \qquad (0 \leqq \gamma < 1; z \in \mathbb{U})$$

in Theorem 7, we have Corollary 3. □

Remark 3. In view of (1.2), (1.3) and Lemma 4, we see that $\mathfrak{S}^*_{sc}((1+z)/(1-z)) \subset \mathcal{C}$. Hence, taking $\alpha = \gamma = 0$ in Corollary 3, we obtain a result due to Kim *et al.* [5, Corollary 3.2].

Finally, we prove

Theorem 8. *Let $\alpha \geqq 0$ and $\mu \in \mathbb{R}$. Suppose that $\phi(z) = 1 + A_1 z + A_2 z^2 + \cdots \in \mathcal{N}$ with $\phi'(0) > 0$ and $\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots \in \mathcal{N}$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathfrak{C}_{sc}(\alpha, \phi, \psi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq M\left(\alpha, \mu, A_1, A_2\right) + N\left(\alpha, \mu, B_1, B_2\right),$$

where $M(\alpha, \mu, A_1, A_2)$ and $N(\alpha, \mu, B_1, B_2)$ are given in (3.1) *and* (3.2).

Proof. Let $f(z) \in \mathfrak{C}_{sc}(\alpha, \phi, \psi)$. In view of (1.6), there exists a function $h(z) \in \mathfrak{S}^*_{sc}(\alpha, \phi)$ such that

$$\frac{\mathcal{D}\mathcal{D}_\alpha f(z)}{\mathcal{D}_\alpha \mathcal{T} h(z)} \prec \psi(z) \qquad (z \in \mathbb{U}).$$

We set $h(z) = z + d_2 z^2 + d_3 z^3 + \cdots$ and

$$p(z) = \frac{\mathcal{D}\mathcal{D}_\alpha f(z)}{\mathcal{D}_\alpha \mathcal{T} h(z)} = 1 + c_1 z + c_2 z^2 + \cdots = \psi(w(z)),$$

where w is analytic in $\mathbb{U}$ such that $|w(z)| \leqq |z|$ for $z \in \mathbb{U}$. Then, by simple calculations, we obtain

$$c_1 = (1+\alpha)\left(2a_2 - \frac{1}{2}(d_2 - \overline{d_2})\right)$$

and

$$c_2 = (1+2\alpha)\left(3a_3 - \frac{1}{2}(d_3 + \overline{d_3})\right) - \frac{(1+\alpha)^2}{2}(d_2 - \overline{d_2})\left(2a_2 - \frac{1}{2}(d_2 - \overline{d_2})\right),$$

so that $a_2 = c_1/2(1+\alpha) + (d_2 - \overline{d_2})/4$ and

$$a_3 = \frac{c_2}{3(1+2\alpha)} + \frac{(1+\alpha)c_1}{6(1+2\alpha)}(d_2 - \overline{d_2}) + \frac{1}{6}(d_3 + \overline{d_3}).$$

Thus we have

$$a_3 - \mu a_2^2 = \frac{1}{6}(d_3 + \overline{d_3}) - \frac{\mu}{16}(d_2 - \overline{d_2})^2 + \frac{c_2}{3(1+2\alpha)} - \frac{\mu}{4(1+\alpha)^2}c_1^2 + c_1(d_2 - \overline{d_2})\left(\frac{1+\alpha}{6(1+2\alpha)} - \frac{\mu}{4(1+\alpha)}\right).$$

Since $h(z) \in \mathfrak{S}_{sc}^*(\alpha, \phi)$, by applying Theorem 1, we observe that $\mathcal{D}_\alpha \mathcal{T} h(z) \in \mathfrak{S}^*(\phi)$, which implies

$$\left|\frac{1}{6}(d_3 + \overline{d_3}) - \frac{\mu}{16}(d_2 - \overline{d_2})^2\right| \leqq M\left(\alpha, \mu, A_1, A_2\right), \tag{3.8}$$

where $M(\alpha, \mu, A_1, A_2)$ is given by (3.1). Furthermore, by using similarly way of the proof of Theorem 7, we obtain

$$\left|\frac{c_2}{3(1+2\alpha)} - \frac{\mu}{4(1+\alpha)^2}c_1^2 + c_1(d_2 - \overline{d_2})\left(\frac{1+\alpha}{6(1+2\alpha)} - \frac{\mu}{4(1+\alpha)}\right)\right| \leqq N(\alpha, \mu, B_1, B_2), \tag{3.9}$$

where $N(\alpha, \mu, B_1, B_2)$ is given by (3.2). Hence, from (3.8) and (3.9) we have the desired result. □

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