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Space Time with Generalized Covariant Recurrent Energy Momentum Tensor*

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Abstract

In this paper generalized covariant recurrent energy momentum tensor has been defined and it is shown that a general relativistic spacetime with this type of tensor is generalized Ricci-recurrent and in a Ricci-recurrent Riemannian manifold energy momentum tensor is generalized recurrent if and only if the Einstein tensor, with and without cosmological constant is generalized Ricci-recurrent. Further definition of semi-Einstein manifold is given with an example and it is found that if Einstein tensor is generalized Ricci-recurrent then the manifold becomes semi-Einstein.

Keywords and Phrases: *Generalized covariant recurrent energy momentum tensor, Generalized Ricci-recurrent tensor, Einstein tensor, Semi-Einstein space.*

1. Introduction

In the general theory of relativity, energy momentum tensor plays an important role and the condition on energy momentum tensor for a perfect fluid

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space time changes the nature of space time [5]. In 1996 M.C.Chaki and S.Roy in [2] have shown that if energy momentum tensor is covariant constant, the space time is Ricci-symmetric. In 2004, A.Konar and B.Biswas studied energy momentum tensor which is covariant recurrent in [6]. Also generalized Ricci-recurrent spaces were introduced in [3] by U.C.De, N.Guha and D.Kamilya. In [1] and [3] authors have studied Kenmotsu and Sasakian manifolds which are generalized Ricci-recurrent. So in this paper we have defined generalized covariant recurrent energy momentum tensor and studied in a general relativistic space time. It is shown that the space time becomes generalized Ricci-recurrent. Further we have defined a semi-Einstein space, given an example and proved that in a Ricci-recurrent Riemannian manifold energy momentum tensor T is generalized recurrent if and only if the Einstein tensor G is generalized Ricci-recurrent and G will be generalized Ricci-recurrent if and only if the manifold is semi-Einstein.

We know an energy momentum tensor T will be covariant recurrent [6] if

$$(\nabla_Z T)(X, Y) = \psi(Z)T(X, Y), \tag{1}$$

where ψ is non-zero 1-form such that, $\psi(Z) = g(Z, \rho)$.

So we like to define generalized covariant recurrent energy momentum tensor as follows:

Definition. *An energy momentum tensor T is said to be generalized covariant recurrent if*

$$(\nabla_Z T)(X, Y) = \psi(Z)T(X, Y) + \varphi(Z)g(X, Y) \tag{2}$$

where ψ and φ are two non-zero 1-forms such that,

$$\psi(Z) = g(Z, \rho) \qquad \text{and} \qquad \varphi(Z) = g(Z, \overline{\rho}) \tag{3}$$

ρ and $\overline{\rho}$ being unit orthogonal vector fields, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Next we define a semi-Einstein manifold which is the generalization of Einstein manifold.

Definition. *A Riemannian manifold M is said to be semi-Einstein if Ricci tensor S, which is non-zero, satisfies*

$$S(X, Y) = P(Z)g(X, Y) \quad \forall X, Y, Z \in TM \tag{4}$$

and P is a non-zero 1-form, such that the vector fields are not scalar multiple of a single vector.

Example of semi-Einstein space: Let M be an odd dimensional Riemannian manifold with a Riemannian metric g defined as follows–

$$g = \begin{pmatrix} 1 & 0 \\ 0 & f^2G \end{pmatrix},$$

where G is Kaehlerian metric for a Kaehlerian manifold F of dimension $2n$ and $f(x) = ce^x$,which is a function on real line for a non-zero constant c.

We define *a 1-form $P(Z) = g(Z, W)$, where Z is a vector field, not a scalar multiple of W. Then M is a semi-Einstein manifold, when G satisfies the relation, $G = e^{QX-X-Z}$ where Q is a symmetric endomorphism of the tangent space of M.*

We also state some other definitions which are given in [3],[6] and will be required in the next section.

A non-flat Riemannian manifold is said to be generalized Ricci-recurrent [3], if the non-zero Ricci-tensor S satisfies

$$(\nabla_Z S)(X, Y) = \alpha(Z)S(X, Y) + \beta(Z)g(X, Y), \tag{5}$$

where α, β are non-zero 1-forms, such that

$$\alpha(Z) = g(Z, \gamma) \qquad and \qquad \beta(Z) = g(Z, \overline{\gamma}), \tag{6}$$

γ and $\overline{\gamma}$ being unit orthogonal vector fields and ∇ has its usual means.

An Einstein equation without any cosmological constant is given by

$$S(X, Y) - \frac{1}{2}rg(X, Y) = KT(X, Y) = G(X, Y), \quad \forall X, Y \in TM, \tag{7}$$

where r is the scalar curvature at any point of the space time and K is the gravitational constant [6]. G is called Einstein's tensor.

Again *an Einstein equation with cosmological constant is given by*

$$\begin{aligned} S(X, Y) - \frac{1}{2}rg(X, Y) + \lambda g(X, Y) &= KT(X, Y) \\ &= K[(\sigma + \rho)\omega(X)\omega(Y) + pg(X, Y)] \end{aligned} \tag{8}$$

where λ is the cosmological constant, σ is the energy density and p is isotropic pressure of the fluid.

2. Generalized recurrent energy momentum tensor in a general relativistic space time

In this section it is assumed that the Einstein equation in a general relativistic space time is without cosmological constant. Here we prove,

Theorem 1. *A necessary and sufficient condition that the energy momentum tensor T without cosmological constant in a general relativistic space time is generalized recurrent is that space time is generalized Ricci-recurrent.*

Proof. Differentiating (7) covariantly with respect to Z and using (2) we get

$$(\nabla_Z S)(X,Y) - \frac{1}{2}dr(Z)g(X,Y) = K\{\psi(Z)T(X,Y) + \varphi(Z)g(X,Y)\}. \quad (9)$$

Replacing (7) in (9) and simplifying we have

$$(\nabla_Z S)(X,Y) = A(Z)S(X,Y) + B(Z)g(X,Y), \quad (10)$$

where,

$A(Z) = \psi(Z)$ and $K\varphi(Z) + \frac{1}{2}dr(Z) - \frac{1}{2}r\psi(Z) = B(Z)$ which are 1-forms. Equation (10) shows that the space time is generalized Ricci-recurrent.

Again in (10) if we take $X = Y = e_i$, $i = 1,2,3,4$, where $\{e_i\}$ are orthonormal basis of space time, we obtain

$$dr(Z) = A(Z)r + 4B(Z). \quad (11)$$

Conversely, we assume that the space time is generalized Ricci-recurrent. Then from (2),(9) and (10) we get

$$K(\nabla_Z T)(X,Y) = A(Z)S(X,Y) + B(Z)g(X,Y) - \frac{1}{2}dr(Z)g(X,Y). \quad (12)$$

Using (11) in (12) and by (7) we have

$$(\nabla_Z T)(X,Y) = A(Z)T(X,Y) - \frac{1}{K}B(Z)g(X,Y). \tag{13}$$

which implies T is generalized recurrent. □

Theorem 2. *In a 4-dimensional generalized Ricci recurrent Riemannian manifold energy momentum tensor is generalized recurrent if and only if the Einstein tensor is generalized Ricci-recurrent.*

Proof. We assume that a Riemannian manifold M of dimension 4 is generalized Ricci-recurrent. Using theorem-1. we have energy momentum tensor T is generalized recurrent.
From (7) we have

$$T(X,Y) = \frac{1}{K}[S(X,Y) - \frac{1}{2}rg(X,Y)]. \tag{14}$$

Differentiating (14) covariantly, using (2), (5) and after a brief calculation, we obtain

$$KT(X,Y) = \frac{A(Z)}{\psi(Z)}S(X,Y) + \frac{1}{\psi(Z)}[-\frac{r}{2}A(Z) - B(Z) - K\varphi(Z)]g(X,Y). \tag{15}$$

(15) can be written as

$$G(X,Y) = \alpha(Z)S(X,Y) + \beta(Z)g(X,Y), \tag{16}$$

where $\alpha(Z) = \frac{A(Z)}{\psi(Z)}$ *and* $\beta(Z) = \frac{1}{\psi(Z)}[-\frac{r}{2}A(Z) - B(Z) - K\varphi(Z)]$

are the 1-forms, as $\psi(z) \neq 0$.

By virtue of (5) and equation (16) we can assert that the Einstein tensor G is generalized Ricci-recurrent.

Conversely, if G is generalized Ricci-recurrent, then

$$G(X,Y) = \omega(Z)S(X,Y) + \delta(Z)g(X,Y), \tag{17}$$

where $\omega(Z)$ and $\delta(Z)$ are non-zero 1-forms.

Taking $\omega(Z)=\frac{\alpha(Z)}{\beta(Z)}$ *and* $\delta(Z)=\frac{1}{\beta(Z)}[m(Z)-\frac{1}{2}dr(Z)-KN(Z)]$, where $\alpha,\ \beta,\ m,\ N$ are also non-zero 1-forms, and substituting in (17) we get

$$\beta(Z)G(X,Y)=[\alpha(Z)S(X,Y)+m(Z)g(X,Y)]+[-\frac{1}{2}dr(Z)-KN(Z)]g(X,Y). \tag{18}$$

As the Riemannian manifold M is generalized Ricci-recurrent, using (5) and (7) in (18) we have

$$\beta(Z)T(X,Y)+N(Z)g(X,Y)=\frac{1}{K}[(\nabla_Z S)(X,Y)-\frac{1}{2}dr(Z)g(X,Y)]. \tag{19}$$

By virtue of covariant differentiation of (7) with respect to Z, (19) reduces to

$$(\nabla_Z T)(X,Y)=\beta(Z)T(X,Y)+N(Z)g(X,Y) \tag{20}$$

which shows that T is generalized recurrent. □

Theorem 3. *In a 4-dimensional generalized Ricci recurrent Riemannian manifold with generalized recurrent energy momentum tensor which is without cosmological constant, an Einstein tensor is generalized Ricci-recurrent if and only if the manifold is semi-Einstein.*

Proof. Further, if G is generalized Ricci-recurrent,then

$$G(X,Y)=(\nabla_Z S)(X,Y). \tag{21}$$

By virtue of covariant derivative of (7) with respect to Z we get

$$G(X,Y)=K(\nabla_Z T)(X,Y)+\frac{1}{2}dr(Z)g(X,Y). \tag{22}$$

Using (2), (7) and (11) in (22) we obtain

$$KT(X,Y)=E(Z)g(X,Y), \tag{23}$$

where $E(Z)=\frac{1}{1-\psi(Z)}[K\varphi(Z)+\frac{1}{2}\{rA(Z)+4B(Z)\}]$ provided $\psi(Z)\neq 1$. From (7) and (23) we have

$$S(X,Y)=P(Z)g(X,Y), \tag{24}$$

where $P(Z)=E(Z)+\frac{1}{2}r$, is a 1-form.

By virtue of definition (3), we assert that the manifold is semi-Einstein. □

Theorem 4. *A necessary and sufficient condition that the energy momentum tensor T with cosmological constant in a general relativistic space time is generalized recurrent is that space time is generalized Ricci-recurrent.*

Proof. Differentiating equation (8) covariantly with respect to Z, we obtain

$$(\nabla_Z S)(X,Y) - \frac{1}{2}dr(Z)g(X,Y) + d\lambda(Z)g(X,Y) = K(\nabla_Z T)(X,Y)$$
$$= K\{\psi(Z)T(X,Y) + \varphi(Z)g(X,Y)\}. \quad (25)$$

Using (2), (8) we get

$$(\nabla_Z S)(X,Y) = M(Z)S(X,Y) + N(Z)g(X,Y), \quad (26)$$

where M and N are 1-forms such that
$M(Z) = \psi(Z)$ *and* $N(Z) = \frac{1}{2}dr(Z) + K\varphi(Z) + (\lambda - \frac{1}{2}r)\psi(Z) - d\lambda(Z)$.

Equation (26) shows that the space time is generalized Ricci-recurrent.

Putting $X = Y = e_i$ in (26) we have

$$dr(Z) = rM(Z) + 4N(Z). \quad (27)$$

Conversely, we assume that the space time is generalized Ricci-recurrent, then from (2), (25) and (26) we obtain

$$K(\nabla_Z T)(X,Y) = M(Z)S(X,Y) + \{N(Z) - \frac{1}{2}dr(Z) + d\lambda(Z)\}g(X,Y). \quad (28)$$

From equations (8), (27) and (28) we have

$$(\nabla_Z T)(X,Y) = A(Z)T(X,Y) - \frac{1}{K}B(Z)g(X,Y), \quad (29)$$

where $A(Z) = M(Z)$ *and* $B(Z) = \{\lambda M(Z) + N(Z) - d\lambda(Z)\}$ and $M,\ N$ are 1-forms. Equation (29) shows that T is generalized recurrent. Hence the proof. □

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