Tamsui Oxford Journal of Mathematical Sciences **25(3)** (2009) 259-267
Aletheia University

Almost Everywhere Convergence of Inverse Dunkl Transform on the Real Line*

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Received November 3, 2007, Accepted February 19, 2008.

Abstract

We show that the spherical partial sums maximal operator S_*^α, associated to Dunkl transform on $\mathbb{R}$ is bounded on $L^p\big(\mathbb{R}, |x|^{2\alpha+1}dx\big)$, $p_0 < p < p_1$, where $p_0 = \dfrac{4(\alpha+1)}{2\alpha+3}$ and $p_1 = \dfrac{4(\alpha+1)}{2\alpha+1}$. This implies that, for every $f \in L^p\big(\mathbb{R}, |x|^{2\alpha+1}dx\big)$, $S_R^\alpha f$ converges to f a. e., as $R \to \infty$. On the other hand we obtain a sharp version by showing that S_*^α is bounded from the Lorentz space $L^{p_i,1}\big(\mathbb{R}, |x|^{2\alpha+1}\big)$ into $L^{p_i,\infty}\big(\mathbb{R}, |x|^{2\alpha+1}\big)$, $\quad i = 0, 1$.

Keywords and Phrases: *Dunkl transform,Maximal function,Almost everywhere convergence,Lorentz space.*

1. Introduction and preliminaries

Given $\alpha \geq -1/2$ and a suitable function f on $\mathbb{R}$, its Dunkl transform D_α is defined by

$$D_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-ixy) d\mu_\alpha(x), \quad y \in \mathbb{R}; \tag{1}$$

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*2000 *Mathematics Subject Classification.* 42B15, 42C10, 42B10, 31B10, 31B20.

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here

$$d\mu_\alpha(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} |x|^{2\alpha+1}\, dx, \tag{2}$$

$$E_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \left\{ \frac{J_\alpha(iz)}{(iz)^\alpha} + z \frac{J_{\alpha+1}(iz)}{(iz)^{\alpha+1}} \right\}, \tag{3}$$

where J_α denotes the Bessel function of the first kind of order α. The inverse Dunkl transform $\check{D_\alpha}$ is given by $\check{D_\alpha} f(\lambda) = D_\alpha f(-\lambda)$ (see [2] and [3]).

In this paper, we are interested in the almost everywhere convergence as $R \to \infty$, of the partial sums $S_R^\alpha f(x)$, where

$$S_R^\alpha f(x) = \int_{|y| \le R} D_\alpha f(y) E_\alpha(ixy) d\mu_\alpha(y).$$

Recall that given $\beta \ge -\frac{1}{2}$, the Hankel transform of order β of a suitable function g on $(0, \infty)$ is defined by

$$\mathcal{H}_\beta g(y) = \int_0^\infty g(x) \frac{J_\beta(yx)}{(yx)^\beta} x^{2\beta+1} dx, \quad y > 0. \tag{4}$$

Nowak and Stempak [6], found an expression of the Dunkl transform D_α in terms of Hankel transform of orders α and $\alpha + 1$.

Lemma 1.1. *(See ([6]) Given $\alpha \ge -\frac{1}{2}$, we have*

$$D_\alpha f(y) = \mathcal{H}_\alpha(f_e)(|y|) - iy\mathcal{H}_{\alpha+1}\left(\frac{f_o(x)}{x}\right)(|y|), \tag{5}$$

where for a function f on $\mathbb{R}$, we denote by f_e and f_o the restrictions to $(0, \infty)$ of its even and odd parts, respectively, i.e. the functions on $(0, \infty)$ defined by

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \quad x > 0.$$

Define, the partial sums $s_R^\beta g(x)$ by

$$s_R^\beta g(x) = \int_0^R \mathcal{H}_\beta g(y) \frac{J_\beta(xy)}{(xy)^\beta} y^{2\beta+1} dy, \quad x > 0, \tag{6}$$

and

$$s_*^\beta g(x) = \sup_{R>0} \left| s_R^\beta g(x) \right|. \tag{7}$$

In 1988, Kanjin [4] and Prestini [7] independently proved the following.

Theorem 1.2. *Let* $\beta \geq -\frac{1}{2}$ *and* $1 \leq p < \infty$.

- *If* $\frac{4(\beta+1)}{2\beta+3} < p < \frac{4(\beta+1)}{2\beta+1}$, *then* s_*^β *is bounded on* $L^p\left((0,\infty), x^{2\beta+1}\right)$.
- *If* $p \leq \frac{4(\beta+1)}{2\beta+3}$ *or* $p \geq \frac{4(\beta+1)}{2\beta+1}$, *then* s_*^β *is not bounded on* $L^p\left((0,\infty), x^{2\beta+1}\right)$.

Throughout this paper we use the convention that c_α denotes a constant, depending on α and p, its value may change from line to line.

2. Almost everywhere convergence

Define linear operators $S_R^\alpha, R > 0$ and S_*^α on the Schwartz space $S(\mathbb{R})$ by

$$S_R^\alpha f(x) = \int_{|y|\leq R} D_\alpha f(y) E_\alpha(ixy) d\mu_\alpha(y), \tag{8}$$

and

$$S_*^\alpha f(x) = \sup_{R>0} \left| S_R^\alpha f(x) \right|, \quad x \in \mathbb{R}. \tag{9}$$

We note that, by Proposition 5 in [5], $S_R^\alpha f$ can be defined for $f \in L^p(\mathbb{R}, d\mu_\alpha)$, $1 < p < p_1$, by

$$S_R^\alpha f(x) = \int_{\mathbb{R}} \phi_R(-y) \tau_x f(-y) d\mu_\alpha(y), \quad x \in \mathbb{R}, \tag{10}$$

where $\phi_R(x) = c_\alpha R^{2(\alpha+1)} j_{\alpha+1}(Rx), \quad x \in \mathbb{R}$, and $\tau_x, x \in \mathbb{R}$ are the so-called Dunkl translation operators on $\mathbb{R}$.

Lemma 2.1. *Given* $\alpha \geq -\frac{1}{2}$, *we have*

$$S_R^\alpha(f)(x) = s_R^\alpha(f_e)(|x|) + x s_R^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|), \tag{11}$$

$$S_*^\alpha f(x) \leq s_*^\alpha(f_e)(|x|) + |x|\, s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|). \tag{12}$$

Proof. Let $x \in \mathbb{R}$. By (3), (8) and Lemma 1.1, we have

$$\begin{aligned} S_R^\alpha f(x) &= \int_{|y|\le R} \left[\mathcal{H}_\alpha(f_e)(|y|) - iy\mathcal{H}_{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|y|)\right] \\ &\quad \left[2^\alpha \Gamma(\alpha+1)\left\{\frac{J_\alpha(yx)}{(yx)^\alpha} + ixy\frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}}\right\}\right] d\mu_\alpha(y) \\ &= \frac{1}{2}\int_{|y|\le R} \mathcal{H}_\alpha(f_e)(|y|)\frac{J_\alpha(yx)}{(yx)^\alpha}\,|y|^{2\alpha+1}\,dy \\ &\quad + \frac{ix}{2}\int_{|y|\le R} y\mathcal{H}_\alpha(f_e)(|y|)\frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}}\,|y|^{2\alpha+1}\,dy \\ &\quad - \frac{i}{2}\int_{|y|\le R} y\mathcal{H}_{\alpha+1}\left(\frac{f_o(r}{r}\right)(|y|)\,\frac{J_\alpha(yx)}{(yx)^\alpha}\,|y|^{2\alpha+1}\,dy \\ &\quad + \frac{x}{2}\int_{|y|\le R} \mathcal{H}_{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|y|)\,\frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}}\,|y|^{2\alpha+3}\,dy. \end{aligned}$$

We note that the second and the third integrals are equal to zero. So

$$\begin{aligned} S_R^\alpha f(x) &= \frac{1}{2}\int_{|y|\le R} \mathcal{H}_\alpha(f_e)(|y|)\frac{J_\alpha(yx)}{(yx)^\alpha}\,|y|^{2\alpha+1}\,dy \\ &\quad + \frac{x}{2}\int_{|y|\le R} \mathcal{H}_{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|y|)\,\frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}}\,|y|^{2\alpha+3}\,dy \\ &= \int_0^R \mathcal{H}_\alpha(f_e)(y)\frac{J_\alpha(|x|\,y)}{(|x|\,y)^\alpha}y^{2\alpha+1}dy \\ &\quad + x\int_0^R \mathcal{H}_{\alpha+1}\left(\frac{f_o(r)}{r}\right)(y)\,\frac{J_{\alpha+1}(|x|\,y)}{(|x|\,y)^{\alpha+1}}y^{2\alpha+3}dy \\ &= s_R^\alpha(f_e)(|x|) + xs_R^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|). \end{aligned}$$

Thus

$$S_*^\alpha f(x) \le s_*^\alpha(f_e)(|x|) + |x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|).$$

Proposition 2.2. *Let* $\alpha > -\frac{1}{2}$.

- *If* $p_0 < p < p_1$, *then* S_*^α *is bounded on* $L^p(\mathbb{R}, d\mu_\alpha(x))$.
- *If* $p \le p_0$ *or* $p \ge p_1$, *then* S_*^α *is not bounded on* $L^p(\mathbb{R}, d\mu_\alpha(x))$.

Proof. S_*^α cannot be bounded for $p \leq p_0$ or $p \geq p_1$ (see: [4] and [7]).
By Theorem 1, we have for $p_0 < p < p_1$,

$$\begin{aligned}\|s_*^\alpha(f_e)(|x|)\|_{L^p(\mathbb{R},d\mu_\alpha(x))} &= 2\,\|s_*^\alpha(f_e)\|_{L^p((0,\infty),x^{2\alpha+1}dx)}\\ &\leq c_\alpha\,\|f_e\|_{L^p((0,\infty),x^{2\alpha+1}dx)}\\ &\leq c_\alpha\,\|f\|_{L^p(\mathbb{R},d\mu_\alpha(x))}\,.\end{aligned}$$

On the other hand, as in ([7],[8]), one gets

$$|x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|) \leq \frac{c_\alpha}{|x|^{\alpha+\frac{1}{2}}}\left[M+H+\widetilde{H}+\widetilde{C}\right]\left[\frac{f_o(r)}{r}r^{\alpha+\frac{3}{2}}\right](|x|), \quad (13)$$

where $M, H, \widetilde{H}$ and $\widetilde{C}$ denotes respectively, the maximal function, the Hilbert integral, the maximal Hilbert transform and the Carleson operator.
Let $K = M + H + \widetilde{H} + \widetilde{C}$ and $w \in A_p(\mathbb{R})\,, p > 1$. It is well known that

$$\|Kf\|_{L^p(\mathbb{R},w(x)dx)} \leq c_p\,\|f\|_{L^p(\mathbb{R},w(x)dx)}\,. \quad (14)$$

Hence

$$\begin{aligned}&\left\||x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|)\right\|_{L^p(\mathbb{R},d\mu_\alpha(x))}\\ \leq\;& c_\alpha\left\||x|^{-\alpha-\frac{1}{2}}\,K\left[\frac{f_o(r)}{r}r^{\alpha+\frac{3}{2}}\right](|x|)\right\|_{L^p(\mathbb{R},d\mu_\alpha(x))}\\ \leq\;& c_\alpha\left\|K\left[\frac{f_o(r)}{r}r^{\alpha+3/2}\right](|x|)\right\|_{L^p(\mathbb{R},w(x)dx)},\end{aligned}$$

with $w(x) = |x|^{2\alpha+1-p(\alpha+1/2)}$.

Since $p_0 < p < p_1$ if and only if $-1 < 2\alpha + 1 - p(\alpha + 1/2) < p - 1$,

then $w \in A_p(\mathbb{R})$, and by (13)

$$\begin{aligned}\left\||x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|)\right\|_{L^p(\mathbb{R},d\mu_\alpha(x))} &\leq c_\alpha\left\|\frac{f_0(|x|)}{|x|}\,|x|^{\alpha+3/2}\right\|_{L^p(\mathbb{R},w(x)dx)}\\ &\leq c_\alpha\,\|f_o(x)\|_{L^p(\mathbb{R},d\mu_\alpha(x))}\\ &\leq c_\alpha\,\|f(x)\|_{L^p(\mathbb{R},d\mu_\alpha(x))}\,.\end{aligned}$$

We conclude by Lemma 2.1.

Using Proposition 2.2, and since almost everywhere convergence holds for functions on $S(\mathbb{R})$, which is a dense subset of $L^p(\mathbb{R}, d\mu_\alpha)$, (see [3]), we obtain

Corollary 2.3. *For every $f \in L^p(\mathbb{R}, d\mu_\alpha)$, if $p_0 < p < p_1$, then*

$$S_R^\alpha f(x) \to f(x) \quad a.e. \quad as \quad R \to \infty.$$

3. Endpoint estimates

We recall that the Lorentz space $L^{p,q}(X, \mu)$, is the set of all measurable functions f on X satisfying

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

when, $1 \leq p < \infty$, $1 \leq q < \infty$, and

$$\|f\|_{p,q} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\lambda>0} \lambda \left(d_f(\lambda)\right)^{\frac{1}{p}} < \infty$$

when, $1 \leq p \leq \infty$ and $q = \infty$. Here f^* denotes the nonincreasing rearrangement of f, i.e.

$$f^*(t) = \inf\left\{s > 0/d_f(s) \leq t\right\}, \qquad d_f(s) = \mu\left\{x \in X/\left|f(x)\right| > s\right\}.$$

In 1991, Romera and Soria [8] (see also Colzani and all [1]) proved the following

Theorem 3.1. *Let $\alpha > -\frac{1}{2}$, then s_*^α is bounded from the Lorentz space $L^{p_i,1}\left((0,\infty), x^{2\alpha+1}dx\right)$ into $L^{p_i,\infty}\left((0,\infty), x^{2\alpha+1}dx\right)$, i=0,1.*

Using this result, we will see that Proposition 2.2 can be strengthened. More precisely we obtain

Proposition 3.2. *Let $\alpha > -\frac{1}{2}$, then S_*^α is bounded from the Lorentz space $L^{p_i,1}(\mathbb{R}, d\mu_\alpha)$ into $L^{p_i,\infty}(\mathbb{R}, d\mu_\alpha)$, $i = 0, 1$.*

Using Marcinkiewicz's interpolation theorem in terms of Lorentz space we retrieve Proposition 2.2 as a corollary.

Proof. By Lemma 2.1, we have

$$\begin{aligned}\mu_\alpha\left\{x\in\mathbb{R}/S_*^\alpha f(x)>\lambda\right\} &\leq \mu_\alpha\left\{x\in\mathbb{R}/s_*^\alpha f_e(|x|)>\frac{\lambda}{2}\right\}\\ &\quad+\mu_\alpha\left\{x\in\mathbb{R}/\,|x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|)>\frac{\lambda}{2}\right\}.\\ &= I+II.\end{aligned}$$

By Theorem 2.4, we get :

$$\begin{aligned}\mu_\alpha\left\{x\in\mathbb{R}/s_*^\alpha f_e(|x|)>\frac{\lambda}{2}\right\} &= 2\mu_\alpha\left\{x\in(0,\infty)/s_*^\alpha f_e(x)>\frac{\lambda}{2}\right\}\\ &\leq \frac{c_\alpha}{\lambda^{p_i}}\left\|f_e\right\|_{p_i,1}\leq\frac{c_\alpha}{\lambda^{p_i}}\left\|f\right\|_{p_i,1}.\end{aligned}$$

To estimate II, we follow closely [8] and we sketch a proof for completeness. We decompose the set

$$\begin{aligned}&\left\{x\in\mathbb{R}/\,|x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|)>\frac{\lambda}{2}\right\}\\ =&\bigcup_{k\in\mathbb{Z}}\left\{x\in\mathbb{R}/\,|x|\in I_k,|x|\,s_*^{\alpha+1}\left(\frac{f_o(r)}{r}\right)(|x|)>\frac{\lambda}{2}\right\},\end{aligned}$$

where $I_k=[2^k,2^{k+1}[$.

Put $g(r):=\dfrac{f_o(r)}{r}=g_k^1(r)+g_k^2(r)$, with $g_k^1=g\chi_{I_k^*}$, $g_k^2=g\chi_{(I_k^*)^c}$, where $I_k^*=]2^{k-1},2^{k+2}[$.

By (12), we have

$$|x|\,s_*^{\alpha+1}\left(g_k^1(r)\right)(|x|)\leq\frac{c_\alpha}{|x|^{\alpha+1/2}}K\left(g_k^1(r)r^{\alpha+3/2}\right)(|x|).$$

By ([8], p: 1021), we have for $1 < p < \infty$,

$$\sum_{k\in\mathbb{Z}} \mu_\alpha \left\{ x \in \mathbb{R}/ \left|x\right| \in I_k, \frac{1}{\left|x\right|^{\alpha+1/2}} K\left(g_k^1(r) r^{\alpha+3/2}\right)(\left|x\right|) > \frac{\lambda}{2} \right\}$$

$$\leq \frac{c_\alpha}{\lambda^p} \left\|f_o\right\|^p_{L^p(\mathbb{R},d\mu_\alpha(x))} \leq \frac{c_\alpha}{\lambda^p} \left\|f\right\|^p_{L^p(\mathbb{R},d\mu_\alpha(x))} \leq \frac{c_\alpha}{\lambda^p} \left\|f\right\|^p_{p,1}.$$

On the other hand as in ([8], p: 1021), we have

$$\begin{aligned} \left|x\right| s_*^{\alpha+1}\left(g_k^2(r)\right)(\left|x\right|) &\leq \frac{c_\alpha}{\left|x\right|^{\alpha+1/2}} \int_0^\infty \frac{s^{\alpha+3/2}\left|f_o(s)\right|}{s(\left|x\right|+s)} ds \\ &\leq \frac{c_\alpha}{\left|x\right|^{\alpha+3/2}} \int_0^\infty \left|f_o(s)\right| s^{\alpha+1/2} ds \\ &\leq \frac{c_\alpha}{\left|x\right|^{\alpha+3/2}} \int_{\mathbb{R}} \left|f_o(s)\right| \frac{1}{\left|s\right|^{\alpha+1/2}} d\mu_\alpha(s). \end{aligned}$$

Remark that we have considered f_0 as a function defined on $\mathbb{R}$.
As the same we get,

$$\begin{aligned} \left|x\right| s_*^{\alpha+1}\left(g_k^2(r)\right)(\left|x\right|) &\leq \frac{c_\alpha}{\left|x\right|^{\alpha+1/2}} \int_0^\infty \left|f_o(s)\right| s^{\alpha-1/2} ds \\ &\leq \frac{c_\alpha}{\left|x\right|^{\alpha+1/2}} \int_{\mathbb{R}} \left|f_o(s)\right| \frac{1}{\left|s\right|^{\alpha+3/2}} d\mu_\alpha(s). \end{aligned}$$

Using the following facts

$$\frac{1}{\left|x\right|^{\alpha+\frac{1}{2}}} \in L^{p_1,\infty}\left(\mathbb{R}, d\mu_\alpha(x)\right),$$

$$\frac{1}{\left|x\right|^{\alpha+\frac{3}{2}}} \in L^{p_0,\infty}\left(\mathbb{R}, d\mu_\alpha(x)\right),$$

and Holder's inequality for the Lorentz spaces, we arrive to

$$\mu_\alpha \left\{ x \in \mathbb{R}/ \left|x\right| s_*^{\alpha+1}\left(g_k^2(r)\right)(\left|x\right|) > \frac{\lambda}{2} \right\} \leq \frac{c_\alpha}{\lambda^{p_i}} \left\|f_o\right\|^{p_i}_{p_i,1} \leq \frac{c_\alpha}{\lambda^{p_i}} \left\|f\right\|^{p_i}_{p_i,1},$$

which completes the proof.

Acknowledgment

We are grateful to Professor K. Stempak for sending us the preprint [6].

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