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New Sharp Bound for a General Ostrowski Type Inequality *

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Abstract

The main purpose of this paper is to give a new sharp bound for a general Ostrowski type inequality which provides some improvement of a previous result.

Keywords and Phrases: *Ostrowski type inequality, Absolutely continuous, Sharp bound.*

1. Introduction

In [1], the author has proved a general sharp Ostrowski-Grüss type inequality as follows:

Theorem 1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 1$ and there exist constants $\gamma_n, \Gamma_n \in \mathbf{R}$ such that $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$ for a.e.$t \in [a,b]$. Then for all $x \in [a,b]$, we have*

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$$
\begin{aligned}
& |f(x) - \frac{(b-x)^n + (-1)^{n-1}(x-a)^n}{(n+1)!(b-a)} f^{(n-1)}(x) \\
& + \sum_{k=1}^{n-1} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!(b-a)} f^{(k)}(x) \\
& + \frac{(b-x)^n f^{(n-1)}(b) + (-1)^{n-1}(x-a)^n f^{(n-1)}(a)}{(n+1)!(b-a)} \\
& - \frac{1}{b-a} \int_a^b f(t)\, dt| \\
\leq \;& \frac{n}{(n+1)(n+1)!\sqrt[n]{n+1}} ((x-a)^{n+1} + (b-x)^{n+1})(\Gamma_n - \gamma_n).
\end{aligned} \tag{1}
$$

The equality in (1) is attained by choosing $f(t) = \int_a^t (\int_a^{y_n} (\cdots \int_a^{y_2} j(y_1)\, dy_1 \cdots)\, dy_{n-1})\, dy_n$, *where*

$$
j(t) = \begin{cases} \gamma_n, & a \leq t < t_1 = a + \frac{1}{\sqrt[n]{n+1}}(x-a), \\ \Gamma_n, & t_1 \leq t < x, \\ \gamma_n, & x \leq t < t_2 = b - \frac{1}{\sqrt[n]{n+1}}(b-x), \\ \Gamma_n, & t_2 \leq t \leq b. \end{cases}
$$

if n *is odd, and*

$$
j(t) = \begin{cases} \gamma_n, & a \leq t < t_1, \\ \Gamma_n, & t_1 \leq t < x, \\ \Gamma_n, & x \leq t < t_2, \\ \gamma_n, & t_2 \leq t \leq b. \end{cases}
$$

if n *is even.*

The main purpose of this note is to give a new sharp bound in (1) in terms of the Euclidean norm of $f^{(n)}$ *which is valid also for a larger class of mappings, i.e., for the mappings* f *for which* $f^{(n)}$ *is unbounded on* (a, b) *but* $f^{(n)} \in L_2[a, b]$*. Some special cases are also considered.*

2. The Results

Theorem 2. *Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ for some $n \geq 1$ and $f^{(n)} \in L_2[a, b]$. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} & |\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \\ & + \frac{(b-x)^n + (-1)^{n-1}(x-a)^n}{(n+1)!} f^{(n-1)}(x) \\ & - \frac{(b-x)^n f^{(n-1)}(b) + (-1)^{n-1}(x-a)^n f^{(n-1)}(a)}{(n+1)!}| \\ \leq \; & \frac{n}{(n+1)!\sqrt{2n+1}} \sqrt{(x-a)^{2n+1} + (b-x)^{2n+1}} \sqrt{\sigma(f^{(n)})} \end{aligned} \tag{2}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}(\int_a^b f(t)\,dt)^2 \tag{3}$$

and

$$\|f\|_2 := [\int_a^b f^2(t)\,dt]^{\frac{1}{2}}. \tag{4}$$

The inequality (2) is sharp in the sense that the constant $\frac{n}{(n+1)!\sqrt{2n+1}}$ cannot be replaced by a smaller one.

Proof. From Lemma 3 in [1], for all $x \in [a, b]$, we have the identity:

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\\ &-\frac{(b-x)^n+(-1)^{n-1}(x-a)^n}{(n+1)!}f^{(n-1)}(x)\\ &+\frac{(b-x)^n f^{(n-1)}(b)+(-1)^{n-1}(x-a)^n f^{(n-1)}(a)}{(n+1)!}\\ &+(-1)^n\int_a^b H_n(x,t)f^{(n)}(t)\,dt,\end{aligned} \tag{5}$$

where the kernel $H_n : [a,b]^2 \to \mathbf{R}$ is given by

$$H_n(x,t) := \begin{cases} \frac{(t-a)^n}{n!} - \frac{(x-a)^n}{(n+1)!}, & a \le t < x, \\ \frac{(t-b)^n}{n!} - \frac{(x-b)^n}{(n+1)!}, & x \le t \le b. \end{cases} \tag{6}$$

By elemently calculus, it is not difficult to get

$$\int_a^b H_n(x,t)\,dt = 0 \tag{7}$$

and

$$\int_a^b H_n^2(x,t)\,dt = \frac{n^2}{[(n+1)!]^2(2n+1)}[(x-a)^{2n+1}+(b-x)^{2n+1}]. \tag{8}$$

From (3)-(8),we can easily get

$$
\begin{aligned}
& |\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \\
& + \frac{(b-x)^n+(-1)^{n-1}(x-a)^n}{(n+1)!} f^{(n-1)}(x) \\
& - \frac{(b-x)^n f^{(n-1)}(b)+(-1)^{n-1}(x-a)^n f^{(n-1)}(a)}{(n+1)!}| \\
= \ & |\int_a^b H_n(x,t) f^{(n)}(t)\,dt| \\
= \ & |\int_a^b H_n(x,t)[f^{(n)}(t) - \frac{1}{b-a}\int_a^b f^{(n)}(s)\,ds]\,dt| \\
\leq \ & (\int_a^b H_n^2(x,t)\,dt)^{\frac{1}{2}} (\int_a^b [f^{(n)}(t) - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}]^2\,dt)^{\frac{1}{2}} \\
= \ & (\frac{n^2}{(2n+1)[(n+1)!]^2}[(x-a)^{2n+1}+(b-x)^{2n+1}])^{\frac{1}{2}} \\
& \times (\|f^{(n)}\|_2^2 - \frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a})^{\frac{1}{2}} \\
= \ & \frac{n}{(n+1)!\sqrt{2n+1}} \sqrt{(x-a)^{2n+1}+(b-x)^{2n+1}} \sqrt{\sigma(f^{(n)})}.
\end{aligned}
$$

We now suppose that (2) holds with a constant $C > 0$ as

$$
\begin{aligned}
& |\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \\
& + \frac{(b-x)^n+(-1)^{n-1}(x-a)^n}{(n+1)!} f^{(n-1)}(x) \\
& - \frac{(b-x)^n f^{(n-1)}(b)+(-1)^{n-1}(x-a)^n f^{(n-1)}(a)}{(n+1)!}| \\
\leq \ & C\sqrt{(x-a)^{2n+1}+(b-x)^{2n+1}} \sqrt{\sigma(f^{(n)})}
\end{aligned} \tag{9}
$$

We may find a function $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(t) := \begin{cases} \frac{(t-a)^{n+1}}{(n+1)!} - \frac{(x-a)^n}{(n+1)!}(t-a), & a \le t < x, \\ \frac{(t-b)^{n+1}}{(n+1)!} - \frac{(x-b)^n}{(n+1)!}(t-b), & x \le t \le b. \end{cases}$$

It follows

$$f^{(n)}(t) := \begin{cases} \frac{(t-a)^n}{n!} - \frac{(x-a)^n}{(n+1)!}, & a \le t < x, \\ \frac{(t-b)^n}{n!} - \frac{(x-b)^n}{(n+1)!}, & x \le t \le b. \end{cases} \tag{10}$$

By (3)-(6), (8) and (10), it is not difficult to find that the left-hand side of the inequality (9) becomes

$$L.H.S.(9) = \frac{n^2}{(2n+1)[(n+1)!]^2}[(x-a)^{2n+1} + (b-x)^{2n+1}]. \tag{11}$$

and the right-hand side of the inequality (9) is

$$R.H.S.(9) = C\frac{n}{(n+1)!\sqrt{2n+1}}[(x-a)^{2n+1} + (b-x)^{2n+1}]. \tag{12}$$

From (9), (11) and (12), we get that $C \ge \frac{n}{(n+1)!\sqrt{2n+1}}$, proving that the constant $\frac{n}{(n+1)!\sqrt{2n+1}}$ is the best possible in (2).

Remark 1. If we take $n = 1$ in Theorem 2. Then for all $x \in [a, b]$, we have

$$\begin{aligned} & |\int_a^b f(t)\,dt - \frac{b-a}{2}f(x) - \frac{(b-x)f(b) + (x-a)f(a)}{2}| \\ \le \; & \frac{1}{2\sqrt{3}}\sqrt{(x-a)^3 + (b-x)^3}\sqrt{\sigma(f')}. \end{aligned} \tag{13}$$

The constant $\frac{1}{2\sqrt{3}}$ is sharp.

If we take $x = a$ or $x = b$ in (13), then we recapture the sharp trapezoid type inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)]| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}}\sqrt{\sigma(f')}$$

which has been appeared in [2].

If we further take $x = \frac{a+b}{2}$ in (13), then we recapture the sharp averaged midpoint-trapezoid type inequality

$$|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a) + 2f(\frac{a+b}{2}) + f(b)]| \leq \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}}\sqrt{\sigma(f')}$$

which also has been appeared in [2].

Remark 2. If we take $n = 2$ in Theorem 2. Then for all $x \in [a, b]$, we have

$$\begin{aligned} & |\int_a^b f(t)\,dt - (b-a)f(x) + \frac{2(b-a)}{3}(x - \frac{a+b}{2})f'(x) - \frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{2}| \\ \leq & \frac{1}{3\sqrt{5}}\sqrt{(x-a)^5 + (b-x)^5}\sqrt{\sigma(f'')}. \end{aligned} \tag{14}$$

The constant $\frac{1}{3\sqrt{5}}$ is sharp.

If we further take $x = \frac{a+b}{2}$ in (13), then we recapture the sharp perturbed midpoint type inequality

$$|\int_a^b f(t)\,dt - (b-a)f(\frac{a+b}{2}) - \frac{(b-a)^2}{24}[f'(b) - f'(a)]| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}$$

which has been appeared in [3].

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