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On the Hermite-hadamard Type Inequalities for Convex Functions and Convex Functions on the Co-Ordinates in a Rectangle from the Plane*

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Abstract

In this paper, we establish some inequalities of Hermite-Hadamard type for convex functions and convex functions on the co-ordinates defined in a rectangle from the plane.

Keywords and Phrases: *Hermite-Hadamard inequality, Convex function, Convex function on the co-ordinates.*

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1. Introduction

Let I be an interval in R, $f : I \subseteq R \to R$ be a convex function and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hermite-Hadamard inequality for convex function (see for example [6]).

For some results which generalize, improve and extend Hermite-Hadamard inequality (1) see $[1-18]$.

In [3], Dragomir and Buşe established the following refinements of the inequality (1):

Theorem A. Let n be a natural number, $q_i \ge 0$ $(i = 1, \cdots, n)$ and $Q_n = \sum_{i=1}^{n} q_i > 0$. If I is an interval in R, $f : I \subseteq R \to R$ is convex and $a, b \in I$ with $a < b$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n} dx_i \\ &\le \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right)\prod_{i=1}^{n} dx_i \\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned} \tag{2}$$

In [18] , Yang and Wang established the following two theorems which give some refinements of the inequality (2):

Theorem B. Let $0 < \alpha_i < 1$ $(i = 1, \cdots, n;\ n \ge 2)$ with $\sum_{i=1}^{n} \alpha_i = 1$ and, let f

be defined as in Theorem A. Then

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\sum_{i=1}^n \alpha_i x_i\right)\prod_{i=1}^n dx_i \\ \leq\ & \sum_{i=1}^n \frac{1-\alpha_i}{n-1}\frac{1}{(b-a)^{n-1}}\left[\int_a^b\cdots\int_a^b f\left(\sum_{j=1,j\neq i}^n \frac{\alpha_j x_j}{1-\alpha_i}\right)\prod_{k=1,k\neq i}^n dx_k\right] \\ \leq\ & \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned} \tag{3}$$

Theorem C. Let n be a natural number, $0 \leq \alpha_i \leq 1$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^n \alpha_i = 1$ and let $\Phi : [0,1] \to R$ be defined by

$$\Phi(t) = \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(t\sum_{i=1}^n \alpha_i x_i + (1-t)\frac{a+b}{2}\right)\prod_{i=1}^n dx_i,\ t \in [0,1].$$

Then Φ is convex, increasing on $[0,1]$ and, for $t \in [0,1]$,

$$f\left(\frac{a+b}{2}\right) = \Phi(0) \leq \Phi(t) \leq \Phi(1) = \int_a^b\cdots\int_a^b f\left(\sum_{i=1}^n \alpha_i x_i\right)\prod_{i=1}^n dx_i. \tag{4}$$

Let $D \subseteq R^2$. In [2], a function $F : D \to R$ will be called *convex on the co-ordinates on* D if the partial mapping $F_y(x) := F(x,y)$ is convex in x for each fixed y, and the partial mapping $F_x(y) := F(x,y)$ is convex in y for each fixed x where $(x,y) \in D$. In [7], the author calls such a function convex separately with respect to each coodinate.

In [7], Lanina established the following two theorems:

Theorem D. Suppose
(*a*) $F : D \to R$ is convex on the co-ordinates on D where $D \subseteq R^2$;
(*b*) $\Delta = [a,b] \times [c,d] \subseteq D$ with $a < b$ and $c < d$;
(*c*) n, q_i $(i = 1, \cdots n)$ and Q_n are defined as in Theorem A.

Then

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{(b-a)^n(d-c)^n}\int_\Delta\cdots\int_\Delta F\left(\frac{1}{n}\sum_{i=1}^n x_i,\frac{1}{n}\sum_{j=1}^n y_j\right)\prod_{i=1}^n dx_i\prod_{j=1}^n dy_j \\
\leq\ & \frac{1}{(b-a)^n(d-c)^n}\int_\Delta\cdots\int_\Delta F\left(\frac{1}{Q_n}\sum_{i=1}^n q_i x_i,\frac{1}{Q_n}\sum_{j=1}^n q_j y_j\right)\prod_{i=1}^n dx_i\prod_{j=1}^n dy_j \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b F(x,y)\,dxdy. \qquad (5)
\end{aligned}$$

Theorem E. Let D, Δ, n, q_i $(i=1,\cdots n)$ and Q_n be defined as in Theorem D. If $F:D\to R$ is convex, then the inequality (5) also holds.

In [2], Dragomir established the following two theorems:

Theorem F. *Let $\Delta=[a,b]\times[c,d]\subset R^2$, $F:\Delta\to R$ be convex on the co-ordinates on Δ and let $H:[0,1]^2\to R$ be defined by*

$$\begin{aligned}
&H(t,s) \\
=\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\times \\
& dydx. \qquad (6)
\end{aligned}$$

Then:
(a) The function H is convex on the co-ordinates on $[0,1]^2$.
(b) The function H is increasing on the co-ordinates on $[0,1]^2$,

$$\sup_{(t,s)\in[0,1]^2} H(t,s)=H(1,1)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dydx$$

and

$$\inf_{(t,s)\in[0,1]^2} H(t,s)=H(0,0)=F\left(\frac{a+b}{2},\frac{c+d}{2}\right).$$

Theorem G. *Let $\Delta=[a,b]\times[c,d]\subset R^2$, $F:\Delta\to R$ be convex on Δ and let $h:[0,1]\to R$ be defined by $h(t)=H(t,t)$* where H is defined as in (6).

Then:
(a) *The function* H *is convex on* $[0,1]^2$.
(b) *The function* h *is convex and increasing on* $[0,1]$,

$$\sup_{t\in[0,1]} h(t) = h(1) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dydx$$

and

$$\inf_{t\in[0,1]} h(t) = h(0) = F\left(\frac{a+b}{2},\frac{c+d}{2}\right).$$

In this paper, we shall establish some inequalities for convex functions and convex functions on the co-ordinates related to Theorems A-G.

2. Main Results

In this section, we assume that m and n are natural numbers, $D \subseteq R^2$, $\Delta = [a,b]\times[c,d] \subseteq D$, $\Delta_x^n = [a,b]^n$ and $\Delta_y^m = [c,d]^m$ with $a<b$ and $c<d$. A function $H:[0,1]^2 \to R$ will be called *increasing on the co-ordinates on* $[0,1]^2$ if the partial mapping $H_s:[0,1]\to R$, $H_s(t) := H(t,s)$ is *increasing* on $[0,1]$ for each $s\in[0,1]$, and the partial mapping $H_t:[0,1]\to R$, $H_t(s) := H(t,s)$ is *increasing* on $[0,1]$ for each $t\in[0,1]$.

Theorem 1. Let $0\le\alpha_i\le 1$ $(i=1,\cdots,n)$ and $0\le\beta_j\le 1$ $(j=1,\cdots,m)$ with $\sum\limits_{i=1}^{n}\alpha_i = 1$ and $\sum\limits_{j=1}^{m}\beta_j = 1$. If $F: D\to R$ is convex on the co-ordinates on D, then

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\le\;& \frac{1}{(b-a)^n(d-c)^m}\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^{n}x_i,\frac{1}{m}\sum_{j=1}^{m}y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j\\
\le\;& \frac{1}{(b-a)^n(d-c)^m}\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^{n}\alpha_i x_i,\sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j\\
\le\;& \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b F(x,y)\,dxdy. \qquad (7)
\end{aligned}$$

Proof. Since F is convex on the co-ordinates on D, we have

$$
\begin{aligned}
&\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^n x_i, \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
&= \int_{\Delta_y^m}\int_{\Delta_x^n} \frac{1}{2}\left[F\left(\frac{1}{n}\sum_{i=1}^n x_i, \frac{1}{m}\sum_{j=1}^m y_j\right) + \right. \\
&\qquad \left. F\left(\frac{1}{n}\left(a+b-x_1+\sum_{i=2}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\right]\prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
&\geq \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\frac{1}{n}\left(\frac{a+b}{2}+\sum_{i=2}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
&= (b-a)\int_{\Delta_y^m}\int_{\Delta_x^{n-1}} F\left(\frac{1}{n}\left(\frac{a+b}{2}+\sum_{i=2}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{i=2}^n dx_i \prod_{j=1}^m dy_j \\
&= (b-a)\int_{\Delta_y^m}\int_{\Delta_x^{n-1}} \frac{1}{2}\left[F\left(\frac{1}{n}\left(\frac{a+b}{2}+\sum_{i=2}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right) + \right. \\
&\qquad \left. F\left(\frac{1}{n}\left(\frac{a+b}{2}+a+b-x_2+\sum_{i=3}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\right]\prod_{i=2}^n dx_i \prod_{j=1}^m dy_j \\
&\geq (b-a)\int_{\Delta_y^m}\int_{\Delta_x^{n-1}} F\left(\frac{1}{n}\left(2\times\frac{a+b}{2}+\sum_{i=3}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{i=2}^n dx_i \prod_{j=1}^m dy_j \\
&= (b-a)^2\int_{\Delta_y^m}\int_{\Delta_x^{n-2}} F\left(\frac{1}{n}\left(2\times\frac{a+b}{2}+\sum_{i=3}^n x_i\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{i=3}^n dx_i \prod_{j=1}^m dy_j \\
&\qquad \vdots \\
&\geq (b-a)^n\int_{\Delta_y^m} F\left(\frac{1}{n}\left(n\times\frac{a+b}{2}\right), \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{j=1}^m dy_j \\
&= (b-a)^n\int_{\Delta_y^m} F\left(\frac{a+b}{2}, \frac{1}{m}\sum_{j=1}^m y_j\right)\prod_{j=1}^m dy_j. \qquad (8)
\end{aligned}
$$

Using a similar argument as the proof of the inequality (8), we have the following inequality

$$\begin{aligned}
&\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
&\geq (b-a)^n \int_{\Delta_y^m} F\left(\frac{a+b}{2}, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{j=1}^{m} dy_j \\
&\geq (b-a)^n (d-c)^m F\left(\frac{a+b}{2}, \frac{c+d}{2}\right). \qquad (9)
\end{aligned}$$

Multiplying each side of (8) and (9) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain the first inequality of (7).

To proof the second inequality and third inequalities of (7), we define $\alpha_{n+t} = \alpha_t$ $(t = 1, \cdots, n-1)$ and $\beta_{m+s} = \beta_s$ $(s = 1, \cdots, m-1)$. Then, by a simple computation, we have the following two identities

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{t=0}^{n-1}\frac{1}{n}\left(\sum_{l=1}^{n}\alpha_{l+t}x_l\right) \qquad (10)$$

and

$$\frac{1}{m}\sum_{j=1}^{m} y_j = \sum_{s=0}^{m-1}\frac{1}{m}\left(\sum_{r=1}^{m}\beta_{r+s}y_r\right) \qquad (11)$$

where $(x_i, y_j) \in \Delta\,(i = 1, \cdots, n; j = 1, \cdots, m)$. Since F is convex on the coordinates on D, using the identity (10), we obtain

$$\begin{aligned}
&\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
= &\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{t=0}^{n-1}\frac{1}{n}\left(\sum_{l=1}^{n}\alpha_{l+t}x_l\right), \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
\leq &\int_{\Delta_y^m}\left[\sum_{t=0}^{n-1}\frac{1}{n}\int_{\Delta_x^n} F\left(\sum_{l=1}^{n}\alpha_{l+t}x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i\right]\prod_{j=1}^{m} dy_j \\
\leq &\int_{\Delta_y^m}\left[\sum_{t=0}^{n-1}\frac{1}{n}\sum_{l=1}^{n}\alpha_{i+t}\int_{\Delta_x^n} F\left(x_l, \frac{1}{m}\sum_{j=1}^{m} y_j\right)\prod_{i=1}^{n} dx_i\right]\prod_{j=1}^{m} dy_j
\end{aligned}$$

$$
\begin{aligned}
&= (b-a)^{n-1}\int_{\Delta_y^m}\left[\sum_{l=0}^{n-1}\frac{1}{n}\sum_{l=1}^{n}\alpha_{i+l}\int_a^b F\left(x,\frac{1}{m}\sum_{j=1}^{m}y_j\right)dx\right]\prod_{j=1}^{m}dy_j \\
&= (b-a)^{n-1}\int_{\Delta_y^m}\int_a^b F\left(x,\frac{1}{m}\sum_{j=1}^{m}y_j\right)dx\prod_{j=1}^{m}dy_j. \qquad (12)
\end{aligned}
$$

Similarly, using the convexity of F on the co-ordinates on D and the identity (11), we obtain the following inequality

$$
\begin{aligned}
&(b-a)^{n-1}\int_c^d\cdots\int_c^d\int_a^b F\left(x,\frac{1}{m}\sum_{j=1}^{m}y_j\right)dx\prod_{j=1}^{m}dy_j \\
&= (b-a)^{n-1}\int_a^b\left[\int_c^d\cdots\int_c^d F\left(x,\frac{1}{m}\sum_{j=1}^{m}y_j\right)\prod_{j=1}^{m}dy_j\right]dx \\
&\leq (b-a)^{n-1}\int_a^b\left[(d-c)^{m-1}\int_c^d F(x,y)\,dy\right]dx \\
&= (b-a)^{n-1}(d-c)^{m-1}\int_c^d\int_a^b F(x,y)\,dxdy. \qquad (13)
\end{aligned}
$$

Multiplying (12) and (13) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain the second and third inequalities of (7). This completes the proof.

Remark 1. Theorem 1 reduced to Theorem D if we choose $m=n, \beta_i=\alpha_i=\frac{q_i}{Q_n}\,(i=1,\cdots,n)$ where $q_i\,(i=1,\cdots,n)$ and Q_n are defined as in Theorem D.

Remark 2. Let f be defined as in Theorem A. If we choose $F(x,y)=f(x)$ $((x,y)\in\Delta)$, then Theorem 1 reduces to Theorem A.

By convexity, it is clear that all convex functions on D are convex on the co-ordinates on D. Thus, the following corollary is a simple consequence of Theorem 1.

Corollary 1. In Theorem 1, let $F:D\to R$ be convex. Then the inequality (7) also holds.

Remark 3. Corollary 1 reduced to Theorem E if we choose $m=n$, $\beta_i=$

$\alpha_i = \frac{q_i}{Q_n}\,(i = 1, \cdots, n)$ where $q_i\,(i = 1, \cdots, n)$ and Q_n are defined as in Theorem E.

Theorem 2. Let $n, m \geq 2$, $0 < \alpha_i < 1$ $(i = 1, \cdots, n)$ and $0 < \beta_j < 1$ $(j = 1, \cdots, m)$ with $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{j=1}^{m} \beta_j = 1$. If $F : D \to R$ is convex on the co-ordinates on D, then

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j\right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \\
&\leq \frac{1}{(b-a)^{n-1} (d-c)^{m-1}} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{1-\beta_j}{m-1} \frac{1-\alpha_i}{n-1} \\
&\quad \times \left[\int_{\Delta_y^{m-1}} \int_{\Delta_x^{n-1}} F\left(\sum_{s=1, s\neq i}^{n} \frac{\alpha_s x_s}{1-\alpha_i}, \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j}\right) \prod_{k=1, k\neq i}^{n} dx_k \prod_{l=1, l\neq j}^{m} dy_j\right] \\
&\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b F(x, y)\, dx dy. \qquad (14)
\end{aligned}$$

Proof. The first inequality of (14) follows immediately from Theorem 1. By a simple computation, we have the following two identities

$$\sum_{i=1}^{n} \alpha_i x_i = \frac{1}{n-1} \sum_{i=1}^{n} \sum_{s=1, s\neq i}^{n} \alpha_s x_s = \sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \sum_{s=1, s\neq i}^{n} \frac{\alpha_s x_s}{1-\alpha_i} \qquad (15)$$

and

$$\sum_{j=1}^{m} \beta_j y_j = \frac{1}{m-1} \sum_{j=1}^{m} \sum_{t=1, t\neq j}^{m} \beta_t y_t = \sum_{j=1}^{m} \frac{1-\beta_j}{m-1} \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j} \qquad (16)$$

where $(x_i, y_j) \in \Delta\,(i = 1, \cdots, n; j = 1, \cdots, m)$. Since $0 < \frac{1-\alpha_i}{n-1} < 1\,(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} = 1$, by the convexity of F on the co-ordinates on D and the identity (15), we obtain

$$\begin{aligned}
&\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^{n}\alpha_i x_i, \sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j \\
=\ &\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{j=1}^{m}\beta_j y_j\right)\prod_{k=1}^{n}dx_k\prod_{j=1}^{m}dy_j \\
\leq &\int_{\Delta_y^m}\left[\int_{\Delta_x^n}\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{j=1}^{m}\beta_j y_j\right)\prod_{k=1}^{n}dx_k\right]\prod_{j=1}^{m}dy \\
=\ &(b-a)\int_{\Delta_y^m}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\left(\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{j=1}^{m}\beta_j y_j\right)\times\right.\right. \\
&\left.\left.\prod_{k=1,k\neq i}^{n}dx_k\right)\right]\prod_{j=1}^{m}dy_j \qquad (17)
\end{aligned}$$

Similarly, using the convexity of F on the co-ordinates on D and the identity (16), we obtain the following inequality

$$\begin{aligned}
&(b-a)\int_{\Delta_y^m}\left[\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{j=1}^{m}\beta_j y_j\right)\prod_{k=1,k\neq i}^{n}dx_k\right]\prod_{j=1}^{m}dy_j \\
=\ &(b-a)\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\int_{\Delta_x^{n-1}}\left\{\int_{\Delta_y^m}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{j=1}^{m}\beta_j y_j\right)\prod_{j=1}^{m}dy_j\right\}\prod_{k=1,k\neq i}^{n}dx_k \\
\leq\ &(b-a)\sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\int_{\Delta_x^{n-1}}\left\{(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\times\right. \\
&\left.\left[\int_{\Delta_y^{m-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{l=1,l\neq j}^{m}dy_j\right]\right\}\prod_{k=1,k\neq i}^{n}dx_k \\
=\ &(b-a)(d-c)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1-\beta_j}{m-1}\frac{1-\alpha_i}{n-1}\times \\
&\left[\int_{\Delta_y^{m-1}}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i}, \sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\prod_{l=1,l\neq j}^{m}dy_j\right]. \ (18)
\end{aligned}$$

Since $0 < \frac{\alpha_s}{1-\alpha_i} < 1$ $(i, s = 1, \cdots, n; s \neq i)$, $\sum\limits_{s=1, s\neq i}^{n} \frac{\alpha_s}{1-\alpha_i} = 1\,(i = 1, \cdots, n)$ and $\sum\limits_{i=1}^{n} \frac{1-\alpha_i}{n-1} = 1$, by the convexity of F on the co-ordinates on D, we obtain

$$
\begin{aligned}
& (b-a)(d-c) \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{1-\beta_j}{m-1} \frac{1-\alpha_i}{n-1} \times \\
& \qquad \left[\int_{\Delta_y^{m-1}} \int_{\Delta_x^{n-1}} F\left(\sum_{s=1, s\neq i}^{n} \frac{\alpha_s x_s}{1-\alpha_i}, \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j} \right) \prod_{k=1, k\neq i}^{n} dx_k \prod_{l=1, l\neq j}^{m} dy_j \right] \\
= \; & (b-a)(d-c) \sum_{j=1}^{m} \frac{1-\beta_j}{m-1} \left\{ \int_{\Delta_y^{m-1}} \left[\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \times \right.\right. \\
& \qquad \left.\left. \int_{\Delta_x^{n-1}} F\left(\sum_{s=1, s\neq i}^{n} \frac{\alpha_s x_s}{1-\alpha_i}, \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j} \right) \prod_{k=1, k\neq i}^{n} dx_k \right] \prod_{l=1, l\neq j}^{m} dy_j \right\} \\
\leq \; & (b-a)(d-c) \sum_{j=1}^{m} \frac{1-\beta_j}{m-1} \left\{ \int_{\Delta_y^{m-1}} \left[\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \sum_{s=1, s\neq i}^{n} \frac{\alpha_s}{1-\alpha_i} \times \right.\right. \\
& \qquad \left.\left. \int_{\Delta_x^{n-1}} F\left(x_s, \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j} \right) \prod_{k=1, k\neq i}^{n} dx_k \right) \right] \prod_{l=1, l\neq j}^{m} dy_j \right\} \\
= \; & (b-a)^{n-1}(d-c) \sum_{j=1}^{m} \frac{1-\beta_j}{m-1} \left\{ \int_{\Delta_y^{m-1}} \left[\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \sum_{s=1, s\neq i}^{n} \frac{\alpha_s}{1-\alpha_i} \times \right.\right. \\
& \qquad \left.\left. \int_a^b F\left(x_s, \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j} \right) dx_s \right] \prod_{l=1, l\neq j}^{m} dy_j \right\} \\
= \; & (b-a)^{n-1}(d-c) \sum_{j=1}^{m} \frac{1-\beta_j}{m-1} \left\{ \left(\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \sum_{s=1, s\neq i}^{n} \frac{\alpha_s}{1-\alpha_i} \right) \times \right. \\
& \qquad \left. \left[\int_{\Delta_y^{m-1}} \int_a^b F\left(x, \sum_{t=1, t\neq j}^{m} \frac{\beta_t y_t}{1-\beta_j} \right) dx \prod_{l=1, l\neq j}^{m} dy_j \right] \right\}
\end{aligned}
$$

$$= (b-a)^{n-1}(d-c)\left\{\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\int_{\Delta_y^{m-1}}\int_a^b F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\times dx\prod_{l=1,l\neq j}^{m}dy_j\right\} \quad (19)$$

Similarly, using the convexity of F on the co-ordinates on D, we obtain the following inequality

$$\begin{aligned}
&(b-a)^{n-1}(d-c)\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\left[\int_{\Delta_y^{m-1}}\int_a^b F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)dx\prod_{l=1,l\neq j}^{m}dy_j\right]\\
=\ &(b-a)^{n-1}(d-c)\int_a^b\left[\sum_{j=1}^{m}\frac{1-\beta_j}{m-1}\int_{\Delta_y^{m-1}}F\left(x,\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{l=1,l\neq j}^{m}dy_j\right]dx\\
\leq\ &(b-a)^{n-1}(d-c)^{m-1}\int_a^b\left[\int_c^d F(x,y)\,dy\right]dx\\
=\ &(b-a)^{n-1}(d-c)^{m-1}\int_c^d\int_a^b F(x,y)\,dxdy. \qquad (20)
\end{aligned}$$

Using the inequalities $(17)-(20)$, we have

$$\begin{aligned}
&\int_{\Delta_y^m}\int_{\Delta_x^n}F\left(\sum_{i=1}^{n}\alpha_i x_i,\sum_{j=1}^{m}\beta_j y_j\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j\\
\leq\ &(b-a)(d-c)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1-\beta_j}{m-1}\frac{1-\alpha_i}{n-1}\times\\
&\left[\int_{\Delta_y^{m-1}}\int_{\Delta_x^{n-1}}F\left(\sum_{s=1,s\neq i}^{n}\frac{\alpha_s x_s}{1-\alpha_i},\sum_{t=1,t\neq j}^{m}\frac{\beta_t y_t}{1-\beta_j}\right)\prod_{k=1,k\neq i}^{n}dx_k\prod_{l=1,l\neq j}^{m}dy_j\right]\\
\leq\ &(b-a)^{n-1}(d-c)^{m-1}\int_c^d\int_a^b F(x,y)\,dxdy. \qquad (21)
\end{aligned}$$

Multiplying (21) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain the second and third inequalities of (14). This completes the proof.

Remark 4. Let f be defined as in Theorem B. If we choose $F(x,y) = f(x)$ $((x,y) \in \Delta)$, then Theorem 2 reduces to Theorem B.

Remark 5. In Theorem 2, the inequality (14) refines the inequality (7).

Theorem 3. Let F, α_i $(i = 1, \cdots, n)$ and β_j $(j = 1, \cdots, m)$ be defined as in Theorem 1 and let $G : [0,1]^2 \to R$ be defined by

$$\begin{aligned} & G(t,s) \\ = & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(t \sum_{i=1}^{n} \alpha_i x_i + (1-t) \frac{a+b}{2}, \right. \\ & \left. s \sum_{j=1}^{m} \beta_j y_j + (1-s) \frac{c+d}{2} \right) \times \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j . \quad (22) \end{aligned}$$

Then:

(a) *The function G is convex on the co-ordinates on $[0,1]^2$.*

(b) *The function G is increasing on the co-ordinates on $[0,1]^2$,*

$$\begin{aligned} & \sup_{(t,s)\in[0,1]^2} G(t,s) = G(1,1) \\ = & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \right) \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \end{aligned}$$

and

$$\inf_{(t,s)\in[0,1]^2} G(t,s) = G(0,0) = F\left(\frac{a+b}{2}, \frac{c+d}{2} \right).$$

Proof. (a) Fix $s \in [0,1]$. Since F is convex on the co-ordinates on Δ, we have for t_1, $t_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$ that

$$\begin{aligned} & \int_{\Delta_y^m} \int_{\Delta_x^n} F\left((\alpha t_1 + \beta t_2) \sum_{i=1}^{n} \alpha_i x_i + (1 - (\alpha t_1 + \beta t_2)) \frac{a+b}{2}, s \sum_{j=1}^{m} \beta_j y_j + (1-s) \frac{c+d}{2} \right) \\ & \times \prod_{i=1}^{n} dx_i \prod_{j=1}^{m} dy_j \end{aligned}$$

$$= \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(\alpha\left(t_1\sum_{i=1}^{n}\alpha_i x_i + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2\sum_{i=1}^{n}\alpha_i x_i + (1-t_2)\frac{a+b}{2}\right),\right.$$
$$\left. s\sum_{j=1}^{m}\beta_j y_j + (1-s)\frac{c+d}{2}\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j$$

$$\leq \int_{\Delta_y^m}\int_{\Delta_x^n}\left[\alpha F\left(t_1\sum_{i=1}^{n}\alpha_i x_i + (1-t_1)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j + (1-s)\frac{c+d}{2}\right) + \right.$$
$$\left.\beta F\left(t_2\sum_{i=1}^{n}\alpha_i x_i + (1-t_2)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j + (1-s)\frac{c+d}{2}\right)\right] \times$$
$$\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j. \quad (23)$$

Multiplying (23) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain

$$G(\alpha t_1 + \beta t_2, s) \leq \alpha G(t_1, s) + \beta G(t_2, s)$$

Similarly, if t is fixed in $[0,1]$, then for $s_1,\ s_2 \in [0,1]$ and $\alpha,\ \beta \geq 0$ with $\alpha + \beta = 1$, we have

$$G(t, \alpha s_1 + \beta s_2) \leq \alpha G(t, s_1) + \beta G(t, s_2)$$

and the statement is proved.

(b) Since F is convex on the co-ordinates on Δ, we have, for all $(t,s) \in [0,1]^2$,

$$\int_{\Delta_y^m}\int_{\Delta_x^n} F\left(t\sum_{i=1}^{n}\alpha_i x_i + (1-t)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j + (1-s)\frac{c+d}{2}\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j$$

$$= \int_{\Delta_y^m}\int_{\Delta_x^n}\frac{1}{2}\left[F\left(t\sum_{i=1}^{n}\alpha_i x_i + (1-t)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j + (1-s)\frac{c+d}{2}\right) + \right.$$
$$\left. F\left(t\left(\alpha_1(a+b-x_1) + \sum_{i=2}^{n}\alpha_i x_i\right) + (1-t)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j + (1-s)\frac{c+d}{2}\right)\right]$$
$$\times\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j$$

$$\begin{aligned}
&\geq \int_{\Delta_y^m}\int_{\Delta_x^n} F\left(t\left(\alpha_1\cdot\frac{a+b}{2}+\sum_{i=2}^{n}\alpha_i x_i\right)+(1-t)\frac{a+b}{2}\right.,\\
&\qquad\qquad \left. s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{i=1}^{n}dx_i\prod_{j=1}^{m}dy_j\\
&= (b-a)\int_{\Delta_y^m}\int_{\Delta_x^{n-1}} F\left(t\left(\alpha_1\cdot\frac{a+b}{2}+\sum_{i=2}^{n}\alpha_i x_i\right)+(1-t)\frac{a+b}{2}\right.,\\
&\qquad\qquad \left. s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{i=2}^{n}dx_i\prod_{j=1}^{m}dy_j\\
&\quad\vdots\\
&\geq (b-a)^2\int_{\Delta_y^m}\int_{\Delta_x^{n-2}} F\left(t\left((\alpha_1+\alpha_2)\frac{a+b}{2}+\sum_{i=3}^{n}\alpha_i x_i\right)+(1-t)\frac{a+b}{2}\right.,\\
&\qquad\qquad \left. s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{i=3}^{n}dx_i\prod_{j=1}^{m}dy_j\\
&\quad\vdots\\
&\geq (b-a)^n\int_{\Delta_y^m} F\left(t\left(\sum_{i=1}^{n}\alpha_i\right)\frac{a+b}{2}+(1-t)\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\\
&\qquad\qquad \times\prod_{j=1}^{m}dy_j\\
&= (b-a)^n\int_{\Delta_y^m} F\left(\frac{a+b}{2}, s\sum_{j=1}^{m}\beta_j y_j+(1-s)\frac{c+d}{2}\right)\prod_{j=1}^{m}dy_j. \qquad (24)
\end{aligned}$$

Multiplying (24) by $\frac{1}{(b-a)^n(d-c)^m}$, we obtain

$$G(t,s)\geq G(0,s). \qquad (25)$$

Similarly, we have, for all $(t,s)\in[0,1]^2$,

$$G(t,s)\geq G(t,0). \qquad (26)$$

If $0 < t_1 < t_2 \le 1$ and $0 < s_1 < s_2 \le 1$, then, for all $(t,s) \in [0,1]^2$, it follows from the convexity of G on the co-ordinates on $[0,1]^2$, (25) and (26) that we have

$$\frac{G(t_2,s) - G(t_1,s)}{t_2 - t_1} \ge \frac{G(t_1,s) - G(0,s)}{t_1 - 0} \ge 0$$

and

$$\frac{G(t,s_2) - G(t,s_1)}{s_2 - s_1} \ge \frac{G(t,s_1) - G(t,0)}{s_1 - 0} \ge 0$$

which show that

$$G(t_1,s) \le G(t_2,s) \tag{27}$$

and

$$G(t,s_1) \le G(t,s_2). \tag{28}$$

By $(25) - (28)$, we obtain that G is increasing on the co-ordinates on $[0,1]^2$. Hence

$$\begin{aligned} &\sup_{(t,s)\in[0,1]^2} G(t,s) = G(1,1) \\ = \;& \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \end{aligned}$$

and

$$\inf_{(t,s)\in[0,1]^2} G(t,s) = G(0,0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

This completes the proof.

Remark 6. Let f be defined as in Theorem F. If we choose $m = n = 1$, then Theorem 3 reduces to Theorem F.

Theorem 4. Let α_i $(i = 1, \cdots, n)$ and β_j $(j = 1, \cdots, m)$ be defined as in Theorem 1 and let $F : \Delta \to R$ be convex. Then:

(a) G is convex on $[0,1]^2$ where G is defined as in (22).

(b) Define $g : [0,1] \to R$ by $g(t) := G(t,t)$. Then g is convex, increasing on $[0,1]$,

$$\begin{aligned} &\sup_{t\in[0,1]} g(t) = g(1) \\ = \;& \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j\right) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \end{aligned} \tag{29}$$

and

$$\inf_{t\in[0,1]} g(t) = g(0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right). \tag{30}$$

Proof.(a) Since F is convex, we have for (t_1, s_1), $(t_2, s_2) \in [0,1]^2$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$ that

$$\begin{aligned}
& G(\alpha(t_1, s_1) + \beta(t_2, s_2)) \\
= \; & G(\alpha t_1 + \beta t_2, \alpha s_1 + \beta s_2) \\
= \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\Bigg((\alpha t_1 + \beta t_2) \sum_{i=1}^n \alpha_i x_i + (1 - (\alpha t_1 + \beta t_2)) \frac{a+b}{2}, \\
& (\alpha s_1 + \beta s_2) \sum_{j=1}^m \beta_j y_j + (1 - (\alpha s_1 + \beta s_2)) \frac{c+d}{2} \Bigg) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
= \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} F\Bigg(\alpha \left(t_1 \sum_{i=1}^n \alpha_i x_i + (1-t_1) \frac{a+b}{2}, s_1 \sum_{j=1}^m \beta_j y_j + (1-s_1) \frac{c+d}{2} \right) \\
& + \beta \left(t_2 \sum_{i=1}^n \alpha_i x_i + (1-t_2) \frac{a+b}{2}, s_2 \sum_{j=1}^m \beta_j y_j + (1-s_2) \frac{c+d}{2} \right) \Bigg) \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
\leq \; & \frac{1}{(b-a)^n (d-c)^m} \int_{\Delta_y^m} \int_{\Delta_x^n} \Bigg[\alpha F \left(t_1 \sum_{i=1}^n \alpha_i x_i + (1-t_1) \frac{a+b}{2}, s_1 \sum_{j=1}^m \beta_j y_j + (1-s_1) \frac{c+d}{2} \right) \\
& + \beta F \left(t_2 \sum_{i=1}^n \alpha_i x_i + (1-t_2) \frac{a+b}{2}, s_2 \sum_{j=1}^m \beta_j y_j + (1-s_2) \frac{c+d}{2} \right) \Bigg] \prod_{i=1}^n dx_i \prod_{j=1}^m dy_j \\
= \; & \alpha G(t_1, s_1) + \beta G(t_2, s_2),
\end{aligned}$$

which shows that G is convex on $[0,1]^2$.

(b) Let t_1, $t_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
& g(\alpha t_1 + \beta t_2) \\
= \; & G(\alpha(t_1, t_1) + \beta(t_2, t_2)) \\
\leq \; & \alpha G(t_1, t_1) + \beta G(t_2, t_2) \\
= \; & \alpha g(t_1) + \beta g(t_2)
\end{aligned}$$

which shows that g is convex on $[0,1]$. By Theorem 3, we have that, for $0 \leq t_1 < t_2 \leq 1$,

$$g(t_1) = G(t_1, t_1) \leq G(t_2, t_1) \leq G(t_2, t_2) = g(t_2)$$

which show that g is increasing on $[0, 1]$. Since g is increasing on $[0, 1]$, (29) and (30) hold. This completes the proof.

Remark 7. Let f be defined as in Theorem G. If we choose $m = n = 1$, then Theorem 4 reduces to Theorem G.

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