Tamsui Oxford Journal of Mathematical Sciences **25(3)** (2009) 291-300
Aletheia University

A New Note on the Increasing Sequences*

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Received February 19, 2008, Accepted January 13, 2009.

Abstract

In the present paper a general theorem on $|\bar{N}, p_n; \delta|_k$ summability factors of infinite series, which generalizes some well known results, has been proved. Also we have obtained a new result dealing with $|C, 1; \delta|_k$ summability factors.

Keywords and Phrases: *Absolute summability, Summability factors, Increasing sequences.*

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

*2000 *AMS Subject Classification.* 40D15, 40F05, 40G99.
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where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{3}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [7], [10])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{\mid t_n^\alpha \mid^k}{n} < \infty. \tag{4}$$

and it is said to be summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [8])

$$\sum_{n=1}^{\infty} n^{\delta k - 1} \mid t_n^\alpha \mid^k < \infty. \tag{5}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{6}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{7}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [9]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2], [3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{8}$$

and it is said to be summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{9}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{10}$$

In the special case $p_n = 1$ for all values of n (resp. $\delta = 0$) $\mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid C, 1; \delta \mid_k$ (resp. $\mid \bar{N}, p_n \mid_k$) summability. Also if we take $\delta = 0$ and $k = 1$, then we get $\mid \bar{N}, p_n \mid$ summability.

2. Known Results

Bor [4] has proved the following theorem for $\mid \bar{N}, p_n \mid_k$ summability factors.

Theorem A. *Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{11}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{13}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{14}$$

If

$$\sum_{v=1}^{n} \frac{\mid s_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{15}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{16}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{17}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.
If we take $k = 1$ in Theorem A, then we get a theorem due to Mishra and Srivastava [12] concerning the $\mid \bar{N}, p_n \mid$ summability factors.
Recently Bor [6] generalized Theorem A for $\mid \bar{N}, p_n; \delta \mid_k$ summability in the following form.

Theorem B. *Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) are such that conditions (11)-(17) of Theorem A are satisfied with the condition (15) replaced by:*

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{18}$$

and

$$\sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} = O((\frac{P_v}{p_v})^{\delta k} \frac{1}{P_v}) \quad as \quad m \to \infty, \tag{19}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Remark. It should be noted that if we take $\delta = 0$, then we get Theorem A. In this case condition (18) reduces to condition (15) and condition (19) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) = O\left(\frac{1}{P_v} \right) \quad as \quad m \to \infty, \tag{20}$$

which always holds. Also it may be noticed that , under the conditions on the sequence (λ_n) we have that (λ_n) is bounded and $\Delta \lambda_n = O(1/n)$ (see [4]).

3. Main Result

The aim of this paper is to prove Theorem B under weaker conditions. For this we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$.
Now, we shall prove the following theorem.

Theorem. *Let* (X_n) *be an almost increasing sequence. If the conditions (11)-(14) and (16)-(19) are satisfied, then the series* $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.
We require the following lemma for the proof of the theorem.

Lemma 1 ([11]). If (X_n) be an almost increasing sequence, then under the conditions (12)-(13) we have that

$$n X_n \beta_n = O(1), \tag{21}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{22}$$

Lemma 2 ([12]). If the conditions (16) and (17) are satisfied, then

$$\Delta \left(\frac{P_n}{n p_n} \right) = O\left(\frac{1}{n} \right) \tag{23}$$

4. Proof of the Theorem

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{vp_v}.$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{vp_v}, \quad n \geq 1, \quad (P_{-1} = 0).$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_v \Delta \left(\frac{P_{v-1} P_v \lambda_v}{vp_v} \right) + \frac{\lambda_n s_n}{n} \\ &= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta \lambda_v}{(v+1) p_{v+1}} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left(\frac{P_v}{vp_v} \right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say. \end{aligned}$$

Since

$$| T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} |^k \leq 4^k (| T_{n,1} |^k + | T_{n,2} |^k + | T_{n,3} |^k + | T_{n,4} |^k)$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,r} |^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{24}$$

Firstly, by Abel transformation, we have

$$
\begin{aligned}
& \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,1} \mid^k \\
= & \sum_{n=1}^{m} \left(\frac{P_n}{np_n}\right)^{k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid \frac{\mid s_n \mid^k}{n} \\
= & O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n} \mid \lambda_n \mid \\
= & O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
& + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{\mid s_n \mid^k}{n} \\
= & O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
= & O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m \\
= & O(1), \quad as \quad m \to \infty,
\end{aligned}
$$

by the hypotheses of the Theorem and Lemma 1.
Now, using the fact that $P_{v+1} = O((v+1)p_{v+1})$ by (16), and applying Hölder's

inequality we have

$$
\begin{aligned}
& \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2} \mid^k \\
= \; & O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k \\
= \; & O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} \mid s_v \mid p_v \mid \Delta\lambda_v \mid \right\}^k \\
= \; & O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \mid \Delta\lambda_v \mid^k \\
& \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1} \\
= \; & O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \mid \Delta\lambda_v \mid^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_n} \\
= \; & O(1) \sum_{v=1}^{m} \left(\frac{P_v \mid \Delta\lambda_v \mid}{p_v}\right)^{k-1} \left(\frac{P_v}{p_v}\right)^{\delta k} \mid s_v \mid^k \mid \Delta\lambda_v \mid \\
= \; & O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \mid s_v \mid^k \mid \Delta\lambda_v \mid \left(\frac{P_v}{v p_v}\right)^{k-1} \\
= \; & O(1) \sum_{v=1}^{m} v\beta_v \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= \; & O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \left(\frac{P_r}{p_r}\right)^{\delta k} \frac{\mid s_r \mid^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= \; & O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
= \; & O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} X_v \mid \beta_v \mid + O(1) m\beta_m X_m = O(1),
\end{aligned}
$$

as $m \to \infty$, by the hypotheses of the Theorem and Lemma 1.
Again, as in $T_{n,1}$, we have

$$\begin{aligned}
&\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,3} \mid^k \\
= &\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left(\frac{P_v}{v p_v}\right) \mid^k \\
= &O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\
= &O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right) p_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\
= &O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{v p_v}\right)^k p_v \mid s_v \mid^k \mid \lambda_v \mid^k \\
&\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= &O(1) \sum_{v=1}^{m} \left(\frac{P_v}{v p_v}\right)^k \mid s_v \mid^k p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \\
= &O(1) \sum_{v=1}^{m} \left(\frac{P_v}{v p_v}\right)^k p_v \mid s_v \mid^k \mid \lambda_v \mid^k \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v} . \frac{v}{v} \\
= &O(1) \sum_{v=1}^{m} \left(\frac{P_v}{v p_v}\right)^{k-1} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= &O(1) \sum_{v=1}^{m} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= &O(1) \sum_{v=1}^{m} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= &O(1) \sum_{v=1}^{m-1} X_v \beta_v + O(1) X_m \mid \lambda_m \mid = O(1), \quad as \quad m \to \infty,
\end{aligned}$$

by the hypotheses of the Theorem, Lemma 1 and Lemma 2.
Finally, using Hölder's inequality, as in $T_{n,3}$, we have

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,4} \mid^k \\
= &\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v \frac{P_v}{v}\lambda_v \mid^k \\
= &\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v \frac{P_v}{vp_v} p_v\lambda_v \mid^k \\
\leq &\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \mid s_v \mid^k \left(\frac{P_v}{vp_v}\right)^k p_v \mid \lambda_v \mid^k \\
&\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1} \\
= &O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k \left(\frac{P_v}{p_v}\right)^{\delta k} \mid s_v \mid^k p_v \mid \lambda_v \mid^k \frac{1}{P_v}.\frac{v}{v} \\
= &O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= &O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid s_v \mid^k}{v} \\
= &O(1)\sum_{v=1}^{m-1} X_v\beta_v + O(1)X_m \mid \lambda_m \mid = O(1), \quad as \quad m \to \infty.
\end{aligned}$$

Therefore we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,r} \mid^k = O(1), \quad for \quad r = 1, 2, 3, 4.$$

This completes the proof of the Theorem.
Finally if we take $p_n = 1$ for all values of n in the Theorem, then we get a new result concerning the $\mid C, 1; \delta \mid_k$ summability factors.

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