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New Bilateral Generating Functions Pertaining to the H-Functions*

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Abstract

The aim of the present paper is to establish a bilateral generating relation, pertaining to the product of Fox H-function ([2]) and the multivariable H-function of Srivastava and Panda ([5]). By suitably specializing the various parameters involved, this formula would yield the corresponding bilateral (or bilinear) generating functions for a variety of simpler special functions.

Keywords and Phrases: *Fox H-function, Generalized Lauricella function, G-function of n-variables, H-function of several complex variables*

1. Introduction and definitions

Let $\Delta(s,\omega)$ and $\nabla(s,\omega)$ stand for the s-parameter sequences $\frac{\omega}{s}, \frac{\omega+1}{s}, ..., \frac{\omega+s-1}{s}$ and $1-\frac{\omega}{s}, 1-\frac{\omega+1}{s}, ..., 1-\frac{\omega+s-1}{s}$ respectively, for an arbitrary complex number ω and for all integers $s \geq 1$.

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The H-function introduced by Fox ([2], p.408) will be represented and defined as follows

$$H_{e,f}^{c,d}\left[x\,\middle|\,\begin{matrix}(a_j,A_j)_{1,e}\\(b_j,B_j)_{1,f}\end{matrix}\right]=\frac{1}{2\pi i}\int_L\theta(\xi)\,x^{\xi}\,d\xi \tag{1.1}$$

where

$$\theta(\xi)=\frac{\prod_{j=1}^{c}\Gamma(b_j-B_j\xi)\prod_{j=1}^{d}\Gamma(1-a_j+A_j\xi)}{\prod_{j=c+1}^{f}\Gamma(1-b_j+B_j\xi)\prod_{j=d+1}^{e}\Gamma(a_j-A_j\xi)} \tag{1.2}$$

where an empty product is interpreted as unity; $0\le c\le f$; $0\le d\le e$; $A_j\,(j=1,...,e)$ and $B_j\,(j=1,...,f)$ are positive numbers. L is a suitable contour of Barnes type such that the poles of $\Gamma\,(b_j-B_j\xi)\,\,(j=1,...,c)$ lie to the right of it and those of $\Gamma\,(1-a_j+A_j\xi)\,\,(j=1,...,d)$ lie to the left of it. Asymptotic expansions and analytic continuations of the H-function have been discussed by Braaksma ([1]). A detailed account of the H-function is available from the book of Srivastava et al. ([4]). The multivariable H-function introduced and studied by Srivastava and Panda ([5]), occurring in this paper will be defined and represented as follows ([4], pp.251-252, eqns. (C.1)-(C.3)).

$$H_{p,q:p_1,q_1;...;p_t,q_t}^{0,n:m_1,n_1;...;m_t,n_t}\left[y_1,...,y_t\,\middle|\,\begin{matrix}\left(c_j:\alpha_j^{(1)},...,\alpha_j^{(t)}\right)_{1,p} & : & \left(g_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1} & ;...; & \left(g_j^{(t)},\gamma_j^{(t)}\right)_{1,p_t}\\ \left(d_j:\beta_j^{(1)},...,\beta_j^{(t)}\right)_{1,q} & : & \left(h_j^{(1)},\delta_j^{(1)}\right)_{1,q_1} & ;...; & \left(h_j^{(t)},\delta_j^{(t)}\right)_{1,q_t}\end{matrix}\right]$$
$$=\frac{1}{(2\pi i)^t}\int_{L_1}...\int_{L_t}\phi_1(\xi_1)...\phi_t(\xi_t)\,\psi(\xi_1,...,\xi_t)\,y_1^{\xi_1}...y_t^{\xi_t}d\xi_1...d\xi_t, \tag{1.3}$$

where

$$\phi_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma\left(h_j^{(i)}-\delta_j^{(i)}\xi_i\right)\prod_{j=1}^{n_i}\Gamma\left(1-g_j^{(i)}+\gamma_j^{(i)}\xi_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1-h_j^{(i)}+\delta_j^{(i)}\xi_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(g_j^{(i)}-\gamma_j^{(i)}\xi_i\right)},\quad\forall i\in\{1,...,t\} \tag{1.4}$$

$$\psi(\xi_1,...,\xi_t)=\frac{\prod_{j=1}^{n}\Gamma\left(1-c_j+\sum_{i=1}^{t}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(c_j-\sum_{i=1}^{t}\alpha_j^{(i)}\xi_i\right)\prod_{j=1}^{q}\Gamma\left(1-d_j+\sum_{i=1}^{t}\beta_j^{(i)}\xi_i\right)} \tag{1.5}$$

All the Greek letters occurring on the left hand side of (1.3) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable H-function will, however, be meaningful even if some of these quantities are zero. For the convergence and existence conditions of the multivariable H-function, we refer to the book of Srivastava et al. ([4], pp.252-253, eqns. (C.4)-(C.8)). Throughout the paper it is assumed that this function satisfies the above cited conditions.

2. The Bilateral Generating Function

We prove the following formula which holds true whenever both of its sides have meaning.

$$\sum_{\omega=0}^{\infty} H_{e+s,f+s}^{c,d+s}\left[x \,\middle|\, \begin{matrix} (a_j,A_j)_{1,d},(\nabla(s,-\omega),1)_{1,s},(a_j,A_j)_{d+1,e} \\ (b_j,B_j)_{1,f},(\nabla(s,-\omega),0)_{1,s} \end{matrix}\right]$$
$$\times H_{p+s,q:p_1,q_1;\ldots;p_t,q_t}^{0,n+s:m_1,n_1;\ldots;m_t,n_t}\left[y_1,\ldots,y_t \,\middle|\, \begin{matrix} \left(c_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)}\right)_{1,n};(\nabla(s,\sigma+\omega):1,\ldots,1)_{1,s} \\ \left(d_j:\beta_j^{(1)},\ldots,\beta_j^{(t)}\right)_{1,q} \end{matrix}\right.$$
$$\left.\begin{matrix} ;\left(c_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)}\right)_{n+1,p}: & \left(g_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1} & ;\ldots; & \left(g_j^{(t)},\gamma_j^{(t)}\right)_{1,p_t} \\ -\quad : & \left(h_j^{(1)},\delta_j^{(1)}\right)_{1,q_1} & ;\ldots; & \left(h_j^{(t)},\delta_j^{(t)}\right)_{1,q_t} \end{matrix}\right]\frac{z^\omega}{\omega!}$$
$$=\left(1-\tfrac{z}{s}\right)^{-\sigma} H_{p+s,q:p_1,q_1;\ldots;p_t,q_t;e,f}^{0,n+s:m_1,n_1;\ldots;m_t,n_t;c,d}$$
$$\left[y_1\left(1-\tfrac{z}{s}\right)^{-s},\ldots,y_t\left(1-\tfrac{z}{s}\right)^{-s},x\left(1-\tfrac{s}{z}\right)^{-s} \,\middle|\, \begin{matrix} \left(c_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)}\right)_{1,n}; \\ \left(d_j:\beta_j^{(1)},\ldots,\beta_j^{(t)}\right)_{1,q} \end{matrix}\right.$$
$$\begin{matrix} (\nabla(s,\sigma):1,\ldots,1)_{1,s};\left(c_j:\alpha_j^{(1)},\ldots,\alpha_j^{(t)}\right)_{n+1,p} \\ - \end{matrix}$$
$$\left.\begin{matrix} : & \left(g_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1} & ;\ldots; & \left(g_j^{(t)},\gamma_j^{(t)}\right)_{1,p_t};(a_j,A_j)_{1,e} \\ : & \left(h_j^{(1)},\delta_j^{(1)}\right)_{1,q_1} & ;\ldots; & \left(h_j^{(t)},\delta_j^{(t)}\right)_{1,q_t};(b_j,B_j)_{1,f} \end{matrix}\right], \tag{2.1}$$

where σ is an arbitrary complex number, s an integer ≥ 1.

Proof. To obtain (2.1), express the multivariable H-function occurring in the left-hand side of (2.1) with the help of (1.3) and then interchange the order of summation and integration (which can easily be justified when the integral and the series involved are absolutely convergent), we find that left-hand side of (2.1)

$$= \frac{1}{(2\pi i)^t} \int_{L_1} ... \int_{L_t} \phi_1(\xi_1)...\phi_t(\xi_t)\ \psi(\xi_1,...,\xi_t) \prod_{j=1}^{s} \Gamma\left(\Delta(s,\sigma) + \sum_{i=1}^{t}\xi_i\right) \left\{ \sum_{\omega=0}^{\infty} \frac{\left(\sigma + s\sum_{i=1}^{t}\xi_i\right)_\omega}{\omega!} \right.$$
$$\left. \times H^{c,d+s}_{e+s,f+s}\left[x \left| \begin{matrix} (a_j,A_j)_{1,d}, (\nabla(s,-\omega),1)_{1,s}, (a_j,A_j)_{d+1,e} \\ (b_j,B_j)_{1,f}, (\nabla(s,-\omega),0)_{1,s} \end{matrix} \right.\right] \left(\frac{z}{s}\right)^\omega \right\} y_1^{\xi_1}...y_t^{\xi_t} d\xi_1...d\xi_t \tag{2.2}$$

now applying the following formula:

$$\sum_{\omega=0}^{\infty} \frac{(\sigma)_\omega}{\omega!} H^{c,d+s}_{e+s,f+s}\left[x \left| \begin{matrix} (a_j,A_j)_{1,d}, (\nabla(s,-\omega),1)_{1,s}, (a_j,A_j)_{d+1,e} \\ (b_j,B_j)_{1,f}, (\nabla(s,-\omega),0)_{1,s} \end{matrix} \right.\right] t^\omega$$
$$= (1-t)^{-\sigma} H^{c,d+s}_{e+s,f+s}\left[x\left(\frac{t}{t-1}\right)^s \left| \begin{matrix} (a_j,A_j)_{1,d}, (\nabla(s,\sigma),1)_{1,s}, (a_j,A_j)_{d+1,e} \\ (b_j,B_j)_{1,f}, (\nabla(s,\sigma),0)_{1,s} \end{matrix} \right.\right], \tag{2.3}$$

and then replace the H-function by its Mellin-Barnes contour integral, given by (1.1), and interpret the resulting Mellin-Barnes contour integral in terms of H-function of t-variables by means of (1.3). We are thus led to the second member of formula (2.1), and the final result follows by an appeal to the principle of analytic continuation.

3. Special Cases

On putting $n = p,\ m_i = 1,\ n_i = p_i$ and replacing q_i by $q_i + 1$ in the result (2.1), the H-function of several complex variables will reduce to the generalized Lauricella function of several complex variables ([3], p.454), we get the following result:

$$\sum_{\omega=0}^{\infty} \left\{ \prod_{j=1}^{s} \Gamma(\Delta(s,\sigma+\omega)) \right\} H^{c,d+s}_{e+s,f+s}\left[x \left| \begin{matrix} (a_j,A_j)_{1,d}, (\nabla(s,-\omega),1)_{1,s}, (a_j,A_j)_{d+1,e} \\ (b_j,B_j)_{1,f}, (\nabla(s,-\omega),0)_{1,s} \end{matrix} \right.\right]$$
$$\times F^{p+s:p_1,...,p_t}_{q:q_1,...,q_t}\left[-y_1,...,-y_t \left| \begin{matrix} \left(1-c_j:\alpha_j^{(1)},...,\alpha_j^{(t)}\right)_{1,p}; (\Delta(s,\sigma+\omega):1,...,1)_{1,s}; \\ \left(1-d_j:\beta_j^{(1)},...,\beta_j^{(t)}\right)_{1,q} \end{matrix} \right.\right.$$
$$\left. \begin{matrix} : \left(1-g_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1} ;...; \left(1-g_j^{(t)},\gamma_j^{(t)}\right)_{1,p_t} \\ : \left(1-h_j^{(1)},\delta_j^{(1)}\right)_{1,q_1} ;...; \left(1-h_j^{(t)},\delta_j^{(t)}\right)_{1,q_t} \end{matrix} \right] \frac{z^\omega}{\omega!}$$

$$= \left(1-\frac{z}{s}\right)^{-\sigma}\left\{\frac{\prod_{j=1}^{s}\Gamma(\Delta(s,\sigma))\prod_{j=1}^{e}\Gamma(1-a_j)}{\prod_{j=1}^{f}\Gamma(1-b_j)}\right\} F^{p+s:p_1,...,p_t,e}_{q:q_1,...,q_t,f}\left[-y_1\left(1-\frac{z}{s}\right)^{-s},...,-y_t\left(1-\frac{z}{s}\right)^{-s},\right.$$

$$-x\left(1-\frac{s}{z}\right)^{-s}\left|\begin{array}{c}\left(1-c_j:\alpha_j^{(1)},...,\alpha_j^{(t)}\right)_{1,p};\left(\Delta(s,\sigma):1,...,1\right)_{1,s}\\ \left(1-d_j:\beta_j^{(1)},...,\beta_j^{(t)}\right)_{1,q}\end{array}\right.$$

$$\left.\begin{array}{l}:\left(1-g_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};...;\left(1-g_j^{(t)},\gamma_j^{(t)}\right)_{1,p_t};\left(1-a_j,A_j\right)_{1,e}\\ :\left(1-h_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};...;\left(1-h_j^{(t)},\delta_j^{(t)}\right)_{1,q_t};\left(1-b_j,B_j\right)_{1,f}\end{array}\right], \tag{3.1}$$

where σ is an arbitrary complex number, s an integer ≥ 1 .
Again on taking $A_t = B_v = \alpha_j^{(i)} = \beta_k^{(i)} = \gamma_l^{(i)} = \delta_r^{(i)} = 1$ $v = 1,...,f;, (t = 1,...,e;$ in (2.1), we arrive at the following result:

$$\sum_{\omega=0}^{\infty}\left\{\prod_{j=1}^{s}\Gamma\left(\Delta\left(s,-\omega\right)\right)\right\}^{-1} G^{c,d+s}_{e+s,f+s}\left[x\left|\begin{array}{c}(a_j)_{1,d},(\nabla(s,-\omega))_{1,s},(a_j)_{d+1,e}\\(b_j)_{1,f}\end{array}\right.\right]$$

$$\times G^{0,n+s:m_1,n_1;...;m_t,n_t}_{p+s,q:p_1,q_1;...;p_t,q_t}\left[y_1,...,y_t\left|\begin{array}{c}(c_j)_{1,n};(\nabla(s,\sigma+\omega))_{1,s};(c_j)_{n+1,p}\\(d_j)_{1,q}\end{array}\right.\right.$$

$$\left.\begin{array}{l}:\left(g_j^{(1)}\right)_{1,p_1};...;\left(g_j^{(t)}\right)_{1,p_t}\\ :\left(h_j^{(1)}\right)_{1,q_1};...;\left(h_j^{(t)}\right)_{1,q_t}\end{array}\right]\frac{z^{\omega}}{\omega!} \tag{3.2}$$

$$= \left(1-\frac{z}{s}\right)^{-\sigma} G^{0,n+s:m_1,n_1;...;m_t,n_t;c,d}_{p+s,q:p_1,q_1;...;p_t,q_t;e,f}\left[y_1\left(1-\frac{z}{s}\right)^{-s},...,y_t\left(1-\frac{z}{s}\right)^{-s},x\left(1-\frac{s}{z}\right)^{-s}\left|\begin{array}{c}(c_j)_{1,n};\\(d_j)_{1,q}\end{array}\right.\right.$$

$$\left.\begin{array}{c}(\nabla(s,\sigma))_{1,s};(c_j)_{n+1,p}\\ -\end{array}\begin{array}{l}:\left(g_j^{(1)}\right)_{1,p_1};...;\left(g_j^{(t)}\right)_{1,p_t};(a_j)_{1,e}\\ :\left(h_j^{(1)}\right)_{1,q_1};...;\left(h_j^{(t)}\right)_{1,q_t};(b_j)_{1,f}\end{array}\right],$$

where σ is an arbitrary complex number, s an integer ≥ 1 .
If we set $n = p = q = 0,\ c = 1,\ d = e,\ A_i = B_j = 1\ (i = 1,...,e;\ j = 1,...,f),\ f \to f+1,\ ,x \to -x,\ a_j \to 1-a_j\ (j = 1,...,e), b_j \to 1-b_j\ (j = 1,...,f)$;(2.1) reduces to a known result obtained by Srivastava and Panda ([5], p.273, eq.(4.10)).

Further on taking in (2.1), we arrive at the result of Srivastava and Panda ([5], p.267, eq.(2.1)).

It is interesting to note that on giving suitable values to parameters in the result (2.1), we get the known results obtained by Srivastava and Panda ([5], eqns. (3.4) and (3.5) on p.269; eqns. (3.7), (3.8) and (3.11) on p.270; eqns. (3.13) and (3.15) on p.271).

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