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A General Interpolating Formula and Error Bounds*

Josip Pečarić

Faculty of Textile Technology, University of Zagreb

Pierottijeva 6, 10000 Zagreb, Croatia

and

Nenad Ujević†

Department of Mathematics, University of Split

Teslina 12/III, 21000 Split, Croatia

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Abstract

A general interpolating formula and few particular corrected interpolating polynomials are derived. It is shown that the Lagrange interpolating polynomial is only a special case of the general formula. Error inequalities of Ostrowski-like type for the general interpolating formula and for the corrected interpolating polynomials are established.

Keywords and Phrases: *General interpolating formula, Corrected interpolating polynomial, Representation of remainder, Ostrowski-like inequalities.*

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†Corresponding author. E-mail:ujevic@pmfst.hr

1. Introduction

Polynomial interpolation is the interpolation of a given data set by a polynomial. As we know there is a unique interpolating polynomial which can be written in different forms. Here we use the Lagrange form.

The main result of this paper is a general interpolating formula. We show that the Lagrange interpolating polynomial is only a special case of this general formula. Using this formula we can obtain many particular interpolating formulas. In this paper we use the formula to obtain few corrected interpolating polynomials, as an illustration. Some corrected interpolating polynomials are given in [11] and [12]. In [12] we can find the following corrected interpolating formula

$$f(x) = L_n(x) + \omega_n(x) \sum_{j=1}^{\left[\frac{k+1}{2}\right]} \frac{2}{2^{2j-1}(2j-1)!} g_{2j-1}\left[x_0; x_1; ...; x_n\right] + \bar{R}_k(x),$$

where $L_n(x)$ is the Lagrange interpolating polynomial, $\omega_n(x) = (x-x_0)\cdots(x-x_n)$, $g_m(t) = (x-t)^{m-1} f^{(m)}(\frac{x+t}{2})$, $g_m\left[x_0; x_1; ...; x_n\right]$ denote the divide difference of order m and $\bar{R}_k(x)$ is a remainder term. We also suppose that we have a given partition of the interval $[a, b]$ ($\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$) and that $f \in C^{k+1}(a, b)$.

In [11] we can find the following formula

$$f(x) = L_n(x) - \omega_n(x) \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \frac{E_{2j+1}(0)}{(2j+1)!} g_{2j+1}\left[x_0; x_1; ...; x_n\right] + R_k(x),$$

where $E_j(x)$ are Euler polynomials, $g_m(t) = (x-t)^{m-1} f^{(m)}(t)$ and $R_k(x)$ is a remainder term. (Note that $g_m(t) = g_m(t, x)$. We use this simplification in the rest of this paper.) We show that the above formula is a special case of the mentioned general interpolating formula. In [11] many error inequalities for the above formula are derived. Here we improve some of these inequalities.

In section 2 we give a general interpolating formula. In section 3 we give few particular corrected interpolating polynomials. We specially mention Corollary 5 in which we show that the Lagrange interpolating polynomial is only a special case of the general formula. In section 4 we give some general error inequalities for the general interpolating formula. In section 5 we give various error bounds for these corrected interpolating polynomials. Similar error inequalities are obtained in numerical integration. For example see [3]–[6], [9] and [10]. In some of these paper it is shown that the obtained error inequalities are superior to bounds obtained in more standard ways. Thus we use these kind of inequalities in this paper to obtain error bounds for interpolating polynomials.

Finally, we emphasize that the usual error inequalities in polynomial interpolation (for the Lagrange interpolating polynomial $L_n(x)$) are given by means of the $(n+1)$th derivative while in this paper we can find these error inequalities expressed by means of the kth derivative for $k = 1, 2, ..., n$.

2. A General Corrected Interpolating Polynomial

Let $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ and let $f : [a, b] \to R$ be a given function. The Lagrange interpolation polynomial is given by

$$L_n(x) = \sum_{i=0}^{n} p_{ni}(x) f(x_i), \tag{1}$$

where

$$p_{ni}(x) = \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}, \tag{2}$$

for $i = 0, 1, ..., n$. Here we always use the notation $L_n(x)$ for the Lagrange interpolating polynomial and $p_{ni}(x)$, $i = 0, 1, 2, ..., n$, denote the basic Lagrange interpolating polynomials. We have the Cauchy relations ([7, pp. 160-161]),

$$\sum_{i=0}^{n} p_{ni}(x) = 1 \tag{3}$$

and

$$\sum_{i=0}^{n} p_{ni}(x)(x - x_i)^j = 0, \;\; j = 1, 2, ..., n. \tag{4}$$

Let $\Delta = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given uniform subdivision of the interval $[a, b]$, i.e. $x_i = x_0 + ih$, $h = (b - a)/n$, $i = 0, 1, 2, ..., n$. Then the Lagrange interpolating polynomial is given by

$$\begin{aligned} L_n(x) &= L_n(x_0+th) \\ &= (-1)^n \frac{t(t-1)\cdots(t-n)}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{f(x_i)}{t-i}, \end{aligned}$$

where $t \notin \{0,1,2,...,n\}$, $0<t<n$.

As we know the divided difference of the first order of the function f is given by

$$f[x_0;x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}.$$

The divided difference of order n is defined via the divided differences of order $n-1$ by the recurrence formula

$$f[x_0;x_1;...;x_n] = \frac{f[x_1;x_2;...;x_n]-f[x_0;x_1;...;x_{n-1}]}{x_n-x_0}.$$

The following lemma is valid ([2, p. 68]).

Lemma 1. *The nth-order divided difference satisfies the relation*

$$f[x_0;x_1;...;x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

The interpolating polynomial can be written in the Newton form as

$$\begin{aligned} L_n(x) &= f(x_0) + \sum_{i=0}^{n-1} (x-x_0)\cdots(x-x_i) f[x_0;...;x_{i+1}] \\ &= f(x_0) + \sum_{i=0}^{n-1} \omega_i(x) f[x_0;...;x_{i+1}], \end{aligned}$$

where

$$\omega_i(x) = (x-x_0)(x-x_1)\cdots(x-x_i), \tag{5}$$

for $i=0,1,2,...,n$.

Lemma 2. *Let $P_m(t)$ be any polynomial of degree $\le m$ and let Δ be a given partition of the interval $[a,b]$. Then*

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)dt = 0, \tag{6}$$

for $0 \le m \le n-1$, where $p_{ni}(x)$ are basic Lagrange polynomials and $x \in [a,b]$.

Proof. Let x be a real number. Then we have

$$P_m(t) = \sum_{j=0}^{m} c_j (x-t)^j,$$

for some coefficients $c_j = c_j(x)$, $j = 0, 1, ..., m$. (This is a consequence of the Taylor formula.) Thus,

$$\int_{x_i}^{x} P_m(t)dt = \sum_{j=0}^{m} c_j \int_{x_i}^{x} (x-t)^j dt.$$

We have

$$\int_{x_i}^{x} (x-t)^j dt = \frac{(x-x_i)^{j+1}}{j+1}.$$

It follows that

$$\int_{x_i}^{x} P_m(t)dt = \sum_{j=0}^{m} c_j \frac{(x-x_i)^{j+1}}{j+1}.$$

Finally, we get

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)dt = \sum_{j=0}^{m} \frac{c_j}{j+1} \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{j+1} = 0,$$

for $0 \leq m \leq n-1$, since (4) holds. □

In what follows (to simplify notations and records) we denote

$$P_j(t, x_i) = P_j(t)$$

such that

$$P_j^{(m)}(t) = \frac{\partial^m P_j(t, x_i)}{\partial t^m},$$

where $x_i \in [a, b]$.

Theorem 3. *Under the assumptions of Lemma 2 suppose that* $f \in C^{k+1}(a, b)$*. Let* $\{P_k(t)\}$ *be a harmonic (or Appell) sequence of polynomials, i.e.* $P_k'(t) = P_{k-1}(t)$*,* $P_0(t) = 1$*. Then*

$$f(x) = L_n(x) - \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i)\right] + R_{k,j}(x), \tag{7}$$

where

$$R_{k,j}(x)=(-1)^k\sum_{i=0}^{n}p_{ni}(x)\int_{x_i}^{x}\left[f^{(k+1)}(t)-Q_j(t)\right]P_k\left(t\right)dt, \tag{8}$$

for any polynomial $Q_j(t)$ *such that* $0\leq j+k\leq n-1$.

Proof. Integrating by parts, we obtain

$$\begin{aligned}&(-1)^k\int_{x_i}^{x}f^{(k+1)}(t)P_k\left(t\right)dt\\ =\ &(-1)^k\left[P_k(x)f^{(k)}(x)-P_k(x_i)f^{(k)}(x_i)\right]+(-1)^{k-1}\int_{x_i}^{x}f^{(k)}(t)P_{k-1}\left(t\right)dt.\end{aligned}$$

In a similar way we have

$$\begin{aligned}&(-1)^{k-1}\int_{x_i}^{x}f^{(k)}(t)P_{k-1}\left(t\right)dt\\ =\ &(-1)^{k-1}\left[P_{k-1}(x)f^{(k-1)}(x)-P_{k-1}(x_i)f^{(k-1)}(x_i)\right]\\ &+(-1)^{k-2}\int_{x_i}^{x}f^{(k-1)}(t)P_{k-2}\left(t\right)dt.\end{aligned}$$

Continuing in this way we get

$$\begin{aligned}&(-1)^k\int_{x_i}^{x}f^{(k+1)}(t)P_k\left(t\right)dt\\ =\ &\sum_{m=1}^{k}(-1)^m\left[P_m(x)f^{(m)}(x)-P_m(x_i)f^{(m)}(x_i)\right]+\int_{x_i}^{x}f'(t)dt\\ =\ &\sum_{m=1}^{k}(-1)^m\left[P_m(x)f^{(m)}(x)-P_m(x_i)f^{(m)}(x_i)\right]+f(x)-f(x_i).\end{aligned}$$

Then we have

$$\begin{aligned} R_k(x) &= (-1)^k \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} f^{(k+1)}(t) P_k(t)\, dt \\ &= \sum_{i=0}^{n} p_{ni}(x) \left[f(x) - f(x_i)\right] \\ &\quad + \sum_{i=0}^{n} p_{ni}(x) \sum_{m=1}^{k} (-1)^m \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i)\right] \\ &= f(x) - L_n(x) \\ &\quad + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i)\right]. \end{aligned}$$

If we define $S_m(t) = Q_j(t) P_k(t)$, $m = j + k$, then we also have

$$(-1)^k \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} Q_j(t) P_k(t)\, dt = (-1)^k \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} S_m(t)\, dt = 0,$$

for $0 \le m \le n - 1$, since (6) holds.
From the above two relations we easily conclude that (7) is valid. □

Remark 4. Note that we can choose $k = 0$ in the above theorem. In this case we have no perturbation of the Lagrange interpolating polynomial. (Of course, the corresponding sum in (7) is empty, if $k = 0$.) If $Q_j(t) = 0$ then we can set $k \le n$.

3. Particular Corrected Interpolating Polynomials

If we substitute the polynomials

$$P_k(t) = \frac{(t - x_i)^k}{k!}$$

in Theorem 3 then we get the following result. (Recall that we have adopted the notation $P_k(t) = P_k(t, x_i)$ - mostly to simplify records.)

Corollary 5. *Under the assumptions of Theorem 3 we have*

$$f(x) = L_n(x) + R_{k,j}(x), \tag{9}$$

where $R_{k,j}(x)$ *is given by (8).*

Hence, we got the Lagrange interpolating polynomial (without perturbation). See also Remark 4.

If we substitute the polynomials

$$P_k(t) = \frac{(x-x_i)^k}{k!} E_k\left(\frac{t-x_i}{x-x_i}\right), \tag{10}$$

($E_k(t)$ are Euler polynomials) in Theorem 3 then we get the following corrected interpolating polynomial.

Corollary 6. *Under the assumptions of Theorem 3 we have*

$$f(x) = L_n(x) + \omega_n(x) \sum_{m=1}^{k} \frac{(-1)^m E_m(0)}{m!} g_m\left[x_0; x_1; ...; x_n\right] + R_{k,j}(x), \tag{11}$$

where $\omega_n(x) = (x-x_0)\cdots(x-x_n)$,

$$R_{k,j}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^k \int_{x_i}^{x} \left[f^{(k+1)}(t) - Q_j(t)\right] E_k\left(\frac{t-x_i}{x-x_i}\right) dt \tag{12}$$

and

$$g_m(t) = (x-t)^{m-1} f^{(m)}(t), \quad m = 1, 2, ..., k.$$

This result is proved in [11]. (Recall that we have adopted the simplification $g_m(t,x) = g_m(t)$.)

It is obvious that we can substitute infinitely many harmonic sequences of polynomials in Theorem 3 to obtain various corrected interpolating formulas.

Here we consider the polynomials

$$P_k(t) = \frac{(x-x_i)^k}{k!} B_k\left(\frac{t-x_i}{x-x_i}\right), \tag{13}$$

where $B_k(t)$ are Bernoulli polynomials. For these polynomials we can get applicable results.

We now recall some properties of Bernoulli polynomials. The Bernoulli polynomials are defined by the relation

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!},\quad |t|<2\pi,$$

such that

$$B_0(x)=1,\ B_1(x)=x-\frac{1}{2},\ B_2(x)=x^2-x+\frac{1}{6},... \tag{14}$$

We have

$$B_k'(x)=kB_{k-1}(x)\text{ or }\int B_k(x)dx=\frac{B_{k+1}(x)}{k+1},\ k=1,2,..., \tag{15}$$

$$\int_0^1 B_n(t)B_m(t)dt=(-1)^n\frac{m!n!}{(m+n)!}B_{m+n} \tag{16}$$

$$\int_0^1 B_n(t)dt=0,\ n=1,2,... \tag{17}$$

$$|B_{2n}(t)|\le|B_{2n}|,\ n=1,2,... \tag{18}$$

$$0<(-1)^nB_{2n+1}(t)<\frac{2(2n+1)!}{(2\pi)^{2n+1}}\left(\frac{1}{1-2^{-2n}}\right),\ \ 0<x<\frac{1}{2},\ n=1,2,... \tag{19}$$

The numbers $B_k=B_k(0)$ are Bernoulli numbers. We have $B_0=1$, $B_1=-1/2$, $B_2=1/6$,..., $B_{2j+1}=0$ for $j=1,2,...$ and

$$B_k(0)=(-1)^kB_k(1). \tag{20}$$

Further properties of Bernoulli polynomials can be found in [1].

If we substitute the polynomials (13) in Theorem 3 then we get the following result.

Corollary 7. *Under the assumptions of Theorem 3 we have*

$$f(x)=L_n(x)+\omega_n(x)\sum_{m=1}^{k}\frac{(-1)^mB_m}{m!}g_m\left[x_0;x_1;...;x_n\right]+R_{k,j}(x), \tag{21}$$

where

$$R_{k,j}(x)=\frac{(-1)^k}{k!}\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^k\int_{x_i}^{x}\left[f^{(k+1)}(t)-Q_j(t)\right]B_k\left(\frac{t-x_i}{x-x_i}\right)dt \tag{22}$$

and

$$g_m(t)=(x-t)^{m-1}f^{(m)}(t),\quad m=1,2,...,k.$$

Proof. If $P_m(t)$ are defined by (13) then we have

$$\begin{aligned}&P_m(x)f^{(m)}(x)-P_m(x_i)f^{(m)}(x_i)\\ =\ &\frac{(x-x_i)^m}{m!}\left[B_m(1)f^{(m)}(x)-B_m(0)f^{(m)}(x_i)\right]\end{aligned}$$

such that

$$\begin{aligned}&\sum_{m=1}^{k}(-1)^m\sum_{i=0}^{n}p_{ni}(x)\left[P_m(x)f^{(m)}(x)-P_m(x_i)f^{(m)}(x_i)\right]\\ =\ &\sum_{m=1}^{k}\frac{(-1)^m}{m!}\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^m\left[B_m(1)f^{(m)}(x)-B_m(0)f^{(m)}(x_i)\right]\\ =\ &\sum_{m=1}^{k}\frac{(-1)^m}{m!}B_m(0)\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^mf^{(m)}(x_i),\end{aligned}$$

since

$$\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^mB_m(1)f^{(m)}(x)=0,$$

for $1\leq m\leq n$. We have

$$\begin{aligned}&\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^mf^{(m)}(x_i)\\ =\ &\omega_n(x)\sum_{i=0}^{n}\frac{(x-x_i)^{m-1}f^{(m)}(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\end{aligned}$$

and

$$g_m\left[x_0;x_1;...;x_n\right]=\sum_{i=0}^{n}\frac{(x-x_i)^{m-1}f^{(m)}(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)},$$

by Lemma 1, such that

$$\begin{aligned}
&\sum_{m=1}^{k}(-1)^m\sum_{i=0}^{n}p_{ni}(x)\left[P_m(x)f^{(m)}(x)-P_m(x_i)f^{(m)}(x_i)\right]\\
=\ &\omega_n(x)\sum_{m=1}^{k}\frac{(-1)^m}{m!}B_m(0)g_m\left[x_0;x_1;...;x_n\right].
\end{aligned}$$

From (7), (8) and the above relation we see that (21) holds. □

Remark 8. Since $B_{2m+1}=0$, $m=1,2,...$, and $B_1=-1/2$ we can write

$$\begin{aligned}
f(x) &= L_n(x)+\frac{1}{2}\omega_n(x)g_1\left[x_0;x_1;...;x_n\right]+\omega_n(x)\sum_{m=1}^{\left[\frac{k}{2}\right]}\frac{B_{2m}}{(2m)!}g_{2m}\left[x_0;x_1;...;x_n\right]\\
&\quad+R_{k,j}(x).
\end{aligned}$$

We also consider the polynomials

$$P_k(t)=\frac{(t-x)^k}{k!}. \tag{23}$$

Corollary 9. *Under the assumptions of Theorem 3 we have*

$$f(x)=L_n(x)+\omega_n(x)\sum_{m=1}^{k}\frac{1}{m!}g_m\left[x_0;x_1;...;x_n\right]+R_{k,j}(x), \tag{24}$$

where

$$R_{k,j}(x)=\frac{(-1)^k}{k!}\sum_{i=0}^{n}p_{ni}(x)\int_{x_i}^{x}\left[f^{(k+1)}(t)-Q_j(t)\right](t-x)^k dt \tag{25}$$

and

$$g_m(t)=(x-t)^{m-1}f^{(m)}(t),\quad m=1,2,...,k.$$

Proof. If we substitute the polynomials (23) in Theorem 3 then we get

$$\begin{aligned}
f(x) &= L_n(x)+\sum_{m=1}^{k}(-1)^m\sum_{i=0}^{n}p_{ni}(x)P_m(x_i)f^{(m)}(x_i)+R_{k,j}(x)\\
&= L_n(x)+\sum_{m=1}^{k}\frac{(-1)^m}{m!}\sum_{i=0}^{n}p_{ni}(x)(x_i-x)^m f^{(m)}(x_i)+R_{k,j}(x),
\end{aligned}$$

since $P_m(x) = 0$ and

$$R_{k,j}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} \left[f^{(k+1)}(t) - Q_j(t) \right] (t-x)^k dt.$$

The above formula can be written in the form

$$\begin{aligned} f(x) &= L_n(x) + \sum_{m=1}^{k} \frac{1}{m!} \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^m f^{(m)}(x_i) + R_{k,j}(x) \\ &= L_n(x) + \omega_n(x) \sum_{m=1}^{k} \frac{1}{m!} g_m \left[x_0; x_1; ...; x_n \right], \end{aligned}$$

where

$$g_m(t) = (x-t)^{m-1} f^{(m)}(t), \quad m = 1, 2, ..., k.$$

□

4. General Error Inequalities

Let $g \in C(a,b)$. As we know among all algebraic polynomials of degree $\leq j$ there exists the only polynomial $Q_j^*(t)$ having the property that

$$\left\| g - Q_j^* \right\|_\infty \leq \left\| g - Q_j \right\|_\infty,$$

where $Q_j \in \Pi_j$ is an arbitrary polynomial of degree $\leq j$. We define

$$G_j(g) = \left\| g - Q_j^* \right\| = \inf_{Q_j \in \Pi_j} \left\| g - Q_j \right\|_\infty. \tag{26}$$

Here we use the above notation G_j without referring to its definition.

We also introduce the notations

$$\left\| g \right\|_{\infty,i} = \begin{cases} \max\limits_{t \in [x_i,x]} |g(t)|, & x > x_i \\ \max\limits_{t \in [x,x_i]} |g(t)|, & x < x_i \\ |g(x_i))|, & x = x_i \end{cases}, \tag{27}$$

$$\left\| g \right\|_{1,i} = \left| \int_{x_i}^{x} |g(t)| \, dt \right|. \tag{28}$$

Specially, if $[x_i, x] = [0,1]$ then we use the notations $\left\| g \right\|_{\infty,1}$ and $\left\| g \right\|_{1,1}$.

Theorem 10. *Under the assumptions of Theorem 3 we have*

$$\left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right| \quad (29)$$

$$\leq G_j(f^{(k+1)}) \sum_{i=0}^{n} |p_{ni}(x)| \, \|P_k\|_{1,i} \, .$$

Proof. Let $Q_j(t) = Q_j^*(t)$, where $Q_j^*(t)$ is defined by (26) for the function $g(t) = f^{(k+1)}(t)$. Then we have

$$\begin{aligned} |R_{k,j}(x)| &\leq \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - Q_j^*(t) \right] P_k(t) \, dt \right| \\ &\leq \left\| f^{(k+1)} - Q_j^* \right\|_{\infty} \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} |P_k(t)| \, dt \right| \\ &= G_j(f^{(k+1)}) \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} |P_k(t)| \, dt \right| . \end{aligned}$$

From (7), (8), (28) and the above relation we see that (29) holds. □

Theorem 11. *Let the assumptions of Theorem 3 hold. If γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}$, $t \in [a,b]$, $k = 0, 1, ..., n-1$, then*

$$\left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right|$$

$$\leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{2} \sum_{i=0}^{n} |p_{ni}(x)| \, \|P_k\|_{1,i} \, ,$$

$$\left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right|$$

$$\leq \sum_{i=0}^{n} |p_{ni}(x)| \, |S_{ki} - \gamma_{k+1}| \, |x - x_i| \, \|P_k\|_{\infty,i}$$

and

$$\begin{aligned}
&\left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right| \\
\leq\ & \sum_{i=0}^{n} |p_{ni}(x)| \, |\Gamma_{k+1} - S_{ki}| \, |x - x_i| \, \|P_k\|_{\infty,i} \, ,
\end{aligned}$$

where $S_{ki} = \left[f^{(k)}(x) - f^{(k)}(x_i) \right] / (x - x_i)$, $i = 0, 1, ..., n$.

Proof. We set $Q_j(t) = (\Gamma_{k+1} + \gamma_{k+1})/2$ in (8). Then we have

$$\begin{aligned}
&\left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right| \\
=\ & |R_{k,j}(x)| \\
\leq\ & \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right] P_k(t) \, dt \right| \\
\leq\ & \left\| f^{(k+1)} - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right\|_{\infty} \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} |P_k(t)| \, dt \right| .
\end{aligned}$$

Since

$$\left\| f^{(k+1)} - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right\|_{\infty} \leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{2}$$

we get

$$\begin{aligned}
&\left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right| \\
\leq\ & \frac{\Gamma_{k+1} - \gamma_{k+1}}{2} \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} |P_k(t)| \, dt \right| .
\end{aligned}$$

The first inequality is proved.

We now have

$$\begin{aligned}
& \left| f(x) - L_n(x) + \sum_{m=1}^{k} (-1)^m \sum_{i=0}^{n} p_{ni}(x) \left[P_m(x) f^{(m)}(x) - P_m(x_i) f^{(m)}(x_i) \right] \right| \\
= & \ |R_{k,j}(x)| \\
\leq & \ \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - \gamma_{k+1} \right] P_k(t)\, dt \right| \\
\leq & \ \sum_{i=0}^{n} |p_{ni}(x)| \, |S_{ki} - \gamma_{k+1}| \, |x - x_i| \, \|P_k\|_{\infty,i} \,,
\end{aligned}$$

since

$$\begin{aligned}
& \left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - \gamma_{k+1} \right] dt \right| \\
= & \ \left| f^{(k)}(x) - f^{(k)}(x_i) - \gamma_{k+1}(x - x_i) \right| \\
= & \ (S_{ki} - \gamma_{k+1}) \, |x - x_i| \,.
\end{aligned}$$

The second inequality is proved. The third inequality can be proved in a similar way. □

5. Particular Error Inequalities

In this section we give various error bounds for the particular corrected interpolating polynomials.

For that purpose, we introduce the notations

$$C_k(x) = \sum_{i=0}^{n} \frac{|x - x_i|^k}{|x_i - x_0| \cdots |x_i - x_{i-1}| \, |x_i - x_{i+1}| \cdots |x_i - x_n|}, \tag{30}$$

$$F_k(x) = \sum_{i=0}^{n} \frac{(S_{ki} - \gamma_{k+1}) \, |x - x_i|^k}{|x_i - x_0| \cdots |x_i - x_{i-1}| \, |x_i - x_{i+1}| \cdots |x_i - x_n|}, \tag{31}$$

$$D_k(x) = \sum_{i=0}^{n} \frac{(\Gamma_{k+1} - S_{ki}) \, |x - x_i|^k}{|x_i - x_0| \cdots |x_i - x_{i-1}| \, |x_i - x_{i+1}| \cdots |x_i - x_n|}, \tag{32}$$

where $S_{ki} = \left[f^{(k)}(x) - f^{(k)}(x_i) \right] / (x - x_i)$, $i = 0, 1, ..., n$ and γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}$, $t \in [a, b]$, $k = 0, 1, ..., n - 1$.

We also introduce notations for the remainders of the corrected interpolating polynomials. These interpolating polynomials are obtained in Section 3. We define

$$\begin{aligned} & R_{k,j}^{B}(x) \qquad (33) \\ = & \frac{(-1)^k}{k!}\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^k \int_{x_i}^{x} \left[f^{(k+1)}(t)-Q_j(t)\right] B_k\left(\frac{t-x_i}{x-x_i}\right)dt \\ = & f(x)-L_n(x)-\omega_n(x)\sum_{m=1}^{k}\frac{(-1)^m B_m}{m!} g_m\left[x_0;x_1;...;x_n\right], \end{aligned}$$

$$\begin{aligned} & R_{k,j}^{E}(x) \qquad (34) \\ = & \frac{(-1)^k}{k!}\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^k \int_{x_i}^{x} \left[f^{(k+1)}(t)-Q_j(t)\right] E_k\left(\frac{t-x_i}{x-x_i}\right)dt \\ = & f(x)-L_n(x)-\omega_n(x)\sum_{m=1}^{k}\frac{(-1)^m E_m(0)}{m!} g_m\left[x_0;x_1;...;x_n\right], \end{aligned}$$

$$\begin{aligned} R_{k,j}^{T}(x) &= \frac{(-1)^k}{k!}\sum_{i=0}^{n} p_{ni}(x)\int_{x_i}^{x}\left[f^{(k+1)}(t)-Q_j(t)\right](t-x)^k dt \qquad (35) \\ &= f(x)-L_n(x)-\omega_n(x)\sum_{m=1}^{k}\frac{1}{m!} g_m\left[x_0;x_1;...;x_n\right]. \end{aligned}$$

In this section we use the above notations without referring to their definitions.

We now consider the interpolation formula given in Corollary 7.

Theorem 12. *Under the assumptions of Corollary 7 we have*

$$\left|R_{k,j}^{B}(x)\right| \le \frac{\|B_k\|_{1,1}}{k!} G_j(f^{(k+1)})C_k(x)\left|\omega_n(x)\right|.$$

Proof. We substitute the polynomials (13) in Theorem 10. If we use the fact that

$$\left|\int_{x_i}^{x}\left|B_k\left(\frac{t-x_i}{x-x_i}\right)\right|dt\right| = |x-x_i|\int_0^1 |B_k(t)|\,dt \qquad (36)$$

then we easily find that the above inequality holds. □

Corollary 13. *Under the assumptions of Theorem 12 let k be odd, $k = 2r - 1, r \geq 1$. Then we have*

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{2\left|B_{2r}\right|}{(2r-1)!}\frac{1-2^{-2r}}{r}G_j(f^{(2r)})C_{2r-1}(x)\left|\omega_n(x)\right|,$$

If k is even then we have

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{2\Lambda_k}{(k+1)!}G_j(f^{(k+1)})C_k(x)\left|\omega_n(x)\right|, \tag{37}$$

where

$$\Lambda_k = 2\left|B_{k+1}(t_0)\right| \tag{38}$$

and $t_0 \in \left(0, \frac{1}{2}\right)$ is a unique zero point (in the interval $\left(0, \frac{1}{2}\right)$) of the Bernoulli polynomial $B_k(\cdot)$, $k = 2r$.

Proof. To prove the first inequality it is sufficient to note that

$$\int_0^1 \left|B_k(t)\right| dt = 2\frac{1-2^{-2r}}{r}\left|B_{2r}\right|, \tag{39}$$

if $k \geq 2$ is odd, $k = 2r - 1$. (The last integral is calculated in [6].)

To prove the second inequality we calculate

$$\begin{aligned} \left\|B_k\right\|_{1,1} &= \int_0^1 \left|B_k(t)\right| dt = 2\left[\left|\int_0^{t_0} B_k(t)dt\right| + \left|\int_{t_0}^{1/2} B_k(t)dt\right|\right] \\ &= \frac{2}{k+1}\left[\left|B_{k+1}(t_0) - B_{k+1}\right| + \left|B_{k+1}(t_0) - B_{k+1}(\frac{1}{2})\right|\right]. \end{aligned} \tag{40}$$

□

Remark 14. The above estimates have only theoretical importance, since it is difficult to find the polynomial Q^* in practice. In fact, we can find Q^* only for some special cases of functions. However, we can use the estimates to obtain some practical estimations.

We also have the estimation

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{\sqrt{\left|B_{2k}\right|}}{\sqrt{(2k)!}}G_j(f^{(k+1)})C_k(x)\left|\omega_n(x)\right|.$$

This estimation follows from Theorem 12 and the fact that

$$\left(\int_0^1 |B_k(t)|\, dt\right)^2 \leq \int_0^1 (B_k(t))^2\, dt = \frac{(k!)^2}{(2k)!} |B_{2k}| . \tag{41}$$

We also emphasize that this inequality would be used only if k is even and we cannot find (or we simply don't know) the zero point t_o.

Finally, we specially note that if k is even then the corresponding Bernoulli polynomial $B_k(t)$ has only one zero point t_0 in $\left(0, \frac{1}{2}\right)$. For example, for $k = 2$ this zero point is $t_0 = \frac{1}{2} - \frac{1}{6}\sqrt{3}$, for $k = 4$ the zero point is $t_0 = \frac{1}{2} - \frac{1}{30}\sqrt{\left(225 - 30\sqrt{30}\right)}$ and generally, for large $k = 2r$, it is (very) close to $\frac{1}{4}$.

Theorem 15. *Let the assumptions of Corollary 7 hold. If γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}$, $t \in [a,b]$, $k = 0, 1, ..., n-1$, then*

$$\left|R_{k,j}^B(x)\right| \leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{2k!} C_k(x) \left|\omega_n(x)\right| \left\|B_k\right\|_{1,1},$$

where $\omega_n(x)$ is defined by (5). We also have

$$\left|R_{k,j}^B(x)\right| \leq \frac{|\omega_n(x)|}{k!} F_k(x) \left\|B_k\right\|_{\infty,1}$$

and

$$\left|R_{k,j}^B(x)\right| \leq \frac{|\omega_n(x)|}{k!} D_k(x) \left\|B_k\right\|_{\infty,1} .$$

Proof. We substitute the polynomials (13) in Theorem 11 and use the relation (36). □

Corollary 16. *Let the assumptions of Theorem 15 hold. If k is odd, $k = 2r-1, r \geq 2$, then*

$$\left|R_{k,j}^B(x)\right| \leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{(2r-1)!} \frac{1 - 2^{-2r}}{r} \left|B_{2r}\right| C_k(x) \left|\omega_n(x)\right| .$$

If k is even then we have

$$\left|R_{k,j}^B(x)\right| \leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{(k+1)!} \Lambda_k C_k(x) \left|\omega_n(x)\right|$$

where Λ_k is defined by (38).

Proof. The proof follows from the above theorem and the relations (39) and (40). □

Remark 17. We additionally have the estimate

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{\Gamma_{k+1}-\gamma_{k+1}}{2\sqrt{(2k)!}}\sqrt{|B_{2k}|}C_k(x)\left|\omega_n(x)\right|. \tag{42}$$

which can be proved using the relation (41). See also Remark 14.

Corollary 18. *Let the assumptions of Theorem 15 hold. Then*

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{2}{(2\pi)^k}\frac{1}{1-2^{1-k}}\left|\omega_n(x)\right| F_k(x),$$

if $k = 4r+3,\ r = 0, 1, 2, ...,$

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{|B_k|}{k!}\left|\omega_n(x)\right| F_k(x),$$

if $k = 2r,\ r = 1, 2, ...,$

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{2}{(2\pi)^k}\frac{1}{1-2^{1-k}}\left|\omega_n(x)\right| D_k(x)$$

if $k = 4r+3,\ r = 0, 1, 2, ...,$

$$\left|R_{k,j}^{B}(x)\right| \leq \frac{|B_k|}{k!}\left|\omega_n(x)\right| D_k(x),$$

if $k = 2r,\ r = 1, 2, ...$

Proof. The proof follows from the above theorem and the following properties of Bernoulli polynomials:

$$\begin{aligned}\|B_k\|_{\infty,1} &= \max_{t\in[0,1]}|B_k(t)| \leq |B_k|\,, \quad k = 2r,\ r = 1, 2, ...\\ \|B_k\|_{\infty,1} &\leq \frac{2k!}{(2\pi)^k}\frac{1}{1-2^{1-k}},\ k = 4r+3,\ r = 0, 1, 2, ...\end{aligned}$$

□

Lemma 19. *Let* $\Delta = \{x_0 = a < x_1 < \cdots < x_n = b\}$ *be a given uniform subdivision of the interval* $[a,b]$*, i.e.* $x_i = x_0 + ih$*,* $h = (b-a)/n$*,* $i = 0,1,2,...,n$*. If* $x \in (x_{j-1}, x_j)$*, for some* $j \in \{1,2,...,n\}$*, then*

$$|\omega_n(x)| \leq j!(n-j+1)!h^{n+1}, \tag{43}$$

$$C_k(x) \leq \frac{2^n}{n!}\left\{\frac{1}{2}\left[n+1+|n-2j+1|\right]\right\}^k h^{k-n}, \tag{44}$$

and

$$C_k(x)\,|\omega_n(x)| \leq \alpha_{jnk}\frac{n-j+1}{n}\frac{2^n(b-a)^{k+1}}{\binom{n}{j}}, \tag{45}$$

where

$$\alpha_{jnk} = \left[\frac{1}{2n}\left(n+1+|2j-n-1|\right)\right]^k. \tag{46}$$

This lemma is proved in [11].

Remark 20. Note that

$$\alpha_{jnk} \leq 1$$

and $\alpha_{jnk} = 1$ if and only if $j = 1$ or $j = n$. If we choose $x \in [x_j, x_{j+1}]$, $j = 0,1,...,n-1$, then we get the factor $(j+1)/n$ instead of the factor $(n-j+1)/n$ in (45).

Theorem 21. *Under the assumptions of Lemma 19 and Theorem 15 we have*

$$\left|R^B_{k,j}(x)\right| \leq \frac{|B_{2r}|\,\alpha_{jnk}}{k!}\frac{1-2^{-2r}}{r}\frac{n-j+1}{n}\frac{2^n(b-a)^{k+1}}{\binom{n}{j}}\left(\Gamma_{k+1}-\gamma_{k+1}\right),$$

if k *is odd,* $k = 2r-1, r \geq 1$ *and*

$$\left|R^B_{k,j}(x)\right| \leq \frac{\Lambda_k\alpha_{jnk}}{(k+1)!}\frac{n-j+1}{n}\frac{2^n(b-a)^{k+1}}{\binom{n}{j}}\left(\Gamma_{k+1}-\gamma_{k+1}\right),$$

if k *is even and where* Λ_k *is defined by (38).*

Proof. The proof follows immediately from Corollary 16 and Lemma 19. □

Remark 22. Here we also have an additional estimate

$$\left|R^B_{k,j}(x)\right| \leq \frac{\Gamma_{k+1}-\gamma_{k+1}}{2\sqrt{(2k)!}}\sqrt{|B_{2k}|}\alpha_{jnk}\frac{n-j+1}{n}\frac{2^n(b-a)^{k+1}}{\binom{n}{j}}.$$

We now consider the interpolation formula given in Corollary 6.

First we recall some properties of the Euler polynomials. The Euler polynomials are defined by the relation

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!},\quad |t|<\pi,$$

such that

$$E_0(x)=1,\ E_1(x)=x-\frac{1}{2},\ E_2(x)=x^2-x,...$$

The numbers $E_k=2^kE_k(\frac{1}{2})$ are Euler numbers; $E_0=1,E_2=-1,E_4=5$, $E_{2k+1}=0$, $k=1,2,....$

We have the properties

$$E_k'(x)=kE_{k-1}(x),\ k=1,2,...,$$

$$E_k(1-x)=(-1)^kE_k(x),\ \ k=0,1,2,...,$$

$$\int_a^x E_k(t)dt=\frac{E_{k+1}(x)-E_{k+1}(a)}{k+1},$$

$$\begin{aligned}(-1)^kE_{2k}(x) &> 0,\quad k=1,2,...,\ 0<x<\frac{1}{2},\\ (-1)^kE_{2k-1}(x) &> 0,\quad k=1,2,...,\ 0<x<\frac{1}{2},\end{aligned}$$

$$0<(-1)^kE_{2k}(x)<4^{-k}\left|E_{2k}\right|,\ k=1,2,...,\ 0<x<\frac{1}{2},$$

$$0<(-1)^kE_{2k-1}(x)<\frac{4(2k-1)!}{\pi^{2k}}\left(1+\frac{1}{2^{2k}-2}\right),\ k=1,2,...,\ 0<x<\frac{1}{2}.$$

Further properties of these polynomials can be found in [1].

We specially have

$$\begin{aligned}\int_0^1\left|E_k(t)\right|dt &= 2\left|\int_0^{\frac{1}{2}}E_k(t)dt\right|=2\left|\frac{E_{k+1}(\frac{1}{2})-E_{k+1}(0)}{k+1}\right| \qquad (47)\\ &= 2\left|\frac{2^{-k-1}E_{k+1}-E_{k+1}(0)}{k+1}\right|.\end{aligned}$$

We also introduce the notation

$$\theta_k=2\left|2^{-k-1}E_{k+1}-E_{k+1}(0)\right|. \qquad (48)$$

Theorem 23. *Under the assumptions of Corollary 6 we have*

$$\left|R_{k,j}^{E}(x)\right| \leq \frac{\theta_k}{(k+1)!} G_j(f^{(k+1)}) C_k(x) \left|\omega_n(x)\right|,$$

where θ_k *is defined by (48).*

Proof. We substitute the polynomials (10) in Theorem 10 and note that

$$\left|\int_{x_i}^{x} \left|E_k\left(\frac{t-x_i}{x-x_i}\right)\right| dt\right| = |x-x_i| \int_0^1 |E_k(t)|\, dt = \frac{\theta_k}{k+1} |x-x_i|. \tag{49}$$

□

Theorem 24. *Let the assumptions of Corollary 6 hold. If* γ_{k+1}, Γ_{k+1} *are real numbers such that* $\gamma_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}$, $t \in [a,b]$, $k = 0, 1, ..., n-1$, *then*

$$\left|R_{k,j}^{E}(x)\right| \leq \frac{\theta_k}{2(k+1)!} \left(\Gamma_{k+1} - \gamma_{k+1}\right) \left|\omega_n(x)\right| C_k(x),$$

where $\omega_n(x)$ *is defined by (5). We also have*

$$\begin{aligned} \left|R_{k,j}^{E}(x)\right| &\leq \frac{4^{-k/2}\left|E_k\right|}{k!} \left|\omega_n(x)\right| F_k(x), \\ \left|R_{k,j}^{E}(x)\right| &\leq \frac{4^{-k/2}\left|E_k\right|}{k!} \left|\omega_n(x)\right| D_k(x), \end{aligned}$$

if $k = 4r$, $r = 1, 2, ...,$ *and*

$$\begin{aligned} \left|R_{k,j}^{E}(x)\right| &\leq \frac{4}{\pi^{k+1}} \left(1 + \frac{1}{2^{k+1}-2}\right) \left|\omega_n(x)\right| F_k(x), \\ \left|R_{k,j}^{E}(x)\right| &\leq \frac{4}{\pi^{k+1}} \left(1 + \frac{1}{2^{k+1}-2}\right) \left|\omega_n(x)\right| D_k(x), \end{aligned}$$

if $k = 4r - 1$, $r = 1, 2, ...$

Proof. We substitute the polynomials (10) in Theorem 11 and use the following properties of Euler polynomials:

$$\begin{aligned} \max_{t\in[0,1]} \left|E_k(t)\right| &\leq 4^{-k/2}\left|E_k\right|, \; k = 4r, \; r = 1, 2, ..., \\ \max_{t\in[0,1]} \left|E_k(t)\right| &\leq \frac{4k!}{\pi^{k+1}} \left(1 + \frac{1}{2^{k+1}-2}\right), \; k = 4r-1, r = 1, 2, ... \end{aligned}$$

□

Corollary 25. *Under the assumptions of Lemma 19 and Theorem 24 we have*

$$\left|R^{E}_{k,j}(x)\right| \le \frac{\theta_k \alpha_{jnk}}{(k+1)!}\frac{n-j+1}{n}\frac{2^{n-1}(b-a)^{k+1}}{\binom{n}{j}}\left(\Gamma_{k+1}-\gamma_{k+1}\right).$$

Proof. The proof follows immediately from Theorem 24 and Lemma 19. □

Remark 26. Some of the above inequalities (for this choice of polynomials $P_k(t)$) are improvements of corresponding error inequalities obtained in [11].

Finally, we consider the interpolation formula given in Corollary 9.

Theorem 27. *Under the assumptions of Corollary 9 we have*

$$\left|R^{T}_{k,j}(x)\right| \le \frac{G_j(f^{(k+1)})}{(k+1)!}C_k(x)\left|\omega_n(x)\right|.$$

Proof. We substitute the polynomials (23) in Theorem 10. □

Theorem 28. *Let the assumptions of Corollary 9 hold. If γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \le f^{(k+1)}(t) \le \Gamma_{k+1}$, $t \in [a,b]$, $k = 0,1,...,n-1$, then*

$$\left|R^{T}_{k,j}(x)\right| \le \frac{\Gamma_{k+1}-\gamma_{k+1}}{2(k+1)!}C_k(x)\left|\omega_n(x)\right|,$$

where $\omega_n(x)$ is defined by (5). We also have

$$\left|R^{T}_{k,j}(x)\right| \le \frac{|\omega_n(x)|}{k!}F_k(x)$$

and

$$\left|R^{T}_{k,j}(x)\right| \le \frac{|\omega_n(x)|}{k!}D_k(x).$$

Proof. We substitute the polynomials (23) in Theorem 11. □

Corollary 29. *Under the assumptions of Lemma 19 and Theorem 28 we have*

$$\left|R^{T}_{k,j}(x)\right| \le \frac{\alpha_{jnk}}{(k+1)!}\frac{n-j+1}{n}\frac{2^{n-1}(b-a)^{k+1}}{\binom{n}{j}}\left(\Gamma_{k+1}-\gamma_{k+1}\right).$$

Proof. The proof follows immediately from Theorem 28 and Lemma 19. □

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