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Wavelet Transform Associated with the q-Dunkl Operator

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Abstract

In this paper, we present some new elements of harmonic analysis related to the q-Dunkl operator introduced in [1], we define and study the q-wavelets and the continuous q-wavelet transforms associated with this operator. Next, as an application, we give inversion formulas for the q-Dunkl intertwining operator and its dual using q-wavelets.

1. Introduction

In [11, 12], R. L. Rubin constructed a q^2-analogue Fourier analysis associated with a q^2-analogue differential operator ∂_q. Using this q-harmonic analysis, the

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authors studied in [4] the q-wavelets and the continuous q-wavelet transforms associated with the operator ∂_q.

In [1], the authors introduced a q-analogue of the Dunkl operator on $\mathbb{R}$ and they defined and studied its associated Fourier transform $F_D^{\alpha,q}$, called q-Dunkl transform, which is a q-analogue of the Bessel-Dunkl transform. They, also, studied the q-Dunkl intertwining operator $V_{\alpha,q}$ and its dual ${}^tV_{\alpha,q}$ via the q-analogues of the Riemann-Liouville and Weyl transforms $R_{\alpha,q}$ and ${}^tR_{\alpha,q}$, studied in [5]. In particular, they proved that $V_{\alpha,q}$ and its dual are automorphism of some spaces $\mathcal{E}_q(\mathbb{R}_q)$ and $\mathcal{D}_q(\mathbb{R}_q)$, respectively and they gave their inversion operators using $R_{\alpha,q}$ and ${}^tR_{\alpha,q}$.

In this paper, we define the generalized q-Dunkl translation operator and its related convolution product, we give some of their properties, then, we are interested by studying the q-wavelets and the continuous q-wavelet transforms associated with the q-Dunkl operator. Next, we establish an inversion formulas for the q-Dunkl intertwining operator $V_{\alpha,q}$ and its dual ${}^tV_{\alpha,q}$ using q-wavelets.

This paper is organized as follows: in Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we recall some results and properties concerning the q-Dunkl transform studied in [1], we introduce the generalized q-Dunkl translation operator and its related convolution product and we give some of their properties. In Section 4, we define and study the q-wavelet and the continuous q-wavelet transform associated with the q-Dunkl operator, and we provide for this transform a Plancherel formula and an inversion theorem. Section 5 is devoted to give some inversion formulas for the q-Dunkl intertwining operator and its dual on some new spaces (other than $\mathcal{E}_q(\mathbb{R}_q)$ and $\mathcal{D}_q(\mathbb{R}_q)$). Finally, in Section 6, we give some relations between the continuous q-wavelet transform associated with the q-Dunkl operator and those associated with the q^2-analogue differential operator ∂_q, studied in [4]. Next, by the help of these relations, we derive the inversion formulas of the q-Dunkl intertwining operator and its dual using q-wavelets.

2. Notations and Preliminaries

For the convenience of the reader,we provide in this section a summary of the mathematical notations and definitions used in this paper, we will follow the notations of [11, 12]. We fix $q \in]0, 1[$ and we refer to the book by G. Gasper and M. Rahman [6] for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.

We will write
$\mathbb{R}_q = \{\pm q^n \ : \ n \in \mathbb{Z}\}$, $\widetilde{\mathbb{R}}_q = \{\pm q^n \ : \ n \in \mathbb{Z}\} \cup \{0\}$.
For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a; q)_0 = 1; \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, 2, \ldots; \quad (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k). \tag{1}$$

We also denote

$$[x]_q = \frac{1 - q^x}{1 - q}, \qquad x \in \mathbb{C} \ ; \ [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \qquad n \in \mathbb{N}. \tag{2}$$

The q^2-analogue differential operator is (see [12], [11]),

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1 - q)z} & if \quad z \neq 0, \\ \lim\limits_{x \to 0} \partial_q(f)(x) \qquad (\text{in } \ \mathbb{R}_q) & if \quad z = 0. \end{cases} \tag{3}$$

Note that if f is differentiable at z, then $\lim\limits_{q \to 1} \partial_q(f)(z) = f'(z)$.
The q-trigonometric functions q-cosine and q-sine are defined by (see [11, 12]):

$$\cos(x; q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!} \tag{4}$$

and

$$\sin(x; q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n + 1]_q!}. \tag{5}$$

These functions induce a ∂_q-adapted q^2-analogue exponential function by

$$e(z; q^2) = \cos(-iz; q^2) + i \sin(-iz; q^2). \tag{6}$$

$e(z;q^2)$ is absolutely convergent for all z in the plane since both of its component functions are. $\lim_{q\to 1^-} e(z;q^2) = e^z$ (exponential function) pointwise and uniformly on compacts.
Using the same technique as in [11], one can prove that for all $x \in \mathbb{R}_q$, we have

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty} \quad \text{and} \quad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty},$$

so,

$$\forall x \in \mathbb{R}_q, \quad |e(-ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{7}$$

The q-Jackson integrals from 0 to $a \in \mathbb{R}$ and from $-\infty$ to $+\infty$ are defined by (see [7], [8], [10], [9])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{8}$$

and

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} \{f(q^n) + f(-q^n)\}\, q^n, \tag{9}$$

provided the sums converge absolutely.
The following result can be verified by direct computation.

Lemma 1. *If* $\int_{-\infty}^{\infty} f(t)d_qt$ *exists, then for all* $a \in \mathbb{R}_q$,

$$\int_{-\infty}^{\infty} f(at)d_qt = |a|^{-1}\int_{-\infty}^{\infty} f(t)d_qt.$$

In the sequel, we will need the following spaces:
• $\mathcal{E}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$, satisfying

$$\forall n \in \mathbb{N}, \quad a \geq 0, \qquad P_{n,a}(f) = \sup\left\{|\partial_q^k f(x)|; 0 \leq k \leq n; x \in [-a,a]\cap\mathbb{R}_q\right\} < \infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

We provide it with the topology defined by the semi norms $P_{n,a}$.

- $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n, m \in \mathbb{N}, \qquad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} \mid x^m \partial_q^n f(x) \mid < +\infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

- $\mathcal{D}_q(\mathbb{R}_q)$ the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of functions with compact supports.
- $L^p_{\alpha,q}(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\}, \quad p > 0$ and $\alpha \in \mathbb{R}$.
- $L^\infty_q(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}.$

3. Elements of q-Dunkl Harmonic Analysis

In this section, we collect some basic results and properties from the q-Dunkl operator theory, studied in [1], and we introduce and study a generalized q-Dunkl translation as well as its related convolution product.

For $\alpha \geq -\frac{1}{2}$, the q-Dunkl transform is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ by (see [1])

$$F_D^{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_{-\infty}^{+\infty} f(x) \psi_{-\lambda}^{\alpha,q}(x).|x|^{2\alpha+1} d_q x, \qquad \lambda \in \widetilde{\mathbb{R}}_q, \tag{10}$$

where $c_{\alpha,q} = \dfrac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}$ and $\psi_\lambda^{\alpha,q}$ is the q-Dunkl kernel defined by

$$\psi_\lambda^{\alpha,q} : x \longmapsto j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2), \tag{11}$$

with $j_\alpha(x;q^2)$ is the normalized third Jackson's q-Bessel function given by:
$j_\alpha(x;q^2) = \displaystyle\sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}.$

It was proved in [1] that for all $\lambda \in \mathbb{C}$, the function: $x \longmapsto \psi_\lambda^{\alpha,q}(x)$ is the unique solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{\alpha,q}(f) & = i\lambda f \\ f(0) & = 1, \end{cases} \tag{12}$$

where $\Lambda_{\alpha,q}$ is the q-Dunkl operator defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q \left[f_e + q^{2\alpha+1} f_o \right](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x}, \tag{13}$$

with f_e and f_o are respectively the even and the odd parts of f.
We recall that the q-Dunkl operator $\Lambda_{\alpha,q}$ lives the spaces $\mathcal{D}_q(\mathbb{R}_q)$ and $\mathcal{S}_q(\mathbb{R}_q)$ invariant. Some other properties of the q-Dunkl kernel and the q-Dunkl transform are given in the following results (see [1]).

Proposition 1.
i) $\psi_\lambda^{\alpha,q}(x) = \psi_x^{\alpha,q}(\lambda)$, $\psi_{a\lambda}^{\alpha,q}(x) = \psi_\lambda^{\alpha,q}(ax)$, $\overline{\psi_\lambda^{\alpha,q}(x)} = \psi_{-\lambda}^{\alpha,q}(x)$, $\forall \lambda, x \in \mathbb{R}$, $a \in \mathbb{C}$.
ii) If $\alpha = -\frac{1}{2}$, *then* $\psi_\lambda^{\alpha,q}(x) = e(i\lambda x; q^2)$.
For $\alpha > -\frac{1}{2}$, $\psi_\lambda^{\alpha,q}$ *has the following q-integral representation of Mehler type*

$$\psi_\lambda^{\alpha,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}(1+t)e(i\lambda xt;q^2)d_qt. \tag{14}$$

iii) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q}$ *is bounded on* $\widetilde{\mathbb{R}}_q$ *and we have*

$$| \psi_\lambda^{\alpha,q}(x) | \leq \frac{4}{(q;q)_\infty}, \quad \forall x \in \widetilde{\mathbb{R}}_q. \tag{15}$$

iv) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q} \in \mathcal{S}_q(\mathbb{R}_q)$.
v) The function $\psi_\lambda^{\alpha,q}$ *verifies the following orthogonality relation: For all* $x, y \in \mathbb{R}_q$,

$$\int_{-\infty}^{+\infty} \psi_\lambda^{\alpha,q}(x)\overline{\psi_\lambda^{\alpha,q}(y)}|\lambda|^{2\alpha+1}dq\lambda = \frac{4(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)\delta_{x,y}}{(1-q)|xy|^{\alpha+1}}. \tag{16}$$

vi) If $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ *then* $F_D^{\alpha,q}(f) \in L_q^\infty(\mathbb{R}_q)$,

$$\|F_D^{\alpha,q}(f)\|_{\infty,q} \leq \frac{4c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,\alpha,q}, \tag{17}$$

and

$$\lim_{\substack{|\lambda|\to+\infty \\ \lambda\in\mathbb{R}_q}} F_D^{\alpha,q}(f)(\lambda)=0, \quad \lim_{\substack{|\lambda|\to 0 \\ \lambda\in\widetilde{\mathbb{R}}_q}} F_D^{\alpha,q}(f)(\lambda)=F_D^{\alpha,q}(f)(0). \tag{18}$$

vii) For $f\in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$F_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda)=i\lambda F_D^{\alpha,q}(f)(\lambda). \tag{19}$$

viii) For $f,g\in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$\int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)g(\lambda)|\lambda|^{2\alpha+1}d_q\lambda = \int_{-\infty}^{+\infty} f(x)F_D^{\alpha,q}(g)(x)|x|^{2\alpha+1}d_qx. \tag{20}$$

Theorem 1. *For all* $f\in L^1_{\alpha,q}(\mathbb{R}_q)$, *we have*

$$\begin{aligned}\forall x\in\mathbb{R}_q,\quad f(x)&=c_{\alpha,q}\int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x).|\lambda|^{2\alpha+1}d_q\lambda\\ &=\overline{F_D^{\alpha,q}(\overline{F_D^{\alpha,q}(f)})}(x).\end{aligned} \tag{21}$$

Theorem 2. *i) Plancherel formula*
For $\alpha\geq -1/2$, *the q-Dunkl transform* $F_D^{\alpha,q}$ *is an isomorphism from* $\mathcal{S}_q(\mathbb{R}_q)$ *onto itself. Moreover, for all* $f\in\mathcal{S}_q(\mathbb{R}_q)$, *we have*

$$\|F_D^{\alpha,q}(f)\|_{2,\alpha,q}=\|f\|_{2,\alpha,q}. \tag{22}$$

ii) Plancherel theorem
The q-Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. *Its inverse transform* $(F_D^{\alpha,q})^{-1}$ *is given by :*

$$(F_D^{\alpha,q})^{-1}(f)(x)=c_{\alpha,q}\int_{-\infty}^{+\infty} f(\lambda)\psi_\lambda^{\alpha,q}(x).|\lambda|^{2\alpha+1}d_q\lambda=F_D^{\alpha,q}(f)(-x). \tag{23}$$

We are now in a position to define the generalized q-Dunkl translation operator.

Definition 1. *The generalized q-Dunkl translation operator is defined for* $f\in L^2_{\alpha,q}(\mathbb{R}_q)$ *and* $x,y\in\mathbb{R}_q$ *by*

$$T_y^{\alpha;q}(f)(x)=c_{\alpha,q}\int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)|\lambda|^{2\alpha+1}d_q\lambda, \tag{24}$$

$$T_0^{\alpha;q}(f)=f.$$

It verifies the following properties.

Proposition 2.
1) For all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ *and all* $x \in \mathbb{R}_q$,

$$\lim_{\substack{y\to 0\\ y\in\mathbb{R}_q}} T_y^{\alpha;q}(f)(x) = f(x).$$

2) For all $x, y \in \mathbb{R}_q$, $T_y^{\alpha;q}(f)(x) = T_x^{\alpha;q}(f)(y)$.
3) If $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ $(resp.\ \ \mathcal{S}_q(\mathbb{R}_q))$ *then* $T_y^{\alpha;q}(f) \in L^2_{\alpha,q}(\mathbb{R}_q)$ $(resp.\ \ \mathcal{S}_q(\mathbb{R}_q))$ *and we have*

$$\| T_y^{\alpha;q}(f) \|_{2,\alpha,q} \leq \frac{4}{(q;q)_\infty} \| f \|_{2,\alpha,q} . \tag{25}$$

4) For all $x, y, \lambda \in \mathbb{R}_q$, $T_y^{\alpha;q}\left(\psi_\lambda^{\alpha,q}\right)(x) = \psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)$.
5) For $f \in L^2_{\alpha,q}(\mathbb{R}_q), x, y \in \mathbb{R}_q$, *we have*

$$F_D^{\alpha,q}(T_y^{\alpha;q} f)(\lambda) = \psi_\lambda^{\alpha,q}(y) F_D^{\alpha,q}(f)(\lambda). \tag{26}$$

6) For $f \in \mathcal{S}_q(\mathbb{R}_q)$ *and* $y \in \mathbb{R}_q$, *we have*

$$\Lambda_{\alpha,q} T_y^{\alpha;q} f = T_y^{\alpha;q} \Lambda_{\alpha,q} f.$$

Proof.
1) Since $\psi_\lambda^{\alpha,q}$ is bounded on $\mathbb{R}_q$, then the Lebesgue theorem and Theorem 1 give the result.
2) Follows from the definition of the generalized q-Dunkl translation.
3) Since $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ $(resp.\ \ \mathcal{S}_q(\mathbb{R}_q))$, $\psi_y^{\alpha,q}$ is bounded for all y (resp. $\psi_y^{\alpha,q} \in \mathcal{S}_q(\mathbb{R}_q)$) and $F_D^{\alpha,q}$ is an automorphism of $L^2_{\alpha,q}(\mathbb{R}_q)$ $(resp.\ \ \mathcal{S}_q(\mathbb{R}_q))$, then $F_D^{\alpha,q}(f).\psi_y^{\alpha,q}$ is in $L^2_{\alpha,q}(\mathbb{R}_q)$ $(resp.\ \ \mathcal{S}_q(\mathbb{R}_q))$. So, by the help of the Plancherel theorem, we have for all $y \in \mathbb{R}_q$, $T_y^{\alpha;q} f = (F_D^{\alpha,q})^{-1}\left(F_D^{\alpha,q}(f).\psi_y^{\alpha,q}\right)$ is in $L^2_{\alpha,q}(\mathbb{R}_q)$ $(resp.\ \ \mathcal{S}_q(\mathbb{R}_q))$. Moreover, using the Plancherel formula and the relation (15), we obtain

$$\begin{aligned} \| T_y^{\alpha;q} f \|_{2,\alpha,q} &= \| (F_D^{\alpha,q})^{-1}(F_D^{\alpha,q}(f).\psi_y^{\alpha,q}) \|_{2,\alpha,q} \\ &= \| F_D^{\alpha,q}(f).\psi_y^{\alpha,q} \|_{2,\alpha,q} \leq \frac{4}{(q;q)_\infty} \| f \|_{2,\alpha,q} . \end{aligned}$$

4) Using the orthogonality relation (16), we obtain

$$F_D^{\alpha,q}\left(\psi_\lambda^{\alpha,q}\right)(y) = \frac{2(1+q)^\alpha \Gamma_{q^2}(\alpha+1)}{(1-q)|\lambda y|^{\alpha+1}} \quad \delta_{\lambda,y}, \quad \lambda, y \in \mathbb{R}_q,$$

witch implies that $T_y^{\alpha;q}\left(\psi_\lambda^{\alpha,q}\right)(x) = \psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)$.
5) From the fact that for $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ and $y \in \mathbb{R}_q$, $T_y^{\alpha;q} f = (F_D^{\alpha,q})^{-1}\left[F_D^{\alpha,q}(f).\psi_y^{\alpha,q}\right]$, we get

$$F_D^{\alpha,q}(T_y^{\alpha;q} f)(\lambda) = \left[F_D^{\alpha,q}(f).\psi_y^{\alpha,q}\right](\lambda) = \psi_\lambda^{\alpha,q}(y).F_D^{\alpha,q}(f)(\lambda).$$

6) For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$F_D^{\alpha,q}\left(\Lambda_{\alpha,q} T_y^{\alpha;q} f\right)(\lambda) = i\lambda F_D^{\alpha,q}\left(T_y^{\alpha;q} f\right)(\lambda) = i\lambda \psi_\lambda^{\alpha,q}(y) F_D^{\alpha,q}(f)(\lambda)$$

and

$$F_D^{\alpha,q}\left(T_y^{\alpha;q} \Lambda_{\alpha,q} f\right)(\lambda) = \psi_\lambda^{\alpha,q}(y) F_D^{\alpha,q}\left(\Lambda_{\alpha,q} f\right)(\lambda) = i\lambda \psi_\lambda^{\alpha,q}(y) F_D^{\alpha,q}(f)(\lambda).$$

The result follows, then, from the fact that $F_D^{\alpha,q}$ is an automorphism of $\mathcal{S}_q(\mathbb{R}_q)$.
□

Definition 2. *The q-convolution product is defined for $f, g \in \mathcal{S}_q(\mathbb{R}_q)$ by:*

$$f * g(x) = c_{\alpha,q} \int_{-\infty}^{\infty} T_x^{\alpha;q} f(-y) g(y) |y|^{2\alpha+1} d_q y. \tag{27}$$

In the following proposition, we present some of its properties.

Proposition 3. *For $f, g \in \mathcal{S}_q(\mathbb{R}_q)$, we have*
i) $F_D^{\alpha,q}(f * g) = F_D^{\alpha,q}(f).F_D^{\alpha,q}(g)$.
ii) $f * g = g * f$.
iii) $(f * g) * h = f * (g * h)$.

Proof.
i) Let $f, g \in \mathcal{S}_q(\mathbb{R}_q)$. Then, with the help of the relation (15), we have for all $x, y \in \mathbb{R}_q$,

$$\begin{aligned} |T_x^{\alpha;q} f(-y)| &\leq c_{\alpha,q} \int_{-\infty}^{\infty} |F_D^{\alpha,q}(f)(\lambda) \psi_\lambda^{\alpha,q}(x) \psi_\lambda^{\alpha,q}(-y)| \, |\lambda|^{2\alpha+1} d_q \lambda \\ &\leq c_{\alpha,q} \left(\frac{4}{(q;q)_\infty} \right)^2 \|F_D^{\alpha,q}(f)\|_{1,\alpha,q}. \end{aligned}$$

So, since for $\lambda \in \mathbb{R}_q$, $\psi^{\alpha,q}_{-\lambda} \in \mathcal{S}_q(\mathbb{R}_q)$, we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |T^{\alpha;q}_x f(-y)g(y)\psi^{\alpha,q}_{-\lambda}(x)||y|^{2\alpha+1}|x|^{2\alpha+1}d_qyd_qx \\ \leq & \frac{16c_{\alpha,q}}{(q;q)^2_\infty}\|F^{\alpha,q}_D(f)\|_{1,\alpha,q}\|g\|_{1,\alpha,q}\|\psi^{\alpha,q}_{-\lambda}\|_{1,\alpha,q}. \end{aligned}$$

Hence, using the Fubini's theorem and the properties of the generalized q-Dunkl translation, we get

$$\begin{aligned} F^{\alpha,q}_D(f*g)(\lambda) &= c_{\alpha,q}\int_{-\infty}^{\infty}(f*g)(x)\psi^{\alpha,q}_{-\lambda}(x)|x|^{2\alpha+1}d_qx \\ &= c^2_{\alpha,q}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}T^{\alpha;q}_x f(-y)g(y)|y|^{2\alpha+1}d_qy\right]\psi^{\alpha,q}_{-\lambda}(x)|x|^{2\alpha+1}d_qx \\ &= c^2_{\alpha,q}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}T^{\alpha;q}_{-y} f(x)\psi^{\alpha,q}_{-\lambda}(x)|x|^{2\alpha+1}d_qx\right]g(y)|y|^{2\alpha+1}d_qy \\ &= c_{\alpha,q}\int_{-\infty}^{\infty}F^{\alpha,q}_D(T^{\alpha;q}_{-y}f)(\lambda)g(y)|y|^{2\alpha+1}d_qy \\ &= c_{\alpha,q}\int_{-\infty}^{\infty}F^{\alpha,q}_D(f)(\lambda)\psi^{\alpha,q}_{\lambda}(-y)g(y)|y|^{2\alpha+1}d_qy \\ &= c_{\alpha,q}F^{\alpha,q}_D(f)(\lambda)\int_{-\infty}^{\infty}\psi^{\alpha,q}_{-\lambda}(y)g(y)|y|^{2\alpha+1}d_qy \\ &= F^{\alpha,q}_D(f)(\lambda)F^{\alpha,q}_D(g)(\lambda). \end{aligned}$$

ii) and iii) follows from i). □

Proposition 4. *Let f and g be in $\mathcal{S}_q(\mathbb{R}_q)$. Then*
1) $f*g \in \mathcal{S}_q(\mathbb{R}_q)$,
2)

$$\int_{-\infty}^{\infty} \mid f*g(x)\mid^2 |x|^{2\alpha+1}d_qx = \int_{-\infty}^{\infty}\mid F^{\alpha,q}_D(f)(x)\mid^2\mid F^{\alpha,q}_D(g)(x)\mid^2 |x|^{2\alpha+1}d_qx. \tag{28}$$

Proof. The proof is a direct consequence of Theorem 2 and the fact that $F^{\alpha,q}_D(f*g) = F^{\alpha,q}_D(f)F^{\alpha,q}_D(g)$. □

4. q-wavelet Transforms Associated with the q-Dunkl Operator

Definition 3. *A q-wavelet, associated with the q-Dunkl operator, is a square q-integrable function g on $\mathbb{R}_q$ satisfying the following admissibility condition*

$$0 < C_{\alpha,g} = \int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(g)(a) \mid^2 \frac{d_q a}{|a|} < \infty. \tag{29}$$

Remark 1.
1) For all $\lambda \in \mathbb{R}_q$, we have

$$C_{\alpha,g} = \int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(g)(a\lambda) \mid^2 \frac{d_q a}{|a|}.$$

2) Let f be a nonzero function in $\mathcal{S}_q(\mathbb{R}_q)$. Then $g = \Lambda_{\alpha,q} f$ is a q-wavelet, in $\mathcal{S}_q(\mathbb{R}_q)$ and we have

$$C_{\alpha,g} = \int_{-\infty}^{\infty} |a| \mid F_D^{\alpha,q}(f)(a) \mid^2 d_q a.$$

Proposition 5. *Let $g \neq 0$ be a function in $L^2_{\alpha,q}(\mathbb{R}_q)$ satisfying:*

(1) $F_D^{\alpha,q}(g)$ is continuous at 0.

(2)$\exists \beta > 0$ such that $F_D^{\alpha,q}(g)(x) - F_D^{\alpha,q}(g)(0) = O(x^\beta)$, as $x \to 0$, $x \in \mathbb{R}_q$.

Then, the admissibility condition (29) is equivalent to

$$F_D^{\alpha,q}(g)(0) = 0. \tag{30}$$

Proof. Assume that (29) is satisfied.
If $F_D^{\alpha,q}(g)(0) \neq 0$, then there exist $p_0 \in \mathbb{N}$ and $M > 0$, such that

$$\forall n \geq p_0, \quad \mid F_D^{\alpha,q}(g)(\pm q^n) \mid \geq M.$$

So, the q-integral in (29) would be equal to ∞.
- Conversely, assume that $F_D^{\alpha,q}(g)(0) = 0$.
Since $g \neq 0$, we deduce from Theorem 2, that the first inequality in (29) holds.

On the other hand, from the assertion (2), there exist $n_0 \in \mathbb{N}$ and $\epsilon > 0$, such that for all $n \geq n_0$, we have

$$\mid F_D^{\alpha,q}(g)(\pm q^n) \mid \leq \epsilon q^{n\beta}.$$

Then using the definition of the q-integral and Theorem 2, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(g)(a) \mid^2 \frac{d_q a}{|a|} &= (1-q) \sum_{n=-\infty}^{\infty} \left[\mid F_D^{\alpha,q}(g)(q^n) \mid^2 + \mid F_D^{\alpha,q}(g)(-q^n) \mid^2\right] \\ &\leq \frac{\|F_D^{\alpha,q}(g)\|^2_{2,\alpha,q}}{q^{(2\alpha+2)n_0}} + \frac{2(1-q)}{1-q^{2\beta}}\epsilon^2.\end{aligned}$$

This proves the second inequality of (29). □

Remark 2.
Using the relation (18), the continuity assumption in the previous proposition will certainly hold if g is moreover in $L^1_{\alpha,q}(\mathbb{R}_q)$. Then (30) can be equivalently written as

$$\int_{-\infty}^{\infty} g(x)|x|^{2\alpha+1} d_q x = 0,$$

which implies that g must have sign changes on $\mathbb{R}_q$. It will also decay to 0 as t tends to $\pm\infty$ (in $\mathbb{R}_q$). This explains the name "q-wavelet".

Proposition 6. *For $a \in \mathbb{R}_q$ and $g \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$), the function*

$$g_a : x \mapsto \frac{1}{|a|^{2\alpha+2}} g\left(\frac{x}{a}\right)$$

belongs to $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) and we have

$$\|g_a\|_{2,\alpha,q} = \frac{1}{|a|^{\alpha+1}} \|g\|_{2,\alpha,q} \tag{31}$$

and

$$F_D^{\alpha,q}(g_a)(\lambda) = F_D^{\alpha,q}(g)(a\lambda), \quad \lambda \in \widetilde{\mathbb{R}}_q. \tag{32}$$

Proof. The change of variables $u = \dfrac{x}{a}$ gives the result. □

Proposition 7. *Let g be a q-wavelet, associated with he q-Dunkl operator, in $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$). Then, for all $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$, the function $g_{(a,b),\alpha}$ defined by*

$$g_{(a,b),\alpha}(x) = \sqrt{|a|}T_b^{\alpha;q}(g_a)(x), \qquad x \in \mathbb{R}_q, \tag{33}$$

is a q-wavelet associated with he q-Dunkl operator in $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) and we have

$$C_{\alpha,g_{(a,b),\alpha}} = |a| \int_{-\infty}^{\infty} \left|\psi_b^{\alpha,q}\left(\frac{x}{a}\right)\right|^2 \mid F_D^{\alpha,q}(g)(x) \mid^2 \frac{d_q x}{|x|}, \tag{34}$$

with $T_b^{\alpha;q}$ is the generalized q-Dunkl translation operator defined by (24).

Proof. Using Proposition 2, Proposition 6 and the properties of the generalized q-Dunkl translation, we obtain $g_{(a,b),\alpha}$ is in $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$).
From Lemma 1 and the relations (26) and (32), we get for $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$,

$$\begin{aligned} C_{\alpha,g_{(a,b),\alpha}} &= |a| \int_{-\infty}^{\infty} \left|F_D^{\alpha,q}\left[T_b^{\alpha;q}(g_a)\right](x)\right|^2 \frac{d_q x}{|x|} \\ &= |a| \int_{-\infty}^{\infty} |\psi_b^{\alpha,q}\left(\frac{x}{a}\right)|^2 \mid F_D^{\alpha,q}(g)(x) \mid^2 \frac{d_q x}{|x|}. \end{aligned}$$

Thus, since $g \neq 0$, we have, from the Plancherel theorem, $F_D^{\alpha,q}(g) \neq 0$ and

$$0 < C_{\alpha,g_{(a,b),\alpha}} \leq |a| \left(\frac{4}{(q;q)_\infty}\right)^2 \int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(g)(x) \mid^2 \frac{d_q x}{|x|} = \frac{16|a|}{(q;q)_\infty^2} C_{\alpha,g} < +\infty.$$

□

Proposition 8. *Let g be a q-wavelet, associated with the q-Dunkl operator, in $L^2_{\alpha,q}(\mathbb{R}_q)$. Then the mapping*

$$F : (a,b) \mapsto g_{(a,b),\alpha}$$

is continuous from $\mathbb{R}_q \times \widetilde{\mathbb{R}}_q$ into $L^2_{\alpha,q}(\mathbb{R}_q)$, via the induced topology on $\mathbb{R}_q \times \widetilde{\mathbb{R}}_q$ by that of $\mathbb{R} \times \mathbb{R}$.

Proof. It is clear, from the previous proposition, that F is a mapping from $\mathbb{R}_q \times \widetilde{\mathbb{R}}_q$ into $L^2_{\alpha,q}(\mathbb{R}_q)$ and it is continuous on $\mathbb{R}_q \times \mathbb{R}_q$, since every element of

$\mathbb{R}_q \times \mathbb{R}_q$ is an isolated point.
Now, let $a \in \mathbb{R}_q$. For $b \in \widetilde{\mathbb{R}}_q$, we have

$$\begin{aligned}\| F(a,b) - F(a,0) \|_{2,\alpha,q}^2 &= |a| \parallel T_b^{\alpha;q}(g_a) - g_a \|_{2,\alpha,q}^2 \\ &= |a| \parallel F_D^{\alpha,q}\left(T_b^{\alpha;q}(g_a) - g_a\right) \|_{2,\alpha,q}^2 \\ &= |a| \int_{-\infty}^{\infty} | 1 - \psi_b^{\alpha,q}(x) |^2 | F_D^{\alpha,q}(g_a) |^2 (x) |x|^{2\alpha+1} d_q x.\end{aligned}$$

Using the relation (15), the fact that $F_D^{\alpha,q}(g_a) \in L^2_{\alpha,q}(\mathbb{R}_q)$ and the Lebesgue theorem we obtain

$$\lim_{\substack{b \to 0 \\ b \in \widetilde{\mathbb{R}}_q}} \| F(a,b) - F(a,0) \|_{2,\alpha,q} = 0.$$

□

Definition 4. *Let g be a q-wavelet, associated with the q-Dunkl operator, in $\mathcal{S}_q(\mathbb{R}_q)$. We define the continuous q-wavelet transform associated with the q-Dunkl operator for $f \in \mathcal{S}_q(\mathbb{R}_q)$, by*

$$\Psi_{q,g}^{\alpha}(f)(a,b) = c_{\alpha,q} \int_{-\infty}^{\infty} f(x) \overline{g_{(a,b),\alpha}}(-x) |x|^{2\alpha+1} d_q x, \qquad a \in \mathbb{R}_q, \quad b \in \widetilde{\mathbb{R}}_q. \quad (35)$$

Remark that (35) is equivalent to

$$\begin{aligned}\Psi_{q,g}^{\alpha}(f)(a,b) &= \sqrt{|a|} f * \overline{g_a}(b) \\ &= \sqrt{|a|} F_D^{\alpha,q}\left[F_D^{\alpha,q}(f * \overline{g_a})\right](-b) \\ &= \sqrt{|a|} F_D^{\alpha,q}\left[F_D^{\alpha,q}(f) . F_D^{\alpha,q}(\overline{g_a})\right](-b) \\ &= \sqrt{|a|} c_{\alpha,q} \int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(x) . F_D^{\alpha,q}(\overline{g})(ax) \psi_b^{\alpha,q}(x) |x|^{2\alpha+1} d_q x,\end{aligned}$$

where $c_{\alpha,q}$ is given by (10).

The following propositions give some properties of $\Psi_{q,g}^{\alpha}$.

Proposition 9. *Let g be a q-wavelet, associated with the q-Dunkl operator, in $\mathcal{S}_q(\mathbb{R}_q)$ and $f \in \mathcal{S}_q(\mathbb{R}_q)$. Then*
i) For all $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$, we have

$$| \Psi_{q,g}^{\alpha}(f)(a,b) | \leq \frac{4 c_{\alpha,q}}{|a|^{\alpha+\frac{1}{2}} \quad (q;q)_\infty} \|f\|_{2,\alpha,q} \|g\|_{2,\alpha,q}; \qquad (36)$$

ii) For all $a \in \mathbb{R}_q$, the mapping $b \mapsto \Psi^{\alpha}_{q,g}(f)(a,b)$ is continuous on $\widetilde{\mathbb{R}}_q$, via the induced topology on $\widetilde{\mathbb{R}}_q$ by that of $\mathbb{R}$, and we have

$$\lim_{b\to\infty} \Psi^{\alpha}_{q,g}(f)(a,b) = 0. \tag{37}$$

Proof. i) Using the properties of the generalized q-Dunkl translation operator, the Cauchy-Schwartz inequality and Lemma 1, we obtain for $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$,

$$\begin{aligned} \mid \Psi^{\alpha}_{q,g}(f)(a,b) \mid &= c_{\alpha,q} \mid \int_{-\infty}^{\infty} f(x)\overline{g_{(a,b),\alpha}}(-x)|x|^{2\alpha+1} d_q x \mid \\ &\leq \sqrt{|a|} c_{\alpha,q} \int_{-\infty}^{\infty} \mid f(x) \mid\mid T^{\alpha;q}_b g_a(-x) \mid |x|^{2\alpha+1} d_q x \\ &\leq \frac{4c_{\alpha,q}}{|a|^{\alpha+\frac{1}{2}} \ (q;q)_\infty} \|f\|_{2,\alpha,q} \|g\|_{2,\alpha,q}. \end{aligned}$$

ii) Since every element of $\mathbb{R}_q$ is an isolated point, it is sufficient to prove the continuity at 0. For $b \in \widetilde{\mathbb{R}}_q$, we have

$$\Psi^{\alpha}_{q,g}(f)(a,b) = \sqrt{|a|} F^{\alpha,q}_D \left[F^{\alpha,q}_D(f).F^{\alpha,q}_D(\overline{g_a})\right](-b).$$

Since $f, g \in \mathcal{S}_q(\mathbb{R}_q)$, then from Theorem 2, we have $F^{\alpha,q}_D(f)$ and $F^{\alpha,q}_D(\overline{g_a})$ are in $\mathcal{S}_q(\mathbb{R}_q)$ and the product $F^{\alpha,q}_D(f).F^{\alpha,q}_D(\overline{g_a})$ is in $L^1_{\alpha,q}(\mathbb{R}_q)$. Thus, using the relation (15), the Lebesgue theorem, gives

$$\begin{aligned} \lim_{\substack{b\to 0 \\ b\in\widetilde{\mathbb{R}}_q}} \Psi^{\alpha}_{q,g}(f)(a,b) &= \lim_{\substack{b\to 0 \\ b\in\widetilde{\mathbb{R}}_q}} \sqrt{|a|} c_{\alpha,q} \int_{-\infty}^{\infty} F^{\alpha,q}_D(f)(x).F^{\alpha,q}_D(\overline{g_a})(x)\psi^{\alpha,q}_b(x) d_q x \\ &= \Psi^{\alpha}_{q,g}(f)(a,0). \end{aligned}$$

Which proves the continuity of $\Psi^{\alpha}_{q,g}(f)(a,.)$ at 0.
Finally, the relation (18) implies that

$$\Psi^{\alpha}_{q,g}(a,b) = \sqrt{|a|} F^{\alpha,q}_D [F^{\alpha,q}_D(f).F^{\alpha,q}_D(\overline{g_a})](-b)$$

tends to 0 when b tends to ∞. □

Let us now establish a Plancherel and a Parseval formulas for $\Psi^{\alpha}_{q,g}$.

Theorem 3. *Let $g \in \mathcal{S}_q(\mathbb{R}_q)$ be a q-wavelet, associated with the q-Dunkl operator.*
i) **Plancherel formula**
For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$\frac{1}{C_{\alpha,g}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mid \Psi_{q,g}^{\alpha}(f)(a,b) \mid^2 |b|^{2\alpha+1} \frac{d_q a d_q b}{|a|^2} = \|f\|_{2,\alpha,q}^2. \tag{38}$$

ii) **Parseval formula**
For $f_1, f_2 \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{\infty} f_1(x)\overline{f}_2(x)|x|^{2\alpha+1} d_q x = \frac{1}{C_{\alpha,g}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^{\alpha}(f_1)(a,b)\overline{\Psi_{q,g}^{\alpha}(f_2)}(a,b)|b|^{2\alpha+1} \frac{d_q a d_q b}{|a|^2}. \tag{39}$$

Proof. The use of the Fubini's theorem, Theorem 2, the relations (28) and (32) gives

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mid \Psi_{q,g}^{\alpha}(f)(a,b) \mid^2 |b|^{2\alpha+1} \frac{d_q a d_q b}{|a|^2} \\
= & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mid f * \overline{g_a}(b) \mid^2 |b|^{2\alpha+1} d_q b \right) \frac{d_q a}{|a|} \\
= & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |[F_D^{\alpha,q}(f) F_D^{\alpha,q}(\overline{g_a})]\,(b)|^2 |b|^{2\alpha+1} d_q b \right) \frac{d_q a}{|a|} \\
= & \int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(f)(b) \mid^2 |b|^{2\alpha+1} \left(\int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(g)(ab) \mid^2 \frac{d_q a}{|a|} \right) d_q b \\
= & \; C_{\alpha,g} \int_{-\infty}^{\infty} \mid F_D^{\alpha,q}(f)(b) \mid^2 |b|^{2\alpha+1} d_q b = C_{\alpha,g} \|f\|_{2,\alpha,q}^2.
\end{aligned}$$

ii) The result follows from (38). □

Theorem 4. *Let $g \in \mathcal{S}_q(\mathbb{R}_q)$ be a q-wavelet, associated with the q-Dunkl operator. Then for $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have*

$$f(x) = \frac{c_{\alpha,q}}{C_{\alpha,g}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{q,g}^{\alpha}(f)(a,b) g_{(a,b),\alpha}(-x)|b|^{2\alpha+1} \frac{d_q a d_q b}{|a|^2}, \qquad x \in \mathbb{R}_q. \tag{40}$$

Proof. Let $x \in \mathbb{R}_q$ and put $h = \delta_x$. It is easy to see that $h \in \mathcal{S}_q(\mathbb{R}_q)$. According to the relation (39) of the previous theorem and the definition of $\Psi^\alpha_{q,g}$ and the q-Jackson integral, we have,

$$(1-q)|x|^{2\alpha+2} f(x) = \int_{-\infty}^{\infty} f(t)\overline{h}(t)|t|^{2\alpha+1} d_q t$$

$$= \frac{1}{C_{\alpha,g}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi^\alpha_{q,g}(f)(a,b)\overline{\Psi^\alpha_{q,g}}(h)(a,b)|b|^{2\alpha+1}\frac{d_q a d_q b}{|a|^2}.$$

$$= \frac{c_{\alpha,q}}{C_{\alpha,g}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi^\alpha_{q,g}(f)(a,b)\left(\int_{-\infty}^{\infty} \overline{h}(t) g_{(a,b),\alpha}(-t)|t|^{2\alpha+1} d_q t\right)|b|^{2\alpha+1}\frac{d_q a d_q b}{|a|^2}$$

$$= (1-q)|x|^{2\alpha+2}\frac{c_{\alpha,q}}{C_{\alpha,g}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi^\alpha_{q,g}(f)(a,b) g_{(a,b),\alpha}(-x)|b|^{2\alpha+1}\frac{d_q a d_q b}{|a|^2}.$$

□

5. Inversion Formulas for the q-Dunkl Intertwining Operator and its Dual

In the what follows, we will need the following spaces:

- $\mathcal{S}_{q,\alpha}(\mathbb{R}_q) = \left\{ f \in \mathcal{S}_q(\mathbb{R}_q) : \quad \int_{-\infty}^{+\infty} f(x)\, x^k |x|^{2\alpha+1} d_q x = 0,\ k = 0,1,... \right\}.$
- $\mathcal{S}^0_q(\mathbb{R}_q) = \left\{ f \in \mathcal{S}_q(\mathbb{R}_q) : \quad \partial^k_q f(0) = 0, \quad k = 0,1,... \right\}.$

We recall that the q-Dunkl intertwining operator $V_{\alpha,q}$ is defined on $\mathcal{E}_q(\mathbb{R}_q)$ by (see [1])

$$V_{\alpha,q}(f)(x) = \frac{(1+q)}{2}\frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})}\int_{-1}^{1}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}(1+t)f(xt)d_q t. \quad (41)$$

The dual operator of $V_{\alpha,q}$ is defined on $\mathcal{D}_q(\mathbb{R}_q)$ by (see [1])

$$({}^tV_{\alpha,q})(f)(t) = \frac{(1+q)^{-\alpha+\frac{1}{2}}}{2\Gamma_{q^2}(\alpha+\frac{1}{2})}\int_{|x|\geq q|t|}\frac{\left((\frac{t}{x})^2q^2;q^2\right)_\infty}{\left((\frac{t}{x})^2q^{2\alpha+1};q^2\right)_\infty}\left(1+\frac{t}{x}\right)f(x)\frac{|x|^{2\alpha+1}}{x}d_q x. \quad (42)$$

These two operators satisfy the following properties:

Proposition 10. *i)* $V_{\alpha,q}(e(-i\lambda x;q^2)) = \psi^{\alpha,q}_{-\lambda}(x)$, $\lambda, x \in \mathbb{R}_q$.

ii) For $f \in \mathcal{E}_q(\mathbb{R}_q)$ and $g \in \mathcal{D}_q(\mathbb{R}_q)$

$$c_{\alpha,q} \int_{-\infty}^{+\infty} V_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_{-\infty}^{+\infty} f(t)({}^tV_{\alpha,q})(g)(t)d_qt. \tag{43}$$

iii) $V_{\alpha,q}$ and ${}^tV_{\alpha,q}$ verify the following transmutation relations

$$\Lambda_{\alpha,q}V_{\alpha,q}(f) = V_{\alpha,q}(\partial_q f), \qquad V_{\alpha,q}(f)(0) = f(0), \quad f \in \mathcal{E}_q(\mathbb{R}_q), \tag{44}$$

$$\partial_q({}^tV_{\alpha,q})(f) = ({}^tV_{\alpha,q})(\Lambda_{\alpha,q})(f), \quad f \in \mathcal{D}_q(\mathbb{R}_q). \tag{45}$$

iv) The q-Dunkl transform and the q^2-analogue Fourier transform $\mathcal{F}_q$, studied in ([11], [12]), are linked by the following relation (see [1]):

$$\forall f \in \mathcal{D}_q(\mathbb{R}_q), \quad F_D^{\alpha,q}(f) = \mathcal{F}_q \circ \ {}^tV_{\alpha,q}(f), \tag{46}$$

where

$$\mathcal{F}_q(f)(x) = F_D^{-\frac{1}{2},q}(f)(x) = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt. \tag{47}$$

We state the following results, useful in the sequel.

Theorem 5. *The q^2-analogue Fourier transform $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ into $\mathcal{S}_q^0(\mathbb{R}_q)$.*

Proof. The result follows from the fact that $\partial_q e(-ix;q^2) = -ie(-ix;q^2)$. □
Similarly, we have the following result.

Theorem 6. *The q-Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ into $\mathcal{S}_q^0(\mathbb{R}_q)$.*

Proof. On the one hand, we have for all $k \in \mathbb{N}$,

$$\partial_q^k F_D^{\alpha,q}(f)(\lambda) = \int_{-\infty}^{+\infty} f(x)|x|^{2\alpha+1}\partial_{q,\lambda}^k \left[\psi_{-\lambda}^{\alpha,q}(x)\right] d_qx.$$

On the other hand, from the relation (14), we have

$$\partial_{q,\lambda}^k \left[\psi_{-\lambda}^{\alpha,q}\right](x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})}(-i)^k x^k \int_{-1}^{1} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}(1+t)t^k e(-i\lambda xt,q^2)d_qt,$$

which gives the result. □

Corollary 1. *The operator* ${}^tV_{\alpha,q}$ *is an isomorphism from* $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ *into* $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$.

Proof. We deduce the result from the relation $F_D^{\alpha,q} = \mathcal{F}_q \circ {}^tV_{\alpha,q}$ and Theorems 5 and 6. □

Proposition 11. *For f in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ (resp. $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$) and g in $\mathcal{S}_q(\mathbb{R}_q)$ the function $f *_q g$ (resp. $f * g$) belongs to $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ (resp. $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$). where "$*_q$" is the q- convolution product associated with the operator ∂_q studied in [11].*

Proof. The result follows from Theorem 5 (resp. 6) and the fact that $f *_q g = \mathcal{F}_q^{-1}(\mathcal{F}_q(f).\mathcal{F}_q(g))$ (resp. $f * g = (F_D^{\alpha,q})^{-1}(F_D^{\alpha,q}(f).F_D^{\alpha,q}(g))$. □

Using the Taylor formula for the Jackson's q-derivative (see [3, 2]), we provide in the following lemma a Taylor formula for the operator ∂_q.

Lemma 2. *Let f be a function N times continuously q-differentiable on $\widetilde{\mathbb{R}}_q$, $N \in \mathbb{N}$. Then,*

$$f(x) = \sum_{n=0}^{N} q^{(E(\frac{n+1}{2}))^2} \frac{\partial_q^n f(0)}{[n]_q!} x^n + \frac{x^N}{[N]_q!} \int_0^1 (tq;q)_N \quad H_{q,N+1}(f)(xt) d_qt,$$

where for $n \in \mathbb{N}$, $E\left(\frac{n+1}{2}\right)$ is the integer part of $\frac{n+1}{2}$ and $H_{q,n}$ is the operator defined by

$$H_{q,n}(f)(t) = q^{a_n} \partial_q^n f_o(tq^{E(1+\frac{n}{2})}) + q^{b_n} \partial_q^n f_e(tq^{E(\frac{n+1}{2})}),$$

with f_o and f_e are respectively the odd and the even parts of f,

$$a_n = \begin{cases} \frac{n(n+2)}{4}, & \text{if } n \text{ is even,} \\ \frac{(n+1)^2}{4}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad b_n = \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even,} \\ \frac{(n-1)(n+1)}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 12. *The operator $K_{\alpha,q,1}$ defined by*

$$K_{\alpha,q,1}(f) = \frac{\Gamma_{q^2}(1/2)}{(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} \mathcal{F}_q^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_q(f))$$

is an isomorphism from $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ onto itself.

Proof. Using the previous lemma, one can prove that the multiplication operator $f \mapsto \dfrac{\Gamma_{q^2}(1/2)}{(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} |\lambda|^{2\alpha+1} f$ is an isomorphism from $\mathcal{S}_q^0(\mathbb{R}_q)$ onto itself, its inverse is given by $f \mapsto \dfrac{(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)|\lambda|^{2\alpha+1}} f$. The result follows, then, from Theorem 5. □

Proposition 13. *The operator $K_{\alpha,q,2}$ defined by*

$$K_{\alpha,q,2}(f)(x) = \frac{\Gamma_{q^2}(1/2)}{(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} (F_D^{\alpha,q})^{-1}(|\lambda|^{2\alpha+1} F_D^{\alpha,q}(f))(x)$$

is an isomorphism from $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ onto itself.

Proof. From the relation $F_D^{\alpha,q} = \mathcal{F}_q \circ \ {}^tV_{\alpha,q}$ and the definition of $K_{\alpha,q,1}$, we have for all $f \in \mathcal{S}_{q,\alpha}(\mathbb{R}_q)$,

$$K_{\alpha,q,2} = ({}^tV_{\alpha,q})^{-1} \circ K_{\alpha,q,1} \circ \ {}^tV_{\alpha,q}. \tag{48}$$

We deduce the result from Proposition 12 and Corollary 1. □

Proposition 14.
i) For all $f \in \mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ and $g \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$K_{\alpha,q,1}(f *_q g) = K_{\alpha,q,1}(f) *_q g.$$

ii) For all $f \in \mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ and $g \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$K_{\alpha,q,2}(f * g) = K_{\alpha,q,2}(f) * g.$$

Proof. The result follows from the properties of the q-convolution product and the definitions of $K_{\alpha,q,1}$ and $K_{\alpha,q,2}$. □

Theorem 7. *For all $f \in \mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, we have the following inversion formulas for the operator $V_{\alpha,q}$*

$$f = V_{\alpha,q} \circ K_{\alpha,q,1} \circ \ {}^tV_{\alpha,q}(f) \tag{49}$$

and

$$f = V_{\alpha,q} \circ^t V_{\alpha,q} \circ K_{\alpha,q,2}(f). \tag{50}$$

Proof. Using the properties of the operator $V_{\alpha,q}$, studied in [1], Theorem 1 and relation (46), we obtain for $x \in \widetilde{\mathbb{R}}_q$,

$$\begin{aligned} f(x) &= c_{\alpha,q} \int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x).|\lambda|^{2\alpha+1} d_q\lambda \\ &= V_{\alpha,q}\left[c_{\alpha,q}\int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda)e(i\lambda\cdot;q^2)|\lambda|^{2\alpha+1}d_q\lambda\right](x) \\ &= V_{\alpha,q}\left\{\frac{c_{\alpha,q}}{c_{-1/2,q}}\mathcal{F}_q^{-1}\left[|\lambda|^{2\alpha+1}\mathcal{F}_q \circ \ {}^tV_{\alpha,q}(f)\right]\right\}(x) \\ &= V_{\alpha,q} \circ K_{\alpha,q,1} \circ \ {}^tV_{\alpha,q}(f)(x). \end{aligned}$$

We deduce the second from the first relation and the the relation (48). □

Corollary 2. *The operator $V_{\alpha,q}$ is an isomorphism from $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ into $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$.*

Proof. We deduce the result from Proposition 12, Corollary 1 and the relation (49). □

Similarly, we have the following result.

Theorem 8. *For all $f \in \mathcal{S}_{q,-1/2}(\mathbb{R}_q)$, we have the following inversion formulas for the operator ${}^tV_{\alpha,q}$*

$$f = \ {}^tV_{\alpha,q} \circ V_{\alpha,q} \circ K_{\alpha,q,1}(f) \tag{51}$$

and

$$f = \ {}^tV_{\alpha,q} \circ K_{\alpha,q,2} \circ V_{\alpha,q}(f). \tag{52}$$

Proof. For $f \in \mathcal{S}_{q,-1/2}(\mathbb{R}_q)$, Corollary 1 (resp. 2) implies that ${}^tV_{\alpha,q}^{-1}(f)$ (resp. $V_{\alpha,q}(f)$) belongs to $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$. Then by writing the relation (49) (resp. (50)) for ${}^tV_{\alpha,q}^{-1}(f)$ (resp. $V_{\alpha,q}(f)$), we obtain the result. □

Corollary 3. *i) For all $f, g \in \mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, we have*

$${}^tV_{\alpha,q}\,(f * g) = \ {}^tV_{\alpha,q}(f) *_q \ {}^tV_{\alpha,q}(g). \tag{53}$$

ii) For all $f, g \in \mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ we have

$$V_{\alpha,q}\,(f *_q g) = V_{\alpha,q}\,(f) * \ {}^tV_{\alpha,q}^{-1}(g). \tag{54}$$

6. Inversion of the q-Dunkl Intertwining Operator and of its Dual using Wavelets

In this section, we assume that the reader is familiar with the notions and notations presented in [4], where the authors studied the particular case $\alpha = -\frac{1}{2}$. In particular, we recall the following notations

$$H_a(f)(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x}{a}\right), \qquad C_g = \int_{-\infty}^{\infty} |\mathcal{F}_q(g)|^2(a) \frac{d_q a}{|a|},$$

$$g_{a,b} = g_{(a,b),-1/2} \text{ and } \Phi_{q,g} = \Psi_{q,g}^{-1/2}.$$

We begin by the following useful and easily verified result.

Proposition 15. *For all $a \in \mathbb{R}_q$ and all $g \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have*

$$\begin{aligned} g_a = \frac{1}{|a|^{2\alpha+3/2}} H_a(g) &= \frac{1}{\sqrt{|a|}} (F_D^{\alpha,q})^{-1} \circ H_{a^{-1}} \circ F_D^{\alpha,q}(g) \\ &= \frac{1}{\sqrt{|a|}} \ {}^tV_{\alpha,q}^{-1} \circ H_a \circ \ {}^tV_{\alpha,q}(g). \end{aligned}$$

Proposition 16. *Let $g \in \mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ be a q-wavelet, associated with the q-Dunkl operator. Then, for all f in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, we have*

$$\Psi_{q,g}^{\alpha}(f)(a,.) = \ {}^tV_{\alpha,q}^{-1}\left[\Phi_{q,\ {}^tV_{\alpha,q}(g)}\left(\ {}^tV_{\alpha,q}(f)\right)(a,.)\right], \qquad a \in \mathbb{R}_q. \tag{55}$$

Proof. Let $a \in \mathbb{R}_q$, from the properties of the continuous q-wavelet transform (see [4]), the relation (53) and Proposition (15), we have

$$\begin{aligned} \Psi_{q,g}^{\alpha}(f)(a,.) &= \sqrt{|a|} f * \overline{g_a} = \sqrt{|a|} \ {}^tV_{\alpha,q}^{-1}\left[\ {}^tV_{\alpha,q}(f) *_q \ {}^tV_{\alpha,q}(\overline{g_a})\right] \\ &= \ {}^tV_{\alpha,q}^{-1}\left[\ {}^tV_{\alpha,q}(f) *_q \overline{H_a \circ \ {}^tV_{\alpha,q}(g)}\right] \\ &= \ {}^tV_{\alpha,q}^{-1}\left[\Phi_{q,\ {}^tV_{\alpha,q}(g)}\left(\ {}^tV_{\alpha,q}(f)\right)(a,.)\right]. \end{aligned}$$

□

Theorem 9. *Let $g \in \mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ be a q-wavelet, associated with the q-Dunkl operator. Then,*
1) For all f in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, we have for $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$,

$$\Psi_{q,g}^{\alpha}(f)(a,b) = V_{\alpha,q}\left[\Phi_{q,\ {}^tV_{\alpha,q}(g)}\left(V_{\alpha,q}^{-1}(f)\right)(a,.)\right](b), \tag{56}$$

2) For all f in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$, we have for $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$,

$$\Phi_{q,\ {}^tV_{\alpha,q}(g)}(f)(a,b) = \ {}^tV_{\alpha,q}\left[\Psi^{\alpha}_{q,g}\left(\ {}^tV^{-1}_{\alpha,q}(f)\right)(a,.)\right](b). \tag{57}$$

Proof. We deduce the result from Proposition 15, Corollary 3, the properties of the continuous q-wavelet transform (see [4]) and the relation (55). □

Proposition 17. *1) If g is a q-wavelet (associated with the operator ∂_q) in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$, then $K_{\alpha,q,1}(g)$ is a q-wavelet in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$ and we have*

$$K_{\alpha,q,1} \circ H_a(g) = \frac{1}{|a|^{2\alpha+1}} H_a \circ K_{\alpha,q,1}(g), \qquad a \in \mathbb{R}_q. \tag{58}$$

2) If g is a q-wavelet, associated with the q-Dunkl operator, in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, then $K_{\alpha,q,2}(g)$ is a q-wavelet, associated with the q-Dunkl operator, in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$ and we have

$$K_{\alpha,q,2}(g_a) = \frac{1}{|a|^{2\alpha+1}} (K_{\alpha,q,2}(g))_a, \qquad a \in \mathbb{R}_q. \tag{59}$$

Proof. 1) Let g be a q-wavelet in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$. From the definition of $K_{\alpha,q,1}$, we have for $\lambda \in \mathbb{R}_q$, $\mathcal{F}_q(K_{\alpha,q,1}(g))(\lambda) = \dfrac{\Gamma_{q^2}(1/2)}{(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} |\lambda|^{2\alpha+1}\mathcal{F}_q(g)(\lambda)$. Proposition 5 of [4], implies that $K_{\alpha,q,1}(g)$ is a q-wavelet. On the other hand, using the fact $\mathcal{F}_q \circ H_a = H_{a^{-1}} \circ \mathcal{F}_q$, $a \in \mathbb{R}_q$ and the above equality, we obtain

$$\mathcal{F}_q(H_a \circ K_{\alpha,q,1}(g))(\lambda) = |a|^{2\alpha+1} \frac{\Gamma_{q^2}(1/2)}{(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} |\lambda|^{2\alpha+1} \mathcal{F}_q\left(H_a(g)\right)(\lambda),$$

which gives the result.
2) The same way of 1) leads to the result. □

Theorem 10. *Let g be a q-wavelet, associated with the q-Dunkl operator, in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$. Then for $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$, we have:*
1) For all f in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$,

$$\Psi^{\alpha}_{q,g}(f)(a,b) = \frac{1}{|a|^{2\alpha+1}} V_{\alpha,q}\left[\Phi_{q,K_{\alpha,q,1}\circ\ {}^tV_{\alpha,q}(g)}(\ {}^tV_{\alpha,q}(f))(a,.)\right](b); \tag{60}$$

2) For all f in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$,

$$\Phi_{q,\ {}^tV_{\alpha,q}(g)}(f)(a,b) = \frac{1}{|a|^{2\alpha+1}}\ {}^tV_{\alpha,q}\left[\Psi^{\alpha}_{q,K_{\alpha,q,2}(g)}(V_{\alpha,q}(f))(a,.)\right](b). \tag{61}$$

Proof. 1) Let f be in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, $a \in \mathbb{R}_q$ and $b \in \widetilde{\mathbb{R}}_q$. Using Corollary 3, we obtain $\Psi^{\alpha}_{q,g}(f)(a,b) = \sqrt{|a|} f * \overline{g_a}(b) = \sqrt{|a|} V_{\alpha,q}\left[\ {}^tV_{\alpha,q}(f) *_q V^{-1}_{\alpha,q}(\overline{g_a})\right](b)$.
So, Theorem 7, Proposition 17 and the relation (55) achieve the proof.
2) Follows from Corollary 3, Theorem 8, and Propositions 14 and 17. □

Theorem 11. *Let g be a q-wavelet, associated with the q-Dunkl operator, in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$. Then for all $x \in \mathbb{R}_q$,*
1) For all f in $\mathcal{S}_{q,-1/2}(\mathbb{R}_q)$, we have

$$
\begin{aligned}
&{}^tV^{-1}_{\alpha,q}(f)(x)\\
&= \frac{c_{\alpha,q}}{C_{\alpha,g}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} V_{\alpha,q}[\Phi_{q,K_{\alpha,q,1}\circ\ {}^tV_{\alpha,q}(g)}(f)(a,.)](b) \times g_{(a,b),\alpha}(-x) \frac{|b|^{2\alpha+1}}{|a|^{2\alpha+3}} d_q b \right) d_q a;
\end{aligned}
$$

2) For all f in $\mathcal{S}_{q,\alpha}(\mathbb{R}_q)$, we have

$$
V^{-1}_{\alpha,q}(f)(x) = \frac{c_{-\frac{1}{2},q}}{C_g} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} {}^tV_{\alpha,q}\left[\Psi^{\alpha}_{q,K_{\alpha,q,2}(g)}(f)(a,.)\right](b) g_{a,b}(-x) \frac{d_q b}{|a|^{2\alpha+3}} \right) d_q a.
$$

Proof. The result derives from the previous theorem, Theorem 4 and ([4], Theorem 5). □

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