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Inverse Spectral Problem for Some Singular Differential Operators*

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Abstract

In this paper, it has given a uniqueness theorem for the singular Sturm-Liouville problem having the singularity type $\frac{l(l+1)}{x^2}$ and $\frac{2}{x} - \frac{l(l+1)}{x^2}$ on the unit interval. It is given that the particular set of eigenvalues is sufficient to determine the unknown potential. We mentioned that these results were given by McLaughlin and Rundell for regular Sturm-Liouville problem.

Keywords and Phrases: *Eigenvalues, Dirichlet Conditions, Singular Differential Operator.*

1. Introduction

Sturm-Liouville problems, in particularly self-adjoints ones, received extensive studies in last years. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. The first spectral problem was given by Ambarzumyan [1]. In 1946, Borg asserted that to quarantee uniqueness, one needs additional spectral data [2]. He showed that the spectra of two boundary value problems for an

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operator with different boundary conditions at one end uniquely determine the potential. And later, Marchenko, Levitan used the transformation operator to show that the eigenvalues and norming constants uniquely determine potential function [9], [7]. Hochstadt showed that the potential is an even function, then potential is uniquely determined [5]. On the other hand, A finite number of eigenvalues in one spectrum is unknown, $q(x)$ is not uniquely determined by one full spectrum and one partial spectrum. This problem was investigated by Gesztesy and Simon in [4]. In later years, these problems was studied for regular and singular problem by some authors, [3], [6], [8], [11], [12], [13], [14].

In this paper, it is shown that the particular set of eigenvalues is sufficient to determine the potential for differential operator having singularity type $\frac{l(l+1)}{x^2}$ and $\frac{l(l+1)}{x^2} - \frac{2}{x}$ at the point zero. We mentioned that analogous results are given by McLaughlin and Rundell in [10].

We consider the eigenvalue problem

$$y'' - \left[q(x) + \frac{l(l+1)}{x^2}\right] y = \lambda y, \tag{1.1}$$

$$y(0) = 0, \quad y'(1, \lambda) + Hy(1, \lambda) = 0, \tag{1.2}$$

where $q(x) \in L^2(0,1)$ and l is an integer. This type equations arise when separation of variables is used for the study of radial Schrödinger operators $\Delta + q(x)$ on a ball in Euclidean space, zonal Schrödinger operators on spheres or in the study of Laplace operators Δ for a Riemann manifold of revolution. For example, the singularity type $\frac{l(l+1)}{x^2}$ refers to Bessel equation, the singularity type $\frac{2}{x} - \frac{l(l+1)}{x^2}$ refers to hydrogen atom. It is given the eigenvalues of the problem (1.1), (1.2) in the form

$$\lambda_n(q) = \left(n + \frac{l}{2}\right)^2 \pi^2 + \int_0^1 q(x)dx - l(l+1) + a_n,$$

where the series $\sum a_n^2 < \infty$ [12].

If the boundary condition at $x = 1$ is changed, say to the Dirichlet condition $y(1) = 0$, then corresponding set of eigenvalues $\overset{\sim}{\lambda}_n(q)$, $k = 1, 2, ...$ of the problem

$$\left.\begin{array}{c} y'' - \left[q(x) + \frac{l(l+1)}{x^2}\right] y = \lambda y \\ y(0) = y(1) = 0 \end{array}\right\} \tag{1.3}$$

is also given.

2. A Uniqueness Theorem

In this section, it will be given that one potential $q(x)$ can be determined from $\lambda_n(q, H_k)$, where n is fixed and H_k, $k = 1, 2, ...$, are distinct.

Firstly, we give two lemmas that required for proof of the main theorem. First lemma is a oscillation theorem for eigenvalues of a Sturm-Liouville operator, the second one is a statement of the inverse problem of the singular Sturm - Liouville operator.

Lemma 2.1. *Let $q(x) \in L^2(0,1)$. Then, for all $-\infty < H < \infty$,*

$$\widetilde{\lambda}_n(q) < \lambda_n(q, H) < \widetilde{\lambda}_{n+1}(q). \tag{2.1}$$

Lemma 2.2. *$\lambda_n(q_1, H_1)$ and $\lambda_n(q_2, H_2)$ are eigenvalues of the problems*

$$y^{''} - \left[q_1(x) + \frac{l(l+1)}{x^2}\right] y = \lambda y, \tag{2.2}$$

$$y(0) = 0, \quad y^{'}(1, \lambda) + H_1 y(1, \lambda) = 0 \tag{2.3}$$

and

$$y^{''} - \left[q_2(x) + \frac{l(l+1)}{x^2}\right] y = \lambda y, \tag{2.4}$$

$$y(0) = 0, \quad y^{'}(1, \lambda) + H_2 y(1, \lambda) = 0, \tag{2.5}$$

respectively. If these eigenvalues satisfy

$$\lambda_n(q_1, H_1) = \lambda_n(q_2, H_1), \quad n = 0, 1, 2, ...$$

$$\lambda_n(q_1, H_2) = \lambda_n(q_2, H_2), \quad n = 0, 1, 2, ...$$

then $q_1 = q_2$.

Theorem 2.1. *Let $q_1(x), q_2(x) \in L^2(0,1)$. Assume that H_k for $k = 1, 2, ...$ are real distinct numbers and*

$$\lambda_n(q_1, H_k) = \lambda_n(q_2, H_k), \quad k = 1, 2, ... \tag{2.6}$$

then $q_1 = q_2$.

Proof. For each λ, let $\varphi_2(x, q_i, \lambda)$ $(i = 1, 2)$ be the solution of problem

$$y^{''} - \left[q_i(x) + \frac{l(l+1)}{x^2}\right] y = \lambda y, \tag{2.7}$$

$$y(0) = 0, \quad y^{'}(0) = 1. \tag{2.8}$$

Then, by using (2,7) it has been the Sturm identity for Sturm-Liouville problem

$$\begin{aligned} & \varphi_2(x, q_1, \lambda) \left\{ \varphi_2^{''}(x, q_2, \lambda) - \left[q_2(x) + \frac{l(l+1)}{x^2}\right] \varphi_2(x, q_2, \lambda) \right\} \\ & -\varphi_2(x, q_2, \lambda) \left\{ \varphi_2^{''}(x, q_1, \lambda) - \left[q_1(x) + \frac{l(l+1)}{x^2}\right] \varphi_2(x, q_1, \lambda) \right\} \\ = & \ [q_1(x) - q_2(x)] \, \varphi_2(x, q_1, \lambda) \varphi_2(x, q_2, \lambda) \\ & + \left[\varphi_2(x, q_1, \lambda) \varphi_2^{'}(x, q_2, \lambda) - \varphi_2(x, q_2, \lambda) \varphi_2^{'}(x, q_1, \lambda)\right]^{'} \\ = & \ 0. \end{aligned} \tag{2.9}$$

On the other hand, we shall denote by $\tilde{\lambda}_n(q_i)$ the eigenvalues of the problem (1.1) with the Dirichlet condition $\varphi_2(1, q_i, \tilde{\lambda}_n) = 0$.

Now, to facilitate some softwares, we use the the simplified notation

$$\nu_k = \lambda_n(q_1, H_k) = \lambda_n(q_2, H_k), \quad k = 1, 2, ...$$

Inserting $\lambda = \nu_k$ in (2.9) and integrating from 0 to 1, it get that the term in brackets on the final line has zero at $x = 0$ and $x = 1$. So, we get

$$\int_0^1 (q_1 - q_2) \, \varphi_2(x, q_1, \nu_k) \varphi_2(x, q_2, \nu_k) dx = 0, \quad k = 1, 2, ... \tag{2.10}$$

It is observed from Lemma 1.1. that the sequence $\{\nu_k\}_{k=1}^{\infty}$ forms a bounded set on the real line and consequently has at least one finite accumulating point. Furthermore, for fixed x since $\varphi_2(x, q_i, \lambda)$ is analytical function of λ, it can be shown that

$$S(\lambda) = \int_0^1 (q_1 - q_2) \, \varphi_2(x, q_1, \lambda) \varphi_2(x, q_2, \lambda) dx \tag{2.11}$$

is also an analytical function of λ. Then,

$$S(\lambda) \equiv 0. \tag{2.12}$$

Now, we will show that all of the eigenvalues of (1.1) with $H = 0$ and all of the eigenvalues (1.3) are the same for the $q(x)$ set equal to $q_i(x)$, i.e., we will show that

$$\widetilde{\lambda}_n(q_1) = \widetilde{\lambda}_n(q_2), \quad n = 1, 2, ..., \tag{2.13}$$

$$\lambda_m(q_1, 0) = \lambda_m(q_2, 0), \quad m = 1, 2, ... \tag{2.14}$$

Then, from Lemma 2 we would be able to conclude that $q_1 = q_2$.

For proving the (2.13) and (2.14), we return to the identity (2.9) and get that when $\lambda = \lambda_m(q_1, 0)$, then $\varphi_2(1, q_1, \lambda_m(q_1, 0)) \neq 0$ while $\varphi_2^{'}(1, q_1, \lambda_m(q_1, 0)) = 0$. Integrating (2.9) from 0 to 1 and using (2.12) we must have $\varphi_2^{'}(1, q_2, \lambda_m(q_1, 0)) = 0, \quad m = 1, 2, ...$ This implies that each $\lambda_m(q_1, 0)$ is an eigenvalue for (1.1) and (1.2) when $(q, H) = (q_2, 0)$. It follows that (2.13) holds.

Similiarly set $\lambda = \widetilde{\lambda}_n(q_1)$ in the identity (2.9) and doing the above process, we conclude that (2.14) holds. Then, the proof is complete.

Now, we consider the operator

$$-y'' + \left[\frac{l\,(l+1)}{x^2} - \frac{2}{x} + q_i\,(x)\right] y = \lambda y \quad (0 < x \le 1)\,, \tag{2.15}$$

$$y\,(0) = 0, \quad y'\,(1) + Hy\,(1) = 0. \tag{2.16}$$

In quantum mechanics the study of the energy levels of the hydrogen atom leads to the equation (2.15). The $\lambda_n(q, H_i)$ be eigenvalues of the problem (2.15), (2.16). Hence, we would be able to the following theorem like to the Theorem 2.1.

Theorem 2.2. *Let $q_1(x), q_2(x) \in L^2(0, 1)$. Assume that H_k for $k = 1, 2, ...$ are real distinct numbers and*

$$\lambda_n(q_1, H_k) = \lambda_n(q_2, H_k), \quad k = 1, 2, ...$$

then $q_1 = q_2$ for the problem (2.15), (2.16). Proof of the theorem 2.2 is analogous to Theorem 2.1.

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