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On Some Fundamental Integral and Finite Difference Inequalities in Three Variables *

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Abstract

The aim of the present paper is to establish some new integral and finite difference inequalities in three variables which can be used as tools in the analysis of certain new classes of differential, integral and finite difference equations. Some applications to convey the importance of one of our result are also given.

Keywords and Phrases: *Integral and finite difference inequalities, Three variables, Sums and products, Uniqueness, Explicit estimates, Sum-difference equations.*

1. Introduction

During the past few decades, explicit bounds on a number of new integral and finite difference inequalities are considered and used in a variety of applications. For a detailed account on such inequalities and large number of applications, see [1,2,4-6] and the references cited therein. Although, an enormous amount of attention has been given to such inequalities, it is easy to check that the

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bounds available in the literature are not directly applicable to study the qualitative behavior of solutions of equations of the forms

$$u(x,y,z) = h(x,y,z) + \int_0^x \int_y^\infty \int_a^b F(x,y,z,s,t,r,u(s,t,r))\, drdtds, \qquad (1.1)$$

$$u(x,y,z) = h(x,y,z) + \int_x^\infty \int_y^\infty \int_a^b F(x,y,z,s,t,r,u(s,t,r))\, drdtds, \qquad (1.2)$$

and their discrete versions. The origin of equations of the forms (1.1), (1.2) can be traced back in the work of D.L. Lovelady [3] in 1973, who studied the existence and uniqueness of solutions of general form of the partial Fredholm integrodifferential equation

$$\frac{\partial^2}{\partial x \partial y} u(x,y,z) = \int_0^1 K(z,r,u(x,y,r))\, dr, \qquad (1.3)$$

with the given data

$$u(x,0,z) = \sigma(x,z), u(0,y,z) = \tau(y,z), \qquad (1.4)$$

in Banach space setting. From such considerations and the desire to widen the scope of applications of inequalities with explicit bounds, motivated us to discover new integral and finite difference inequalities, which can be applied fairly well to achieve a diversity of desired goals. The main objective of the present paper is to establish explicit bounds on some fundamental integral and finite difference inequalities in three variables, which will be equally important in handling the equations of the forms (1.1),(1.2), (1.3)-(1.4) and their discrete versions. Some applications to illustrate the usefulness of one of our result are also given.

2. Statement of Results

In what follows R denotes the set of real numbers and $I = [a,b]\,(a < b)$, $R_+ = [0,\infty)$, $N_0 = \{0,1,2,...\}$, $N = \{1,2,...\}$, $N_{\alpha,\beta} = \{\alpha, \alpha+1, ..., \alpha+n = \beta\}$

$(\alpha \in N_0, n \in N)$ are the given subsets of R. The partial derivatives of a function $h(x,y)\,(x,y \in R)$ with respect to x and y are denoted by $h_x(x,y)$ and $h_y(x,y)$ and for a function $g(m,n)\,(m,n \in N_0)$, we define the operators $\Delta_1,\ \Delta_2$ by $\Delta_1 g(m,n) = g(m+1,n) - g(m,n),\ \Delta_2 g(m,n) = g(m,n+1) - g(m,n)$. Let $G = R_+^2 \times I, H = N_0^2 \times N_{\alpha,\beta}$ and denote by $C(A,B)$ and $D(A,B)$, the class of continuous functions and the class of discrete functions from the set A to the set B. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively and assume that all the integrals, sums and products involved exist and are finite.

We require the following known inequalities in the proof of our main results.

Lemma 1. *(see [4, p. 440]). Let $u, f, g \in C\left(R_+^2, R_+\right)$.*
(i) Let $f(x,y)$ be nondecreasing in x and nonincreasing in y for $x,y \in R_+$. If

$$u(x,y) \leq f(x,y) + \int_0^x \int_y^\infty g(s,t)\, u(s,t)\, dtds,$$

for $x,y \in R_+$, then

$$u(x,y) \leq f(x,y) \exp\left(\int_0^x \int_y^\infty g(s,t)\, dtds\right),$$

for $x,y \in R_+$.
(ii) Let $f(x,y)$ be nonincreasing in each variable $x,y \in R_+$. If

$$u(x,y) \leq f(x,y) + \int_x^\infty \int_y^\infty g(s,t)\, u(s,t)\, dtds,$$

for $x,y \in R_+$, then

$$u(x,y) \leq f(x,y) \exp\left(\int_x^\infty \int_y^\infty g(s,t)\, dtds\right),$$

for $x,y \in R_+$.

Lemma 2. *(see [6, p. 266]). Let* $u, f, g \in D\left(N_0^2, R_+\right)$.
(i) Let $f(m,n)$ *be nondecreasing in* m *and nonincreasing in* n *for* $m, n \in N_0$. *If*

$$u(m,n) \leq f(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} g(s,t)\, u(s,t),$$

for $m, n \in N_0$, *then*

$$u(m,n) \leq f(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} g(s,t)\right],$$

for $m, n \in N_0$.
(ii) Let $f(m,n)$ *be nonincreasing in each variable* $m, n \in N_0$. *If*

$$u(m,n) \leq f(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} g(s,t)\, u(s,t),$$

for $m, n \in N_0$, *then*

$$u(m,n) \leq f(m,n) \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} g(s,t)\right],$$

for $m, n \in N_0$.
Our main results are given in the following theorems.

Theorem 1. *Let* $u, p, q, f \in C(G, R_+)$.
(a_1) *If*

$$u(x,y,z) \leq p(x,y,z) + q(x,y,z) \int_0^x \int_y^\infty \int_a^b f(s,t,r)\, u(s,t,r)\, dr dt ds, \quad (2.1)$$

for $(x,y,z) \in G$, *then*

$$u(x,y,z) \leq p(x,y,z) + q(x,y,z) \left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, p(s,t,r)\, dr dt ds \right)$$

$$\times \exp\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds\right), \tag{2.2}$$

for $(x,y,z) \in G.$
(a_2) *If*

$$u(x,y,z) \le p(x,y,z) + q(x,y,z) \int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, u(s,t,r)\, drdtds, \tag{2.3}$$

for $(x,y,z) \in G,$ *then*

$$u(x,y,z) \le p(x,y,z) + q(x,y,z) \left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, p(s,t,r)\, drdtds\right)$$

$$\times \exp\left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds\right), \tag{2.4}$$

for $(x,y,z) \in G.$

Theorem 2. *Let* $u,p,q,c,f,g \in C(G,R_+).$
(b_1) *Suppose that*

$$u(x,y,z) \le p(x,y,z) + q(x,y,z) \int_0^x \int_y^\infty \int_a^b f(s,t,r)\, u(s,t,r)\, drdtds$$

$$+c(x,y,z) \int_0^\infty \int_0^\infty \int_a^b g(s,t,r)\, u(s,t,r)\, drdtds, \tag{2.5}$$

for $(x,y,z) \in G.$ *If*

$$\alpha_1 = \int_0^\infty \int_0^\infty \int_a^b g(s,t,r)\, B_1(s,t,r)\, drdtds < 1, \tag{2.6}$$

then

$$u(x,y,z) \leq A_1(x,y,z) + D_1 B_1(x,y,z), \tag{2.7}$$

for $(x,y,z) \in G$, *where*

$$A_1(x,y,z) = p(x,y,z) + q(x,y,z)\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, p(s,t,r)\, drdtds\right)$$

$$\times \exp\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds\right), \tag{2.8}$$

$$B_1(x,y,z) = c(x,y,z) + q(x,y,z)\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, c(s,t,r)\, drdtds\right)$$

$$\times \exp\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds\right), \tag{2.9}$$

and

$$D_1 = \frac{1}{1-\alpha_1} \int_0^\infty \int_0^\infty \int_a^b g(s,t,r)\, A_1(s,t,r)\, drdtds. \tag{2.10}$$

(b_2) *Suppose that*

$$u(x,y,z) \leq p(x,y,z) + q(x,y,z) \int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, u(s,t,r)\, drdtds$$

$$+c(x,y,z) \int_0^\infty \int_0^\infty \int_a^b g(s,t,r)\, u(s,t,r)\, drdtds, \tag{2.11}$$

for $(x,y,z) \in G$. *If*

$$\alpha_2 = \int_0^\infty \int_0^\infty \int_a^b g(s,t,r)\, B_2(s,t,r)\, drdtds < 1, \tag{2.12}$$

then

$$u(x,y,z) \leq A_2(x,y,z) + D_2 B_2(x,y,z), \tag{2.13}$$

for $(x,y,z) \in G,$ *where*

$$A_2(x,y,z) = p(x,y,z) + q(x,y,z)\left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, p(s,t,r)\, drdtds\right)$$

$$\times \exp\left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds\right), \tag{2.14}$$

$$B_2(x,y,z) = c(x,y,z) + q(x,y,z)\left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, c(s,t,r)\, drdtds\right)$$

$$\times \exp\left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds\right), \tag{2.15}$$

and

$$D_2 = \frac{1}{1-\alpha_2} \int_0^\infty \int_0^\infty \int_a^b g(s,t,r)\, A_2(s,t,r)\, drdtds. \tag{2.16}$$

The discrete analogues of the inequalities in Theorems 1 and 2 are given as follows.

Theorem 3. *Let* $u, p, q, f \in D(H, R_+)$.
(c_1) *If*

$$u(m,n,k) \leq p(m,n,k) + q(m,n,k) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} f(s,t,r)\, u(s,t,r), \tag{2.17}$$

for $(m,n,k) \in H,$ *then*

$$u(m,n,k) \leq p(m,n,k) + q(m,n,k)\left(\sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} f(s,t,r)\, p(s,t,r)\right)$$

$$\times \prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\, q(s,t,r)\right], \tag{2.18}$$

for $(m,n,k)\in H$.
(c_2) *If*

$$u(m,n,k) \leq p(m,n,k)+q(m,n,k)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\, u(s,t,r), \tag{2.19}$$

for $(m,n,k)\in H$, *then*

$$u(m,n,k) \leq p(m,n,k)+q(m,n,k)\left(\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\, p(s,t,r)\right)$$

$$\times \prod_{s=m+1}^{\infty}\left[1+\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\, q(s,t,r)\right], \tag{2.20}$$

for $(m,n,k)\in H$.

Theorem 4. *Let* $u,p,q,c,f,g \in D(H,R_+)$.
(d_1) *Suppose that*

$$u(m,n,k) \leq p(m,n,k)+q(m,n,k)\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\, u(s,t,r)$$

$$+c(m,n,k)\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=\alpha}^{\beta} g(s,t,r)\, u(s,t,r), \tag{2.21}$$

for $(m,n,k)\in H$. *If*

$$\beta_1=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=\alpha}^{\beta} g(s,t,r)\, \bar{B}_1(s,t,r) < 1, \tag{2.22}$$

then

$$u(m,n,k) \leq \bar{A}_1(m,n,k)+\bar{D}_1\bar{B}_1(m,n,k), \tag{2.23}$$

for $(m,n,k)\in H$, *where*

$$\bar{A}_1(m,n,k)=p(m,n,k)+q(m,n,k)\left(\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta}f(s,t,r)\,p(s,t,r)\right)$$

$$\times\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta}f(s,t,r)\,q(s,t,r)\right], \tag{2.24}$$

$$\bar{B}_1(m,n,k)=c(m,n,k)+q(m,n,k)\left(\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta}f(s,t,r)\,c(s,t,r)\right)$$

$$\times\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta}f(s,t,r)\,q(s,t,r)\right], \tag{2.25}$$

and

$$\bar{D}_1=\frac{1}{1-\beta_1}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=\alpha}^{\beta}g(s,t,r)\,\bar{A}_1(s,t,r). \tag{2.26}$$

(d_2) *Suppose that*

$$u(m,n,k)\le p(m,n,k)+q(m,n,k)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta}f(s,t,r)\,u(s,t,r)$$

$$+c(m,n,k)\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=\alpha}^{\beta}g(s,t,r)\,u(s,t,r), \tag{2.27}$$

for $(m,n,k)\in H$. *If*

$$\beta_2=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=\alpha}^{\beta}g(s,t,r)\,\bar{B}_2(s,t,r)<1, \tag{2.28}$$

then

$$u(m,n,k)\le\bar{A}_2(m,n,k)+\bar{D}_2\bar{B}_2(m,n,k), \tag{2.29}$$

for $(m,n,k)\in H$, *where*

$$\bar{A}_2(m,n,k)=p(m,n,k)+q(m,n,k)\left(\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta}f(s,t,r)\,p(s,t,r)\right)$$

$$\times \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} f(s,t,r)\, q(s,t,r)\right], \tag{2.30}$$

$$\bar{B}_2(m,n,k) = c(m,n,k) + q(m,n,k) \left(\sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} f(s,t,r)\, c(s,t,r) \right)$$

$$\times \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} f(s,t,r)\, q(s,t,r)\right], \tag{2.31}$$

and

$$\bar{D}_2 = \frac{1}{1-\beta_2} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{r=\alpha}^{\beta} g(s,t,r)\, \bar{A}_2(s,t,r)\,. \tag{2.32}$$

Remark 1. We note that the inequalities developed in this paper can be considered as new variants of the similar inequalities in two variables given in [6, Chapters 2 and 4]. The stricking feature of the inequalities established here is that, they are applicable in situations for which the earlier inequalities do not apply directly.

3. Proofs of Theorems 1-4

We give the details of the proofs of (a_1), (b_2) and (c_1) only. The proofs of other inequalities can be completed by following the proofs of these inequalities.
(a_1) Introducing the notation

$$E(s,t) = \int_a^b f(s,t,r)\, u(s,t,r)\, dr, \tag{3.1}$$

the inequality (2.1) can be restated as

$$u(x,y,z) \le p(x,y,z) + q(x,y,z) \int_0^x \int_y^{\infty} E(s,t)\, dt ds. \tag{3.2}$$

Define

$$w(x,y) = \int_0^x \int_y^{\infty} E(s,t)\, dt ds, \tag{3.3}$$

then $w(0,y)=0$ and

$$u\left(x,y,z\right)\leq p\left(x,y,z\right)+q\left(x,y,z\right)w\left(x,y\right). \tag{3.4}$$

From (3.3), (3.1) and (3.4), we observe that

$$\begin{aligned}
& w_x\left(x,y\right) \\
= & \int_y^\infty E\left(x,t\right)dt \\
= & \int_y^\infty \left\{\int_a^b f\left(x,t,r\right)u\left(x,t,r\right)dr\right\}dt \\
\leq & \int_y^\infty \left\{\int_a^b f\left(x,t,r\right)\left[p\left(x,t,r\right)+q\left(x,t,r\right)w\left(x,t\right)\right]dr\right\}dt \\
= & \int_y^\infty \left\{\int_a^b f\left(x,t,r\right)p\left(x,t,r\right)dr\right\}dt \\
& +\int_y^\infty w\left(x,t\right)\left\{\int_a^b f\left(x,t,r\right)q\left(x,t,r\right)dr\right\}dt.
\end{aligned} \tag{3.5}$$

By taking $x=s$ in (3.5) and integrating both sides with respect to s from 0 to x, we get

$$w\left(x,y\right)\leq e_1\left(x,y\right)+\int_0^x\int_y^\infty\left\{\int_a^b f\left(s,t,r\right)q\left(s,t,r\right)dr\right\}w\left(s,t\right)dtds, \tag{3.6}$$

where

$$e_1\left(x,y\right)=\int_0^x\int_y^\infty\int_a^b f\left(s,t,r\right)p\left(s,t,r\right)drdtds. \tag{3.7}$$

Clearly $e_1\left(x,y\right)$ is nonnegative, continuous, nondecreasing in x and nonincreasing in y for $x,y\in R_+$. Now, a suitable application of the inequality in part (i) given in Lemma 1 to (3.6) yields

$$w(x,y) \le e_1(x,y)\exp\left(\int_0^x\int_y^\infty\int_a^b f(s,t,r)\,q(s,t,r)\,drdtds\right). \tag{3.8}$$

Using (3.8), (3.7) in (3.4), we get (2.2).
(b_2) Let $E(s,t)$ be as in (3.1) and

$$\lambda = \int_0^\infty\int_0^\infty\int_a^b g(s,t,r)\,u(s,t,r)\,drdtds. \tag{3.9}$$

Then (2.5) can be restated as

$$u(x,y,z) \le p(x,y,z) + q(x,y,z)\int_x^\infty\int_y^\infty E(s,t)\,dtds + c(x,y,z)\,\lambda. \tag{3.10}$$

Let

$$v(x,y) = \int_x^\infty\int_y^\infty E(s,t)\,dtds, \tag{3.11}$$

then $v(\infty,y) = 0$ and from (3.10), we have

$$u(x,y,z) \le p(x,y,z) + q(x,y,z)\,v(x,y) + c(x,y,z)\,\lambda. \tag{3.12}$$

From (3.11), (3.1) and (3.12), we have

$$v_x(x,y) = -\int_y^\infty E(x,t)\,dt$$

$$= -\int_y^\infty \left\{ \int_a^b f(x,t,r)\,u(x,t,r)\,dr \right\} dt$$

$$\geq -\int_y^\infty \left\{ \int_a^b f(x,t,r)\left[p(x,t,r) + q(x,t,r)\,v(x,t) + c(x,t,r)\,\lambda\right] dr \right\} dt$$

$$= -\int_y^\infty \left\{ \int_a^b f(x,t,r)\,p(x,t,r)\,dr \right\} dt$$

$$-\int_y^\infty \left\{ \int_a^b f(x,t,r)\left[q(x,t,r)\,v(x,t) + c(x,t,r)\,\lambda\right] dr \right\} dt. \tag{3.13}$$

By taking $x = s$ in (3.13) and integrating both sides with respect to s from x to ∞ for $x \in R_+$, we have

$$v(x,y) \leq e_2(x,y) + \int_x^\infty \int_y^\infty \left\{ \int_a^b f(s,t,r)\,q(s,t,r)\,dr \right\} v(s,t)\,dtds, \tag{3.14}$$

where

$$e_2(x,y) = \int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\left[p(s,t,r) + c(s,t,r)\,\lambda\right] drdtds. \tag{3.15}$$

Clearly $e_2(x,y)$ is nonnegative, continuous, nonincreasing in each variable $x, y \in R_+$. Now, a suitable application of the inequality in part (ii) given in Lemma 1 to (3.14) yields

$$v(x,y) \leq e_2(x,y) \exp\left(\int_x^\infty \int_y^\infty \int_a^b f(s,t,r)\,q(s,t,r)\,drdtds \right). \tag{3.16}$$

Using (3.16), (3.15) in (3.12), we get

$$\begin{aligned} &u(x,y,z) \\ \leq\ & p(x,y,z) + q(x,y,z)\left[\int_x^\infty\int_y^\infty\int_a^b f(s,t,r)\left[p(s,t,r)+c(s,t,r)\lambda\right]drdtds\right] \\ & \times\exp\left(\int_x^\infty\int_y^\infty\int_a^b f(s,t,r)\,q(s,t,r)\,drdtds\right) + c(x,y,z)\lambda \\ =\ & A_2(x,y,z) + \lambda B_2(x,y,z). \end{aligned} \tag{3.17}$$

From (3.9) and (3.17), we have

$$\begin{aligned} \lambda &= \int_0^\infty\int_0^\infty\int_a^b g(s,t,r)\,u(s,t,r)\,drdtds \\ &\leq \int_0^\infty\int_0^\infty\int_a^b g(s,t,r)\left[A_2(s,t,r)+\lambda B_2(s,t,r)\right]drdtds, \end{aligned}$$

which implies

$$\lambda \leq D_2. \tag{3.18}$$

Using (3.18) in (3.17), we get (2.13).
(c_1) Introducing the notation

$$e_0(s,t) = \sum_{r=\alpha}^{\beta} f(s,t,r)\,u(s,t,r), \tag{3.19}$$

the inequality (2.17) can be restated as

$$u(m,n,k) \leq p(m,n,k) + q(m,n,k)\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} e_0(s,t). \tag{3.20}$$

Define

$$\psi(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} e_0(s,t), \tag{3.21}$$

then $\psi(0,n)=0$ and

$$u(m,n,k)\leq p(m,n,k)+q(m,n,k)\psi(m,n). \tag{3.22}$$

From (3.21), (3.19) and (3.22), we observe that

$$\begin{aligned}
\Delta_1\psi(m,n) &= \sum_{t=n+1}^{\infty} e_0(m,t) \\
&= \sum_{t=n+1}^{\infty}\left\{\sum_{r=\alpha}^{\beta} f(m,t,r)\,u(m,t,r)\right\} \\
&\leq \sum_{t=n+1}^{\infty}\left\{\sum_{r=\alpha}^{\beta} f(m,t,r)\left[p(m,t,r)+q(m,t,r)\psi(m,t)\right]\right\} \\
&= \sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(m,t,r)\,p(m,t,r) \\
&\quad+\sum_{t=n+1}^{\infty}\left\{\sum_{r=\alpha}^{\beta} f(m,t,r)\,q(m,t,r)\,\psi(m,t)\right\}.
\end{aligned} \tag{3.23}$$

By taking $m=s$ in (3.23) and then taking sum over s from $s=0$ to $m-1$, $m\in N_0$, we get

$$\begin{aligned}
\psi(m,n) &\leq \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\,p(s,t,r) \\
&\quad+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left\{\sum_{r=\alpha}^{\beta} f(s,t,r)\,q(s,t,r)\,\psi(s,t)\right\} \\
&= \bar{e}(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\psi(s,t)\left\{\sum_{r=\alpha}^{\beta} f(s,t,r)\,q(s,t,r)\right\},
\end{aligned} \tag{3.24}$$

where

$$\bar{e}(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\sum_{r=\alpha}^{\beta} f(s,t,r)\,p(s,t,r). \tag{3.25}$$

Clearly, $\bar{e}(m,n)$ is nonnegative, nondecreasing in m and nonincreasing in n for $m,n \in N_0$. Now, a suitable application of the inequality (i) given in Lemma 2 to (3.24) yields

$$\psi(m,n) \le \bar{e}(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} f(s,t,r)\, q(s,t,r) \right]. \tag{3.26}$$

Using (3.26), (3.25) in (3.22), we get (2.18).

Remark 2. We note that, many new generalizations, extensions and variants of the inequalities given in Theorems 1-4 are possible, which would also be equally important in certain new applications to the equations of the forms (1.1), (1.2) and (1.3)-(1.4), about which almost nothing seems to be known.

4. Some Applications

In this section, we apply the inequality in Theorem 1, part (a_1) to obtain the uniqueness and explicit estimates on the solutions of equation (1.1). One can formulate existence result for the solution of equation (1.1) by using the idea employed in [7], see also [1,3,8].
First, we shall give the following theorem concerning the uniqueness of solutions of equation (1.1).

Theorem 5. *Suppose that* $h \in C(G,R)$, $F \in C(G^2 \times R, R)$ *and*

$$|F(x,y,z,s,t,r,u) - F(x,y,z,s,t,r,v)| \le q(x,y,z)\, f(s,t,r)\, |u-v|, \tag{4.1}$$

where $q, f \in C(G, R_+)$. *Then the equation (1.1) has at most one solution on* G.

Proof. Let $u_1(x,y,z)$ and $u_2(x,y,z)$ be two solutions of equation (1.1). Then by using the hypotheses, we have

$$
\begin{aligned}
& |u_1(x,y,z) - u_2(x,y,z)| \\
\leq\ & \int_0^x \int_y^\infty \int_a^b |F(x,y,z,s,t,r,u_1(s,t,r)) - F(x,y,z,s,t,r,u_2(s,t,r))| drdtds \\
\leq\ & q(x,y,z) \int_0^x \int_y^\infty \int_a^b f(s,t,r) |u_1(s,t,r) - u_2(s,t,r)| drdtds. \qquad (4.2)
\end{aligned}
$$

Now, a suitable application of the inequality in Theorem 1, part (a_1) to (4.2) yields $|u_1(x,y,z) - u_2(x,y,z)| \leq 0$, and hence $u_1(x,y,z) = u_2(x,y,z)$. Thus there is at most one solution to equation (1.1) on G.
The following theorem deals with the estimate on the solution of equation (1.1).

Theorem 6. Suppose that $h \in C(G,R), F \in C(G^2 \times R, R)$ and

$$|F(x,y,z,s,t,r,u)| \leq q(x,y,z) f(s,t,r) |u|, \qquad (4.3)$$

where $q, f \in C(G, R_+)$. If $u(x,y,z)$ is any solution of equation (1.1) on G, then

$$
\begin{aligned}
|u(x,y,z)| \leq |h(x,y,z)| + q(x,y,z) \left(\int_0^x \int_y^\infty \int_a^b f(s,t,r) |h(s,t,r)| drdtds \right) \\
\times \exp\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r) q(s,t,r) drdtds \right), \qquad (4.4)
\end{aligned}
$$

for $(x,y,z) \in G$.

Proof. Using the fact that $u(x,y,z)$ is a solution of equation (1.1) and hypotheses, we have

$$
\begin{aligned}
|u(x,y,z)| \leq |h(x,y,z)| + \int_0^x \int_y^\infty \int_a^b |F(x,y,z,s,t,r,u(s,t,r))| drdtds \\
\leq\ |h(x,y,z)| + q(x,y,z) \int_0^x \int_y^\infty \int_a^b f(s,t,r) |u(s,t,r)| drdtds. \qquad (4.5)
\end{aligned}
$$

Now, an application of the inequality in Theorem 1, part (a_1) to (4.5) yields (4.4).
The next theorem gives the estimation on the solution of equation (1.1) assuming that the function F in equation (1.1) satisfies the Lipschitz type condition.

Theorem 7. Suppose that $h \in C(G, R), F \in C(G^2 \times R, R)$ and the condition (4.1) holds. If $u(x, y, z)$ is any solution of equation (1.1) on G, then

$$|u(x,y,z) - h(x,y,z)| \le d(x,y,z) + q(x,y,z) \left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, d(s,t,r)\, drdtds \right) \times \exp\left(\int_0^x \int_y^\infty \int_a^b f(s,t,r)\, q(s,t,r)\, drdtds \right), \tag{4.6}$$

for $(x, y, z) \in G$, where

$$d(x,y,z) = \int_0^x \int_y^\infty \int_a^b |F(x,y,z,s,t,r,h(s,t,r))|\, drdtds. \tag{4.7}$$

Proof. Let $u(x, y, z)$ be a solution of equation (1.1) on G. Then from the hypotheses, we have

$$\begin{aligned} |u(x,y,z) - h(x,y,z)| &\le \int_0^x \int_y^\infty \int_a^b |F(x,y,z,s,t,r,u(s,t,r))|\, drdtds \\ &\le \int_0^x \int_y^\infty \int_a^b |F(x,y,z,s,t,r,u(s,t,r)) - F(x,y,z,s,t,r,h(s,t,r))|\, drdtds \\ &\quad + \int_0^x \int_y^\infty \int_a^b |F(x,y,z,s,t,r,h(s,t,r))|\, drdtds \\ &\le d(x,y,z) \\ &\quad + q(x,y,z) \int_0^x \int_y^\infty \int_a^b f(s,t,r)\, |u(s,t,r) - h(s,t,r)|\, drdtds, \end{aligned} \tag{4.8}$$

for $(x, y, z) \in G$. Now, an application of the inequality in Theorem 1, part (a_1) to (4.8) gives the required estimate in (4.6).
(a_2) can be used to obtain results similar to those of given in Theorems 5-7 for the solutions of equation (1.2). Furthermore, the inequalities given in Theorem 1 can be used to formulate results on the continuous dependence of solutions of equations (1.1), (1.2) by closely looking at the corresponding results recently given in [7].
In concluding we note that, one can use the inequalities obtained in Theorem 3, to establish results as in Theorems 5-7, to the solutions of sum-difference equations of the forms

$$u(m,n,k) = h(m,n,k) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} L(m,n,k,s,t,r,u(s,t,r)), \quad (4.9)$$

$$u(m,n,k) = h(m,n,k) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \sum_{r=\alpha}^{\beta} L(m,n,k,s,t,r,u(s,t,r)), \quad (4.10)$$

for $(m, n, k) \in H$, under some suitable conditions on the functions involved in equations (4.9), (4.10). Moreover, we note that the inequalities given in Theorems 2 and 4 can be used to study similar properties as noted above for more general versions of equations (1.1), (1.2), (4.9), (4.10). The details of formulation of such results are very close to those of given in Theorems 5-7 (see also [7]) with suitable modifications. Here we omit the details.

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