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A Generalzation of Ostrowski-grüss Type Inequality for First Differentiable Mappings *

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Abstract

In this paper, we improve and further generalize some Ostrowski-Grüss type inequalities involving first differentiable functions and apply them to probability density functions, generalized beta random variable and special means.

Keywords and Phrases: *Ostrowski-Grüss inequality, Euclidean norm.*

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1. Introduction

In 1938, A. Ostrowski [8] proved the following integral inequality.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in I^0 (interior of I), and let $a, b \in I^0$ with $a < b$. If $f^{'} : (a,b) \to \mathbb{R}$ is bounded on (a,b) with $\sup_{t\in[a,b]} \left|f^{'}(t)\right| \leq M$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,M, \tag{1.1}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

This inequality provides an upper bound for the approximation of integral mean of a function f by the functional value $f(x)$ at $x \in [a,b]$ .In 1997, Dragomir and Wang [3], by the use of the Grüss inequality proved the following Ostrowski-Grüss type integral inequality.

Theorem 2. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If $\gamma \leq f^{'}(x) \leq \Gamma$, $x \in [a,b]$ for some constants $\gamma, \Gamma \in \mathbb{R}$, then*

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \\ \leq\ &\frac{1}{4}(b-a)(\Gamma - \gamma), \end{aligned} \tag{1.2}$$

for all $x \in [a,b]$.

This inequality provides a connection between Ostrowski inequality [8] and the Grüss inequality [5]. In 2000, M. Matić, J. Pecarić and N. Ujević [7], by the use of pre-Grüss inequality improved the factor of the right membership of (1.2) with $\frac{1}{4\sqrt{3}}$ as follows.

Theorem 3. *Under the assumption of Theorem 2, we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4\sqrt{3}} (\Gamma - \gamma)(b-a), \tag{1.3}$$

for all $x \in [a,b]$.

In 2000, Barnett et al.[1], by the use of Chebyshev's functional, improved the Matić-Pecarić-Ujević result by providing first membership of the right side of (1.3) in terms of Euclidean norm as follows.

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous function whose derivative* $f' \in L_2[a,b]$. *Then we have the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \\ \leq\ & \frac{(b-a)}{2\sqrt{3}} \left[\frac{1}{b-a} \left\| f' \right\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \\ & \left(\leq \frac{(b-a)(\Gamma - \gamma)}{4\sqrt{3}} \quad if \quad \gamma \leq f'(t) \leq \Gamma \ for\ a.e\ t\ on\ [a,b]. \right) \end{aligned} \tag{1.4}$$

for all $x \in [a,b]$.

We define for two mappings $f, g : [a,b] \to \mathbb{R}$, the Chebyshev functional as

$$T(f,g) = \frac{1}{b-a} \int_a^b f(t)\, g(t)\, dt - \left(\frac{1}{b-a} \int_a^b f(t)\, dt \right) \left(\frac{1}{b-a} \int_a^b g(t)\, dt \right),$$

provided that f, g and fg are integrable on $[a,b]$.

Also in [1] we can find the pre-Grüss inequality as

$$T^2(f,g) \leq T(f,f)\, T(g,g),$$

where $f, g \in L_2[a,b]$ and $T(f,g)$ is the Chebyshev functional as defined above.

In this paper, we give a generalization of (1.4) and then apply it to probability density functions, generalized beta random variable and special means.

2. Main Results

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function whose first derivative $f^{'}\in L_2[a,b]$. Then, we have the inequality*

$$\begin{aligned}
&\left|(1-h)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}} \\
&\times\left[\frac{1}{b-a}\left\|f^{'}\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{2}(\Gamma-\gamma)\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}, \\
&\left(if\quad \gamma\leq f^{'}(t)\leq\Gamma \text{ almost everywhere } t \text{ on } (a,b).\right) \qquad (2.1)
\end{aligned}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ and $h\in[0,1]$.

Proof. We consider the kernel $p:[a,b]^2\to\mathbb{R}$ as defined in [2]:

$$p(x,t)=\begin{cases} t-\left(a+h\frac{b-a}{2}\right), & if\ t\in[a,x] \\ t-\left(b-h\frac{b-a}{2}\right), & if\ t\in(x,b]. \end{cases}$$

Using Korkine's identity

$$T(f,g):=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds,$$

we obtain

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b p(x,t)f^{'}(t)\,dt-\frac{1}{b-a}\int_a^b p(x,t)\,dt\frac{1}{b-a}\int_a^b f^{'}(t)\,dt \\
=\ &\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\left(f^{'}(t)-f^{'}(s)\right)dtds. \qquad (2.2)
\end{aligned}$$

Since,

$$\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt=(1-h)\,f(x)+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt,$$

$$\frac{1}{b-a}\int_a^b p(x,t)\,dt=(1-h)\left(x-\frac{a+b}{2}\right),$$

and

$$\frac{1}{b-a}\int_a^b f'(t)\,dt=\frac{f(b)-f(a)}{b-a},$$

then by (2.2) we get the following identity

$$\begin{aligned}
&(1-h)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]\\
&+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\\
=\;&\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))\left(f'(t)-f'(s)\right)dtds, \qquad (2.3)
\end{aligned}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ and $h\in[0,1]$.

Using the Cauchy-Bunaikowski-Schwartz inequality for double integrals,we may write

$$\begin{aligned}
&\frac{1}{2(b-a)^2}\left|\int_a^b\int_a^b (p(x,t)-p(x,s))\left(f'(t)-f'(s)\right)dtds\right|\\
\leq\;&\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds\right)^{\frac{1}{2}}\\
&\times\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f'(t)-f'(s)\right)^2 dtds\right)^{\frac{1}{2}}. \qquad (2.4)
\end{aligned}$$

However,

$$\begin{aligned}
& \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds \\
= & \frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2 \\
= & \frac{1}{b-a}\left[\frac{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^3+\left(b-h\frac{b-a}{2}-x\right)^3}{3}+\frac{h^3(b-a)^3}{12}\right] \\
& -(1-h)^2\left(x-\frac{a+b}{2}\right)^2. \qquad (2.5)
\end{aligned}$$

In addition simple calculations shows that

$$\begin{aligned}
& \left(x-\left(a+h\frac{b-a}{2}\right)\right)^3+\left(b-h\frac{b-a}{2}-x\right)^3 \\
= & (b-a)(1-h)\left[3\left(x-\frac{a+b}{2}\right)^2+\frac{(1-h)^2(b-a)^2}{4}\right], \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f'(t)-f'(s)\right)^2 dtds \\
= & \frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2. \qquad (2.7)
\end{aligned}$$

Using (2.3)−(2.7),we deduce the first inequality.

Moreover, if $\gamma \le f'(t) \le \Gamma$ almost everywhere t on (a,b), then, by using Grüss inequality, we have

$$0 \le \frac{1}{b-a}\int_a^b \left(f'(t)\right)^2 dt - \left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2 \le \frac{1}{4}(\Gamma-\gamma)^2,$$

which proves the last inequality of (2.1).

Remark 1. Since

$$3h^2 - 3h + 1 \leq 1, \ \forall \ h \in [0,1] .$$

and is minimum for $h = \frac{1}{2}$.

Thus, (2.1) shows an overall improvement in the inequality obtained by Barnett et al. [5].

The following corollary contains some special cases of (2.1).

Corollary 1. *1. For $h = 1$, i.e., $x = \frac{a+b}{2}$, (2.1) gives*

$$\begin{aligned} & \left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t)\, dt \right| \\ \leq \ & \frac{(b-a)^2}{2\sqrt{3}} \left[\frac{1}{b-a} \left\| f' \right\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \\ \leq \ & \frac{1}{4\sqrt{3}} (\Gamma - \gamma)(b-a)^2 . \\ & \left(if \quad \gamma \leq f'(t) \leq \Gamma \ almost\ everywhere\ t\ on\ (a,b) . \right) \end{aligned} \tag{2.8}$$

which is trapezoid inequality.

2. For $h = 0$ and $x = \frac{a+b}{2}$, (2.1) gives

$$\begin{aligned} & \left| (b-a) f\left(\frac{a+b}{2} \right) - \int_a^b f(t)\, dt \right| \\ \leq \ & \frac{(b-a)^2}{2\sqrt{3}} \left[\frac{1}{b-a} \left\| f' \right\|_2^2 - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}, \\ \leq \ & \frac{1}{4\sqrt{3}} (\Gamma - \gamma)(b-a)^2 . \\ & \left(if \quad \gamma \leq f'(t) \leq \Gamma \ almost\ everywhere\ t\ on\ (a,b) . \right) \end{aligned} \tag{2.9}$$

which is mid-point inequality.

3. For $h=\frac{1}{2}$ *and* $x=\frac{a+b}{2}$, *(2.1) gives*

$$\begin{aligned}
&\left|\frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & \frac{(b-a)^2}{4\sqrt{3}}\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{8\sqrt{3}}(\Gamma-\gamma)(b-a)^2. \\
& \left(if\ \ \gamma\leq f'(t)\leq\Gamma \text{ almost everywhere } t \text{ on } (a,b).\right) \qquad (2.10)
\end{aligned}$$

which is an averaged mid-point and trapezoid inequality.

4. For $h=\frac{1}{3}$ *and* $x=\frac{a+b}{2}$, *(2.1) gives*

$$\begin{aligned}
&\left|(b-a)\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}-\int_a^b f(t)\,dt\right| \\
\leq\ & \frac{(b-a)^2}{6}\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \\
\leq\ & \frac{1}{12}(\Gamma-\gamma)(b-a)^2. \\
& \left(if\ \gamma\leq f'(t)\leq\Gamma \text{ almost everywhere } t \text{ on } (a,b).\right) \qquad (2.11)
\end{aligned}$$

which is a variant of Simpson's inequality for first differentiable function f, f' *is integrable and there exist constants* $\gamma,\Gamma\in R$ *such that* $\gamma\leq f'(t)\leq\Gamma$, $t\in(a,b)$.

3. Applications

3.1 Application for P.D.F's

Let X be a random variable having the p.d.f $f : [a, b] \to \mathbb{R}_+$ and the cumulative distribution function $F : [a, b] \to [0, 1]$, i.e.,

$$F(x) = \int_a^x f(t)\, dt, \ \ x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right] \subset [a, b],$$

and

$$E(X) = \int_a^b t f(t)\, dt,$$

is the expectation of the random variable X on the interval $[a, b]$. Then, we may have the following.

Theorem 6. *Under the above assumptions and if the p.d.f belongs to* $L_2[a, b]$, *then, we have the inequality:*

$$\begin{aligned} &\left|(1-h)\left[F(x) - \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)\right] + \frac{h}{2} - \frac{b - E(X)}{b-a}\right| \\ \leq\ & \frac{1}{b-a}\left[\frac{1}{12}\left(3h^2 - 3h + 1\right) + h(1-h)\left(x - \frac{a+b}{2}\right)^2\right]^{\frac{1}{2}} \\ & \times \left[(b-a)\|f\|_2^2 - 1\right]^{\frac{1}{2}}, \\ \leq\ & \frac{(M-m)}{2(b-a)}\left[\frac{1}{12}\left(3h^2 - 3h + 1\right) + h(1-h)\left(x - \frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}, \\ & (if\ \ m \leq f \leq M\ \ almost\ everywhere\ on\ \ [a, b].) \qquad (3.1) \end{aligned}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$.

Proof. Put in (2.1) F instead of f to get (3.1) and the details are omitted.

Corollary 2. *Under the above assumptions, we have*

$$\begin{aligned} & \left|(1-h)\Pr\left(X\leq\frac{a+b}{2}\right)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right| \\ \leq\ & \frac{1}{2\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}}\left[(b-a)\|f\|_2^2-1\right]^{\frac{1}{2}}, \\ \leq\ & \frac{1}{4\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}}(M-m),\ m\leq f\leq M \text{ as above.} \end{aligned} \tag{3.2}$$

3.2 Applications for generalized beta random variable

If X is a beta random variable with parameters $\beta_3 > -1$, $\beta_4 > -1$ and for $\beta_2 > 0$ and any β_1, the generalized beta random variable

$$Y = \beta_1 + \beta_2 X,$$

is said to have a generalized beta distribution [4] and the probability density function of the generalized beta distribution of beta random variable is,

$$f(x) = \begin{cases} \frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1 < x < \beta_1+\beta_2 \\ 0, & \text{otherwise.} \end{cases},$$

where $\beta(l,m)$ is the beta function with $l,\ m > 0$ and is defined as

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

For p, $q > 0$ and $h \in [0,1)$, we choose,

$$\begin{aligned} \beta_1 &= \frac{h}{2}, \\ \beta_2 &= (1-h), \\ \beta_3 &= p-1, \\ \beta_4 &= q-1. \end{aligned}$$

Then, the probability density function associated with generalized beta random variable

$$Y = \frac{h}{2} + (1-h)X,$$

takes the form

$$f(x)=\begin{cases}\frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)(1-h)^{p+q-1}}, & \frac{h}{2}<x<1-\frac{h}{2}\\ 0, & \text{otherwise.}\end{cases}$$

Now,

$$\begin{aligned}E(Y) &= \int_{\frac{h}{2}}^{1-\frac{h}{2}} xf(x)\,dx \\ &= (1-h)\frac{p}{p+q}+\frac{h}{2}. \qquad (3.3)\end{aligned}$$

and

$$\|f(.;p,q)\|_2^2=\frac{1}{(1-h)\beta^2(p,q)}\beta(2p-1,2q-1).$$

Then, by Theorem 6, we may state the following.

Proposition 1. *Let X be a beta random variable with parameters (p,q). Then for generalized beta random variable*

$$Y=\frac{h}{2}+(1-h)X,$$

we have the inequality

$$\begin{aligned}&\left|\left[\Pr(Y\le x)-x+\frac{1}{2}\right]-\frac{q}{p+q}\right| \\ \le\ &\left[\frac{1}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{1}{2}\right)^2\right]^{\frac{1}{2}} \\ &\times\frac{\left[\beta(2p-1,2q-1)-(1-h)\beta^2(p,q)\right]^{\frac{1}{2}}}{(1-h)^{\frac{3}{2}}\beta(p,q)}. \qquad (3.4)\end{aligned}$$

for all $x\in\left[\frac{h}{2},1-\frac{h}{2}\right]$.

In particular, for $x=\frac{1}{2}$ in (3.4), we have:

$$\begin{aligned}&\left|\Pr\left(Y\le\frac{1}{2}\right)-\frac{q}{p+q}\right| \\ \le\ &\frac{1}{2\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}}\frac{\left[\beta(2p-1,2q-1)-(1-h)\beta^2(p,q)\right]^{\frac{1}{2}}}{(1-h)^{\frac{3}{2}}\beta(p,q)}.\end{aligned}$$

3.3 Applications for Special Means

Example 1. Consider the mapping $f(x) = x^p$, $p \in \mathbb{R} \setminus \{-1, 0\}$. Then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\,dt &= L_p^p(a,b),\\ \frac{f(b)-f(a)}{b-a} &= pL_{p-1}^{p-1},\\ \frac{f(a)+f(b)}{2} &= \frac{a^p+b^p}{2} = A(a^p,b^p),\end{aligned}$$

and

$$\frac{1}{b-a}\left\|f'\right\|_2^2 = \frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt = p^2 L_{2(p-1)}^{2(p-1)}.$$

Therefore, (2.1) takes the form

$$\begin{aligned}&\left|(1-h)\left[x^p - pL_{p-1}^{p-1}(x-A(a,b))\right] + hA(a^p,b^p) - L_p^p\right|\\ \leq\ & |p|\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right) + h(1-h)(x-A(a,b))^2\right]^{\frac{1}{2}}\\ &\times\left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}} \qquad (3.5)\end{aligned}$$

Choose $x = A(a,b)$ in (3.5), to get

$$\begin{aligned}&\left|(1-h)A^p(a,b) + hA(a^p,b^p) - L_p^p\right|\\ \leq\ & \frac{(b-a)}{2\sqrt{3}}\left(3h^2-3h+1\right)^{\frac{1}{2}}|p|\left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}}.\end{aligned}$$

which is minimum for $h = \frac{1}{2}$. Moreover for $h = 1$,

$$\begin{aligned}&\left|A(a^p,b^p) - L_p^p\right|\\ \leq\ & \frac{(b-a)}{2\sqrt{3}}|p|\left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}}.\end{aligned}$$

Example 2. Consider the mapping $f(x) = \frac{1}{x}$, $\left(x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right] \subset (0,\infty)\right)$.

Then,

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{L}, \\
\frac{f(b)-f(a)}{b-a} &= -\frac{1}{G^2}, \\
\frac{f(a)+f(b)}{2} &= \frac{A(a,b)}{G^2}, \\
\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt &= \frac{a^2+ab+b^2}{3a^3b^3},
\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2 = \frac{(b-a)^2}{3a^3b^3}.$$

Therefore, (2.1) becomes,

$$\begin{aligned}
&\left|(1-h)\left[\frac{1}{x}+\frac{1}{G^2}(x-A(a,b))\right]+h\frac{A(a,b)}{G^2}-\frac{1}{L}\right| \\
\leq\ &\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)(x-A(a,b))^2\right]^{\frac{1}{2}} \\
&\times\frac{(b-a)}{\sqrt{3}G^3}. \qquad (3.6)
\end{aligned}$$

Choosing $x=A(a,b)$ in (3.6),

$$\begin{aligned}
&\left|(1-h)\frac{1}{A}+h\frac{A(a,b)}{G^2}-\frac{1}{L}\right| \\
\leq\ &\frac{(b-a)^2}{6G^3}\left(3h^2-3h+1\right)^{\frac{1}{2}}.
\end{aligned}$$

If we choose $x = L$ in (3.6),we get

$$\begin{aligned} & \left|(1-h)\frac{L}{G^2}+(2h-1)\frac{A(a,b)}{G^2}-h\frac{1}{L}\right| \\ \leq & \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)(L-A(a,b))^2\right]^{\frac{1}{2}} \\ & \times\frac{(b-a)}{\sqrt{3}G^3}. \end{aligned}$$

Example 3. Finally, let us consider the mapping
$f(x) = \ln x, \ \left(x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right] \subset (0,\infty)\right)$. Then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &= \ln I, \\ \frac{f(b)-f(a)}{b-a} &= \frac{1}{L}, \\ \frac{f(a)+f(b)}{2} &= \ln G, \\ \frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt &= \frac{1}{G^2}, \end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2 = \frac{L^2-G^2}{L^2G^2}.$$

Thus, (2.1) takes the form,

$$\begin{aligned} & \left|\ln\frac{x^{(1-h)}G^h}{I}-(1-h)\frac{x-A(a,b)}{L}\right| \\ \leq & \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)(x-A(a,b))^2\right]^{\frac{1}{2}} \\ & \times\frac{(L^2-G^2)^{\frac{1}{2}}}{LG}. \end{aligned} \tag{3.7}$$

For $x = A(a,b)$ in (3.7), we get:

$$\begin{aligned}&\left|\ln \frac{(A(a,b))^{(1-h)} G^h}{I}\right| \\ \leq\ & \frac{(b-a)(3h^2-3h+1)^{\frac{1}{2}}}{2\sqrt{3}} \frac{(L^2-G^2)^{\frac{1}{2}}}{LG}.\end{aligned}$$

which for $h = 1$, takes the form

$$\begin{aligned}&\left|\ln \frac{G}{I}\right| \\ \leq\ & \frac{(b-a)(L^2-G^2)^{\frac{1}{2}}}{2\sqrt{3}LG}.\end{aligned}$$

Also, choosing $x = I$, we get

$$\begin{aligned}&\left|\ln \frac{G^h}{I^h} - (1-h)\frac{I - A(a,b)}{L}\right| \\ \leq\ & \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right) + h(1-h)(I - A(a,b))^2\right]^{\frac{1}{2}} \\ & \times \frac{(L^2-G^2)^{\frac{1}{2}}}{LG}.\end{aligned}$$

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