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Argument Estimates of Certain Multivalent Analytic Functions Defined by Integral Operators*

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Abstract

In the present paper, we derive certain new argument properties of a class of multivalent analytic functions defined in an open unit disk by using a theorem recently established by A. Y. Lashin in 2004. Certain intersting (known or new) results are derived in the form of corollaries from our main results.

Keywords and Phrases: *Argument estimates, Multivalent analytic functions, Jung-Kim-Srivastava integral operator.*

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in N), \tag{1.1}$$

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which are analytic in the open unit disc $\Delta := \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z) \in \mathcal{A}(p)$, the Hadamard product (or convolution) is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z), \tag{1.2}$$

where

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad (p \in N). \tag{1.3}$$

For $f(z) \in \mathcal{A}(p)$, we consider following p-modification of the familiar Jung-Kim-Srivastava integral operator:

$$\mathcal{I}^\sigma f(z) = \frac{(p+1)^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} f(t)dt, \tag{1.4}$$

$$= z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\sigma a_n z^n \qquad \sigma > 0. \tag{1.5}$$

Obviously

$$\mathcal{I}^0 f(z) \equiv f(z). \tag{1.6}$$

For the p-modified Jung-Kim-Srivastava integral operator, we easily get

$$z[\mathcal{I}^\sigma f(z)]' = (p+1)\mathcal{I}^{\sigma-1} f(z) - \mathcal{I}^\sigma f(z). \tag{1.7}$$

Many classes of analytic functions defined by the p-modified Jung-Kim-Srivastava integral operator (1.4) were studied earlier by Shams *et al.* [6], Liu [4] and Patel and Mohanty [5].
In this paper, we derive certain argument properties of analytic functions defined by means of the p-modified Jung-Kim-Srivastava integral operator (1.4).
In order to prove our main results, we shall require the following result.

Lemma 2.1 [3]. *Let $p(z)$ be analytic in Δ, with $p(0) = 1$, and $p(z) \neq 0$ $(z \in \Delta)$. Further suppose that $\alpha, \beta \in R_+$ and*

$$|arg(p(z) + \beta z p'(z))| < \frac{\pi}{2}(\alpha + \frac{2}{\pi} tan^{-1}\beta), \ \ (\alpha > 0, \beta > 0), \tag{2.1}$$

then

$$|arg(p(z))| < \frac{\pi}{2}\alpha \ \ for \ \ z \in \Delta. \tag{2.2}$$

2. Main Results

Theorem 3.1. *If $f(z) \in \mathcal{A}(p)$ satisfies the condition*

$$\left| \left\{ \frac{\mathcal{I}^{\sigma} f(z)}{\mathcal{I}^{\sigma} g(z)} \right\}^{\gamma} \left\{ 1 + \frac{\lambda}{p} \left(\frac{\mathcal{I}^{\sigma-1} f(z)}{\mathcal{I}^{\sigma} f(z)} - \frac{\mathcal{I}^{\sigma-1} g(z)}{\mathcal{I}^{\sigma} g(z)} \right) \right\} \right| < \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\lambda}{p(p+1)}\alpha \right), \tag{3.1}$$

then

$$\left| \left\{ \frac{\mathcal{I}^{\sigma} f(z)}{\mathcal{I}^{\sigma} g(z)} \right\}^{\gamma} \right| < \frac{\pi}{2}\alpha \tag{3.2}$$

where $\alpha,\ \beta,\ \gamma,\ \sigma\ \in R_+,\ \lambda\ \geq\ 0$ and $z \in \Delta$.

Proof. Define a function

$$p(z) = \left\{ \frac{\mathcal{I}^{\sigma} f(z)}{\mathcal{I}^{\sigma} g(z)} \right\}^{\gamma}, \quad \gamma \neq 0 \tag{3.3}$$

then $p(z) = 1 + c_1 z + c_2 z^2 + \ldots\ldots\ldots\ldots.$ which is analytic in Δ with $p(0) = 1$ and $p(z) \neq 0\ \ (z \in \Delta)$.

Diffrentiating (2.3) logarithmically, we get

$$\frac{z\, p'(z)}{p(z)} = \gamma \left[\frac{z\,[\mathcal{I}^{\sigma} f(z)]'}{\mathcal{I}^{\sigma} f(z)} - \frac{z\,[\mathcal{I}^{\sigma} g(z)]'}{\mathcal{I}^{\sigma} g(z)} \right]. \tag{3.4}$$

Now making use of identity (1.7) in (3.4), we easily get

$$p(z) + \frac{\lambda}{\gamma p(p+1)} z p'(z) = \left\{ \frac{\mathcal{I}^{\sigma} f(z)}{\mathcal{I}^{\sigma} g(z)} \right\}^{\gamma} \left\{ 1 + \frac{\lambda}{p} \left(\frac{\mathcal{I}^{\sigma-1} g(z)}{\mathcal{I}^{\sigma} g(z)} - \frac{\mathcal{I}^{\sigma-1} f(z)}{\mathcal{I}^{\sigma} f(z)} \right) \right\}, \tag{3.5}$$

and the statement of the Theorem 3.1 directly follows from Lemma 2.1.

Setting $\gamma = 1$ and $g(z) = z^p$ i.e. all $b_i = 0\ \ (i = p+1, \ldots\ldots)$ in Theorem 3.1, we easily arrive at the

Corollary 3.2. *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$\left| arg \left\{ \frac{\lambda}{p} \frac{\mathcal{I}^{\sigma-1} f(z)}{z^p} + \frac{(p-\lambda)}{p} \frac{\mathcal{I}^{\sigma} f(z)}{z^p} \right\} \right| < \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\lambda}{p(p+1)}\alpha \right), \tag{3.6}$$

then

$$\left| arg \left(\frac{\mathcal{I}^{\sigma} f(z)}{z^p} \right) \right| < \frac{\pi}{2}\alpha, \tag{3.7}$$

where $\alpha, \beta, \sigma \in R_+, \lambda \geq 0$ *and* $z \in \Delta$.

Again taking $\gamma = p = 1$, we get

Corollary 3.3. *Let* $\alpha,\ \sigma \in R_+$ *and* $\lambda \geq 0$. *If* $f(z) \in \mathcal{A}(1)$ *satisfies*

$$\left| arg \left\{ \lambda \frac{\mathcal{I}^{\sigma-1} f(z)}{z} + (1-\lambda) \frac{\mathcal{I}^{\sigma} f(z)}{z} \right\} \right| < \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\lambda}{2}\alpha\right), \qquad (3.8)$$

then

$$\left| arg \left(\frac{\mathcal{I}^{\sigma} f(z)}{z} \right) \right| < \frac{\pi}{2}\alpha. \qquad (3.9)$$

Further taking $\gamma = 1$, $\lambda = p+1$ and $\sigma \to 0$ in Theorem 3.1, we get result on argument estimate given earlier by Cho et al. [1].

If we put $\gamma = p = 1$, and let $\sigma \to 0$ in Theorem 3.1, and replace λ by 2β therein , we get a result due to Lashin [3].

Lastly taking $\gamma = 1$, and $f(z) = z^p$ *i.e.* $(a_i = 0, i = p+1, ...)$ in Theorem 3.1, we get an interesting result contained in

Corollary 3.4. *Let* $\frac{z^p}{\mathcal{I}^{\sigma} g(z)} \neq 0$, $g(z) \in \mathcal{A}(p)$ *and* $\lambda \geq 0$. *Suppose that*

$$\left| \arg \left[\left(1 + \frac{\lambda}{p}\right) \frac{z^p}{\mathcal{I}^{\sigma} g(z)} - \frac{\lambda}{p} \frac{\mathcal{I}^{\sigma-1} g(z)}{\mathcal{I}^{\sigma} g(z)} \left(\frac{z^p}{\mathcal{I}^{\sigma} g(z)} \right) \right] \right| < \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\lambda}{p(p+1)}\alpha\right), \qquad (3.10)$$

then

$$\left| \frac{z^p}{\mathcal{I}^{\sigma} g(z)} \right| < \frac{\pi}{2}\alpha. \qquad (\alpha \in R_+, z \in \Delta) \qquad (3.11)$$

Theorem 3.5. *Let* $\lambda, \sigma \in R_+$ *and* $0 < \lambda < p$. *Suppose that* $f(z) \in \mathcal{A}(p)$ *satisfies*

$$\left| \frac{\mathcal{I}^{\sigma} f(z)}{z^p} \right| < \frac{\pi}{2}\alpha + tan^{-1}\left(\frac{\lambda\alpha}{p(p+1)}\right) \qquad (3.12)$$

then we have

$$\left| \arg \left(\frac{p(p+1)}{\lambda} z^{-p(p+1)/\lambda} \int_0^z t^{(p+1)(p-\lambda)/\lambda} \mathcal{I}^{\sigma} f(t) dt \right) \right| < \frac{\pi}{2}\alpha. \qquad (3.13)$$

Proof. Consider the function

$$p(z) = \left(\frac{p(p+1)}{\lambda} z^{-p(p+1)/\lambda} \int_0^z t^{(p+1)(p-\lambda)/\lambda} \mathcal{I}^{\sigma} f(t) dt \right). \tag{3.14}$$

Obviously

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots\dots\dots\dots \tag{3.15}$$

and $p(z)$ is an analytic in Δ. Also $p(0) = 1,\ \textit{and}\ p'(z) \neq 0.$

Differentiating (3.14), we get the following result after some computations

$$p(z) + \frac{\lambda}{p(p+1)} z p'(z) = \frac{\mathcal{I}^{\sigma} f(z)}{z^p}. \tag{3.16}$$

Now making use of Lemma 2.1 , the proof of the Theorem 3.5 is complete.

Setting $p = 1,\ \lambda = 2\ \textit{and}\ \sigma \to 0$, in Theorem 3.5, we arrive at the following interesting result contained in

Corollary 3.6. *Let* $f(z) \in \mathcal{A}(1)$ *satisfies*

$$\left| \arg \left(\frac{f(z)}{z} \right) \right| < \frac{\pi}{2}\alpha + \tan^{-1}(\alpha), \tag{3.17}$$

then we have

$$\left| \arg \left(\frac{1}{z} \int_0^z \frac{f(t)}{t} dt \right) \right| < \frac{\pi}{2}\alpha \qquad (\alpha > 0\ \textit{and}\ z \in \Delta). \tag{3.18}$$

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