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New Estimates on Certain Fundamental Finite Difference Inequalities*

B. G. Pachpatte†

57 Shri Niketan Colony, Near Abhinay Talkies,

Aurangabad 431 001 (Maharashtra) India

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Abstract

In the present note we establish new estimates on some fundamental finite difference inequalities which can be used as effective tools in the study of a class of sum-difference equations. Some applications are also given to convey the importance of one of our result.

Keywords and Phrases: *New estimates, Finite difference inequalities, Sum-difference equations, Variant of the inequality, Explicit estimates, Lipschitz type condition.*

1. Introduction

In [7,8] explicit bounds on a number of new finite difference inequalities of the form

$$u(m,n) \le f(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} k(m,n,s,t)\, u(s,t), \tag{1.1}$$

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†E-mail: bgpachpatte@gmail.com

are considered and used in various applications. It is easy to observe that these bounds are not directly applicable to studying sum-difference equations of the form

$$u(x,n) = h(x,n) + \sum_{s=0}^{n-1} \sum_{G} F(x,n,y,s,u(y,s)), \tag{1.2}$$

where G is a bounded subset of R^n. The main aim of the present note is to establish new estimates on certain fundamental finite difference inequalities which can be used as tools in handling the equations of the form (1.2). Some applications are also given to illustrate the usefulness of one of our result. The origin of the results presented here can be traced back to the fact that the integral analogue of equation (1.2) occur in a natural way while studying the parabolic equations which describe diffusion or heat transfer phenomena, see [1, p. 18], [6, Chapter VI] and also [2,3,5].

2. Statement of Results

Let $R_+ = [0,\infty), R_1 = [1,\infty), N_0 = \{0,1,2,...\}$ be the given subsets of R, the set of real numbers and for any function $z : N_0 \to R$ define the operator Δ by $\Delta z(n) = z(n+1) - z(n)$. Let $N_i[\alpha_i, \beta_i] = \{\alpha_i, \alpha_i + 1, ..., \beta_i\} (\alpha_i < \beta_i), \alpha_i, \beta_i \in N_0$, $i = 1,...,n$ and $G = \prod_{i=1}^{n} N_i[\alpha_i, \beta_i]$. For any function $w : G \to R$ we denote the n-fold sum over G with respect to the variable $y = (y_1,...,y_n) \in G$ by $\sum_G w(y) = \sum_{y_1=\alpha_1}^{\beta_1} ... \sum_{y_n=\alpha_n}^{\beta_n} w(y_1,...,y_n)$. Clearly, $\sum_G w(y) = \sum_G w(x)$ for $x, y \in G$. Let $E = G \times N_0$ and denote by $D(S_1, S_2)$ the class of discrete functions from the set S_1 to the set S_2. We use the usual convention that empty sums and products involved exist on the respective domains of their definitions and are finite.

Our main results are given in the following theorems.

Theorem 1. *Let $u, a, b \in D(E, R_+)$ and $L \in D(E \times R_+, R_+)$ be such that*

$$0 \leq L(x,n,u) - L(x,n,v) \leq M(x,n,v)(u-v), \tag{2.1}$$

for $u \geq v \geq 0,$ *where* $M \in D\left(E \times R_{+}, R_{+}\right).$ *If*

$$u\left(x,n\right) \leq a\left(x,n\right) + b\left(x,n\right) \sum_{s=0}^{n-1} \sum_{G} L\left(y,s,u\left(y,s\right)\right), \tag{2.2}$$

for $\left(x,n\right) \in E,$ *then*

$$\begin{aligned} u\left(x,n\right) &\leq a\left(x,n\right) + b\left(x,n\right) \sum_{s=0}^{n-1} \sum_{G} L\left(y,s,a\left(y,s\right)\right) \\ &\quad \times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} M\left(y,\sigma,a\left(y,\sigma\right)\right) b\left(y,\sigma\right)\right], \end{aligned} \tag{2.3}$$

for $\left(x,n\right) \in E.$

Remark 1. By taking (i) $L\left(y,s,u\left(y,s\right)\right) = f\left(y,s\right) u\left(y,s\right)$ $\left(f,u \in D\left(E,R_{+}\right)\right)$ and (ii) in addition to (i), let $a\left(x,n\right) = d, b\left(x,n\right) = 1$ ($d \geq 0$ is a real constant) in (2.2), it is easy to see that the bound obtained in (2.3) reduces respectively to

$$\begin{aligned} u\left(x,n\right) &\leq a\left(x,n\right) + b\left(x,n\right) \sum_{s=0}^{n-1} \sum_{G} f\left(y,s\right) a\left(y,s\right) \\ &\quad \times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} f\left(y,\sigma\right) b\left(y,\sigma\right)\right], \textit{and} \end{aligned} \tag{2.4}$$

$$u\left(x,n\right) \leq d \prod_{s=0}^{n-1} \left[1 + \sum_{G} f\left(y,s\right)\right], \tag{2.5}$$

for $\left(x,n\right) \in E.$

Theorem 2. *Let* $u,a,b \in D\left(E,R_{+}\right)$ *and* $k \geq 0$ *be a real constant. If*

$$u^{2}\left(x,n\right) \leq k^{2} + 2 \sum_{s=0}^{n-1} \sum_{G} \left[a\left(y,s\right) u^{2}\left(y,s\right) + b\left(y,s\right) u\left(y,s\right)\right], \tag{2.6}$$

for $\left(x,n\right) \in E,$ *then*

$$u\left(x,n\right) \leq k \prod_{s=0}^{n-1} \left[1 + \sum_{G} a\left(y,s\right)\right] + \sum_{s=0}^{n-1} \sum_{G} b\left(y,s\right) \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} a\left(y,\sigma\right)\right], \tag{2.7}$$

for $(x, n) \in E$.

Remark 2. If we take $a(y, s) = 0$ in (2.6), then the bound obtained in (2.7) reduces to

$$u(x, n) \leq k + \sum_{s=0}^{n-1} \sum_{G} b(y, s), \tag{2.8}$$

for $(x, n) \in E$, which in turn is a new variant of the inequality given in Theorem 5.4.1 in [7, p. 420].

Theorem 3. *Let* $u \in D(E, R_1)$, $a \in D(E, R_+)$ *and* $c \geq 1$ *be a real constant. If*

$$u(x, n) \leq c + \sum_{s=0}^{n-1} \sum_{G} a(y, s) u(y, s) \log u(y, s), \tag{2.9}$$

for $(x, n) \in E$, *then*

$$u(x, n) \leq c^{\prod_{s=0}^{n-1} \left[1 + \sum_{G} a(y,s)\right]}, \tag{2.10}$$

for $(x, n) \in E$.

Remark 3. We note that the inequality given in Theorem 3 can be considered as a variant of the inequality given in Theorem 5.5.1 in [7, p. 436] and it can be used in some new situations.

3. Proofs of Theorems 1-3

Introducing the notation

$$e(s) = \sum_{G} L(y, s, u(y, s)), \tag{3.1}$$

in (2.2), we get

$$u(x, n) \leq a(x, n) + b(x, n) \sum_{s=0}^{n-1} e(s), \tag{3.2}$$

for $(x, n) \in E$. Define

$$z(n) = \sum_{s=0}^{n-1} e(s), \tag{3.3}$$

for $n \in N_0$, then $z(0) = 0$ and from (3.2), we get

$$u(x, n) \leq a(x, n) + b(x, n) z(n), \tag{3.4}$$

for $(x, n) \in E$. From (3.3), (3.1), (3.4) and (2.1), we observe that

$$\begin{aligned}
\Delta z(n) &= e(n) \\
&= \sum_G L(y, n, u(y, n)) \\
&\leq \sum_G [L(y, n, \{a(y, n) + b(y, n) z(n)\}) - L(y, n, a(y, n))] \\
&\quad + \sum_G L(y, n, a(y, n)) \\
&\leq z(n) \sum_G M(y, n, a(y, n)) b(y, n) + \sum_G L(y, n, a(y, n)). \qquad (3.5)
\end{aligned}$$

Now a suitable application of Theorem 1.2.1 given in [7, p. 11] with $z(0) = 0$ to (3.5), yields

$$\begin{aligned}
z(n) &\leq \sum_{s=0}^{n-1} \sum_G L(y, s, a(y, s)) \\
&\quad \times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_G M(y, \sigma, a(y, \sigma)) b(y, \sigma)\right]. \qquad (3.6)
\end{aligned}$$

Using (3.6) in (3.4), we get the required inequality in (2.3). This completes the proof of Theorem 1.

Introducing the notation

$$r(s) = 2 \sum_G \left[a(y, s) u^2(y, s) + b(y, s) u(y, s)\right], \tag{3.7}$$

in (2.6), we get

$$u^2(x,n) \le k^2 + \sum_{s=0}^{n-1} r(s), \tag{3.8}$$

for $(x,n) \in E$. Let $k > 0$ and define

$$w(n) = k^2 + \sum_{s=0}^{n-1} r(s), \tag{3.9}$$

for $n \in N_0$, then $w(0) = k^2$ and from (3.8), we have

$$u^2(x,n) \le w(n), \tag{3.10}$$

for $(x,n) \in E$. From (3.9), (3.7), (3.10), we observe that

$$\begin{aligned} \Delta w(n) &= r(n) \\ &= 2\sum_G \left[a(y,n)\,u^2(y,n) + b(y,n)\,u(y,n)\right] \\ &\le 2\sqrt{w(n)}\sum_G \left[a(y,n)\sqrt{w(n)} + b(y,n)\right]. \end{aligned} \tag{3.11}$$

Using the facts that $\sqrt{w(n)} > 0$, $\Delta w(n) \ge 0$, $\sqrt{w(n)} \le \sqrt{w(n+1)}$ for $n \in N_0$ and (3.11), we observe that

$$\begin{aligned} \Delta\left(\sqrt{w(n)}\right) &= \frac{\left(\sqrt{w(n+1)} - \sqrt{w(n)}\right)\left(\sqrt{w(n+1)} + \sqrt{w(n)}\right)}{\left(\sqrt{w(n+1)} + \sqrt{w(n)}\right)} \\ &\le \frac{\Delta w(n)}{2\sqrt{w(n)}} \\ &\le \sum_G \left[a(y,n)\sqrt{w(n)} + b(y,n)\right] \\ &= \sqrt{w(n)}\sum_G a(y,n) + \sum_G b(y,n). \end{aligned} \tag{3.12}$$

Now a suitable application of Theorem 1.2.1 given in [7, p. 11] with $w(0) = k^2$ to (3.12), yields

$$\sqrt{w(n)} \le k\prod_{s=0}^{n-1}\left[1 + \sum_G a(y,s)\right]$$

$$+\sum_{s=0}^{n-1}\sum_{G} b\left(y,s\right)\prod_{\sigma=s+1}^{n-1}\left[1+\sum_{G} a\left(y,\sigma\right)\right]. \tag{3.13}$$

Using (3.13) in (3.10), we get the required inequality in (2.7). If $k \geq 0$, we carry out the above procedure with $k+\varepsilon$ instead of k, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.7). This completes the proof of Theorem 2.

The proof of Theorem 3 can be completed by following the proofs of Theorems 1 and 2 given above and closely looking at the idea used in the proof of Theorem 3.5.1 given in [7, p. 243]. We omit the details.

4. Some Applications

In this section we apply the inequality established in Theorem 1 to obtain explicit estimates on the solutions of equation (1.2). Here, we note that one can formulate existence and uniqueness results for the solution of equation (1.2) by modifying the idea employed in [9], see also [4,10].

The following theorem deals with the explicit estimate on the solution of equation (1.2).

Theorem 4. *Suppose that* $h \in D\left(E,R\right)$, $F \in D\left(E^2 \times R, R\right)$ *and*

$$\left|F\left(x,n,y,s,u\right)\right| \leq b\left(x,n\right) L\left(y,s,\left|u\right|\right), \tag{4.1}$$

where $b \in D\left(E,R_+\right)$, L *is as defined in Theorem 1 and verifies the condition (2.1). Then for every solution* $u \in D\left(E,R\right)$ *of equation (1.2) we have the estimate*

$$\begin{aligned}\left|u\left(x,n\right)\right| \quad &\leq \quad \left|h\left(x,n\right)\right| + b\left(x,n\right)\sum_{s=0}^{n-1}\sum_{G} L\left(y,s,\left|h\left(y,s\right)\right|\right)\\ &\times\prod_{\sigma=s+1}^{n-1}\left[1+\sum_{G} M\left(y,\sigma,\left|h\left(y,\sigma\right)\right|\right) b\left(y,\sigma\right)\right],\end{aligned} \tag{4.2}$$

for $(x,n) \in E$, *where* M *is as in Theorem 1.*

Proof. Let $u \in D(E,R)$ be a solution of equation (1.2). Then from the hypotheses, we have

$$|u(x,n)| \leq |h(x,n)| + b(x,n)\sum_{s=0}^{n-1}\sum_{G} L(y,s,|u(y,s)|). \tag{4.3}$$

Now an application of Theorem 1 to (4.3), gives the desired estimate in (4.2).

Next theorem gives the estimation on the solution of equation (1.2) assuming that the function F in equation (1.2) satisfies the Lipschitz type condition.

Theorem 5. *Suppose that* $h \in D(E,R)$, $F \in D(E^2 \times R, R)$ *and*

$$|F(x,n,y,s,u) - F(x,n,y,s,v)| \leq b(x,n) L(y,s,|u-v|), \tag{4.4}$$

where $b \in D(E,R_+)$, L *is as defined in Theorem 1 and verifies the condition (2.1). Then for any solution* $u \in D(E,R)$ *of equation (1.2) we have the estimate*

$$\begin{aligned} |u(x,n) - h(x,n)| \quad &\leq \quad \bar{h}(x,n) + b(x,n)\sum_{s=0}^{n-1}\sum_{G} L\left(y,s,\bar{h}(x,s)\right) \\ &\qquad \times \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_{G} M\left(y,\sigma,\bar{h}(x,n)\right) b(y,\sigma)\right], \end{aligned} \tag{4.5}$$

for $(x,n) \in E$, *where*

$$\bar{h}(x,n) = \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,h(y,s))|, \tag{4.6}$$

and M *is as in Theorem 1.*

Proof. Let $u \in D(E,R)$ be a solution of equation (1.2). Then from the hypotheses, we have

$$\begin{aligned}
|u(x,n)-h(x,n)| &\le \sum_{s=0}^{n-1}\sum_{G}|F(x,n,y,s,u(y,s))| \\
&\le \sum_{s=0}^{n-1}\sum_{G}|F(x,n,y,s,u(y,s))-F(x,n,y,s,h(y,s))| \\
&\quad +\sum_{s=0}^{n-1}\sum_{G}|F(x,n,y,s,h(y,s))| \\
&\le \bar{h}(x,n) \\
&\quad +b(x,n)\sum_{s=0}^{n-1}\sum_{G}L(y,s,|u(y,s)-h(y,s)|). \qquad (4.7)
\end{aligned}$$

Now an application of Theorem 1 to (4.7) yields, (4.5).

We next consider the following two sum-difference equations

$$v(x,n)=h_1(x,n)+\sum_{s=0}^{n-1}\sum_{G}H(x,n,y,s,v(y,s)), \qquad (4.8)$$

$$w(x,n)=h_2(x,n)+\sum_{s=0}^{n-1}\sum_{G}K(x,n,y,s,w(y,s)), \qquad (4.9)$$

where $h_1, h_2 \in D(E,R), H, K \in D(E^2 \times R, R)$.

The following theorem holds.

Theorem 6. *Suppose that* $h_1, h_2 \in D(E,R), H, K \in D(E^2 \times R, R)$ *and*

$$|H(x,n,y,s,v)-H(x,n,y,s,w)| \le b(x,n)L(y,s,|v-w|), \qquad (4.10)$$

where $b \in D(E,R_+)$, L *is as defined in Theorem 1 and verifies the condition (2.1). Then for every solution* $w \in D(E,R)$ *of equation (4.9) and*

$v \in D(E,R)$ a solution of equation (4.8), we have the estimation

$$\begin{aligned} |v(x,n) - w(x,n)| &\leq [h(x,n) + r(x,n)] \\ &+ b(x,n) \sum_{s=0}^{n-1} \sum_{G} L(y,s,[h(y,s) + r(y,s)]) \\ &\times \prod_{\sigma=s+1}^{n-1} [1 + M(y,\sigma,[h(y,\sigma) + r(y,\sigma)])], \quad (4.11) \end{aligned}$$

for $(x,n) \in E$, where

$$h(x,n) = |h_1(x,n) - h_2(x,n)|, \quad (4.12)$$

$$r(x,n) = \sum_{s=0}^{n-1} \sum_{G} |H(x,n,y,s,w(y,s)) - K(x,n,y,s,w(y,s))|, \quad (4.13)$$

and M is as in Theorem 1.

Proof. Using the facts that $v(x,n)$ and $w(x,n)$ are respectively the solutions of equations (4.8) and (4.9) and hypotheses, we have

$$\begin{aligned} |v(x,n) - w(x,n)| &\leq |h_1(x,n) - h_2(x,n)| \\ &+ \sum_{s=0}^{n-1} \sum_{G} |H(x,n,y,s,v(y,s)) - H(x,n,y,s,w(y,s))| \\ &+ \sum_{s=0}^{n-1} \sum_{G} |H(x,n,y,s,w(y,s)) - K(x,n,y,s,w(y,s))| \\ &\leq [h(x,n) + r(x,n)] \\ &+ b(x,n) \sum_{s=0}^{n-1} \sum_{G} L(y,s,|v(y,s) - w(y,s)|). \quad (4.14) \end{aligned}$$

Now an application of Theorem 1 to (4.14), gives the required estimate in (4.11).

Remark 4. We note that, Theorem 1 can be used to establish the results on continuous dependence of solutions of equations of the form (1.2) by closely looking at the results given in [10]. Moreover, many generalizations, extensions, variants and applications of the inequalities given above are also possible. We leave it to the reader to fill in where needed.

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