Tamsui Oxford Journal of Mathematical Sciences **26(1)** (2010) 1-19
Aletheia University

A Class of Multivalent Non-Bazilevič Functions Involving the Cho-Kwon-Srivastava Operator*

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Received June 28, 2008, Accepted October 11, 2008.

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*2000 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C80

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Abstract

The main purpose of this paper is to derive some new properties for a class of p-valent non-Bazilevič functions involving the Cho-Kwon-Srivastava operator, such results as subordination and superordination properties, convolution properties, coefficient estimates, sufficient conditions for starlikeness and sufficient conditions for functions belonging to this class are obtained.

Keywords and Phrases: *Analytic functions, Multivalent functions, Non-Bazilevič functions, Hadamard product (or convolution), Cho-Kwon-Srivastava operator, Jack's Lemma, Subordination between Analytic functions, Superordination, Starlikeness.*

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For simplicity, we write

$$\mathcal{A}_1 =: \mathcal{A}.$$

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\varrho)$ of p-valent starlike functions of order ϱ in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \varrho \qquad (0 \leqq \varrho < p;\ z \in \mathbb{U}).$$

Let $\mathcal{H}[a, n]$ be the class of analytic functions of the form:

$$F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (z \in \mathbb{U}).$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}(\mu)$ if it satisfies the following inequality:

$$\Re\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right) > 0 \qquad (0 < \mu < 1;\ z \in \mathbb{U}).$$

The function class $\mathcal{N}(\mu)$ was introduced recently by Obradović [5, 6], he called this class to be of non-Bazilevič type. Until now, this class was studied in a direction of finding necessary conditions over μ that embeds it into the class of univalent functions or its subclasses, which is still an open problem.

In recent years, Obradović and Owa [7], Tuneski and Darus [15], Wang *et al.* [16] and Shanmugam *et al.* [10, 12, 13, 14] obtained many interesting results associated with different subclasses of non-Bazilevič functions.

Let $f,\ g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} =: (g * f)(z).$$

For parameters

$$a \in \mathbb{R}, \quad c \in \mathbb{R} \setminus \mathbb{Z}_0^- \quad (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}),$$

Saitoh [9] introduced a linear operator:

$$\mathcal{L}_p(a,c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$$

defined by

$$\mathcal{L}_p(a,c) f(z) = \phi_p(a,c;z) * f(z) \qquad (z \in \mathbb{U};\ f \in \mathcal{A}_p),$$

where

$$\phi_p(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \tag{1.2}$$

and $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k := \begin{cases} 1, & (k=0), \\ \lambda(\lambda+1)\cdots(\lambda+k-1), & (k \in \mathbb{N}). \end{cases}$$

In 2004, Cho *et al.* [1] introduced the following family of linear operators $\mathcal{I}_p^\lambda(a,c)$ analogous to $\mathcal{L}_p(a,c)$:

$$\mathcal{I}_p^\lambda(a,c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p,$$

which is defined as

$$\mathcal{I}_p^\lambda(a,c)f(z) := \phi_p^\dagger(a,c;z) * f(z) \qquad (a,c \in \mathbb{R}\setminus\mathbb{Z}_0^-;\ \lambda > -p;\ z \in \mathbb{U};\ f \in \mathcal{A}_p), \tag{1.3}$$

where $\phi_p^\dagger(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition:

$$\phi_p(a,c;z) * \phi_p^\dagger(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}}. \tag{1.4}$$

We can easily find from (1.2), (1.3) and (1.4) that

$$\mathcal{I}_p^\lambda(a,c)f(z) = z^p + \sum_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{k!(a)_k}a_{k+p}z^{k+p} \qquad (z \in \mathbb{U};\ \lambda > -p). \tag{1.5}$$

It is also readily verified from (1.5) that

$$z\left(\mathcal{I}_p^\lambda(a+1,c)f\right)'(z) = a\mathcal{I}_p^\lambda(a,c)f(z) - (a-p)\mathcal{I}_p^\lambda(a+1,c)f(z), \tag{1.6}$$

and

$$z\left(\mathcal{I}_p^\lambda(a,c)f\right)'(z) = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^\lambda(a,c)f(z). \tag{1.7}$$

Also by definition and specializing the parameters λ, a, c, we obtain

$$\mathcal{I}_p^1(p+1,1)f(z) = f(z), \quad \mathcal{I}_p^1(p,1)f(z) = \frac{zf'(z)}{p},$$

and

$$\mathcal{I}_p^n(a,a)f(z) = \mathcal{D}^{n+p-1}f(z) \qquad (n > -p),$$

where $\mathcal{D}^{n+p-1}$ is the well-known Ruscheweyh derivative of $(n+p-1)$-th order.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

By making use of Cho-Kwon-Srivastava operator $\mathcal{I}_p^\lambda(a,c)$ and the above-mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclass of p-valent non-Bazilevič functions.

Definition 1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$ if it satisfies the following inequality:

$$\begin{aligned} &(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu - \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \\ \prec\ & \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \end{aligned} \tag{1.8}$$

where (throughout this paper without any special remark)

$$0<\mu<1;\ \ \alpha \in \mathbb{C};\ \ a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-;\ \ \lambda > -p;\ \ -1 \leqq B \leqq 1;\ \ \mathbb{R} \ni A \neq B \ \text{and}\ \ p \in \mathbb{N},$$

and the powers are understood as principle values.

Clearly, if we set $\lambda = 0$, $\alpha = B = -1$ and $p = a = c = A = 1$ in the class $\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$, then it reduces to the class $\mathcal{N}(\mu)$ of non-Bazilevič functions.

In the present paper, we aim at proving such results as subordination and superordination properties, convolution properties, coefficient estimates, sufficient conditions for starlikeness and sufficient conditions for functions belonging to the class $\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$.

2. Preliminary Results

To derive our main results, we need the following lemmas.

Lemma 1. (Jack's lemma [2]) *Let $\omega(z)$ be a non-constant analytic function in $\mathbb{U}$ with $w(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0\omega'(z_0) = k\omega(z_0),$$

where $k \geqq 1$ is a real number.

Lemma 2. (see [11]) *Let q be a convex univalent function in $\mathbb{U}$ and let $\delta, \gamma \in \mathbb{C}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\delta}{\gamma}\right)\right\}.$$

If $p(z)$ is analytic in $\mathbb{U}$ and

$$\delta p(z) + \gamma z p'(z) \prec \delta q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{\varepsilon \in \partial\mathbb{U} :\ \lim_{z \to \varepsilon} f(z) = \infty\right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

Lemma 3. (see [4]) *Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\Re(\overline{\kappa}) > 0$. If*

$$p(z) \in \mathcal{H}[q(0), 1] \cap Q,$$

and $p(z) + \kappa z p'(z)$ is univalent in $\mathbb{U}$, then

$$q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)$$

implies $q(z) \prec p(z)$ and q is the best dominant.

Lemma 4. (see [3]) *Let*

$$k(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

be analytic in $\mathbb{U}$, $h(z)$ *be analytic and convex (univalent) in* $\mathbb{U}$ *with* $h(0) = 1$. *If*

$$k(z) + \frac{zk'(z)}{\zeta} \prec h(z) \qquad (\Re(\zeta) > 0;\ \zeta \neq 0;\ z \in \mathbb{U}), \tag{2.1}$$

then

$$k(z) \prec \chi(z) = \zeta z^{-\zeta} \int_0^z t^{\zeta-1} h(t)dt \prec h(z) \qquad (z \in \mathbb{U}),$$

and $\chi(z)$ *is the best dominant of* (2.1).

Lemma 5. (see [8]) *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be analytic in $\mathbb{U}$ *and*

$$g(z) = \sum_{k=0}^{\infty} b_k z^k$$

be analytic and convex in $\mathbb{U}$. *If* $f \prec g$, *then*

$$|a_k| \leqq |b_1| \qquad (k \in \mathbb{N}).$$

3. Main Results

We begin by stating the following result with the aid of Jack's lemma.

Theorem 1. *Let* $f \in \mathcal{A}_p$, $\xi \in \mathbb{C} \setminus \{0\}$ *and* $0 \leqq \beta < 1$. *Also let the function* φ *be defined by*

$$\varphi(z) = \frac{z\left(\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu} - 1\right)'}{\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu} - 1} \qquad (z \in \mathbb{U}). \tag{3.1}$$

If $\varphi(z)$ satisfies one of the following conditions:

$$\Re(\varphi(z)) \begin{cases} < \frac{1}{|\xi|^2}\Re(\xi) & (\Re(\xi) > 0), \\ \neq 0 & (\Re(\xi) = 0), \\ > \frac{1}{|\xi|^2}\Re(\xi) & (\Re(\xi) < 0), \end{cases} \tag{3.2}$$

or

$$\Im(\varphi(z)) \begin{cases} > -\frac{1}{|\xi|^2}\Im(\xi) & (\Im(\xi) > 0), \\ \neq 0 & (\Im(\xi) = 0), \\ < -\frac{1}{|\xi|^2}\Im(\xi) & (\Im(\xi) < 0), \end{cases} \tag{3.3}$$

then

$$\left| \left(\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right) \left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right)^{\mu} - 1 \right)^{\xi} \right| < 1 - \beta. \tag{3.4}$$

Proof. We define the function ϕ by

$$\left(\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right) \left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right)^{\mu} - 1 \right)^{\xi} = (1-\beta)\phi(z) \tag{3.5}$$

$$\left(0 \leqq \beta < 1;\ \xi \in \mathbb{C} \setminus \{0\};\ z \in \mathbb{U}\right).$$

It is easy to see that the function ϕ is analytic in $\mathbb{U}$ with $\phi(0) = 0$.

Differentiating both sides of (3.5) with respect to z logarithmically, we get

$$\frac{z\phi'(z)}{\phi(z)} = \xi \frac{z\left(\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right) \left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right)^{\mu} - 1 \right)'}{\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right) \left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right)^{\mu} - 1} \qquad (z \in \mathbb{U};\ \xi \in \mathbb{C} \setminus \{0\}). \tag{3.6}$$

We now consider the function φ defined by

$$\varphi(z) := \frac{\overline{\xi}}{|\xi|^2} \frac{z\phi'(z)}{\phi(z)} \qquad (z \in \mathbb{U};\ \xi \in \mathbb{C} \setminus \{0\}). \tag{3.7}$$

Assume that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |\phi(z)| = |\phi(z_0)| = 1,$$

by Lemma 1, we know that

$$z_0 \phi'(z_0) = k\phi(z_0) \qquad (k \geqq 1). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} \Re(\varphi(z_0)) &= \Re\left(\frac{\overline{\xi}}{|\xi|^2}\frac{z\phi'(z_0)}{\phi(z_0)}\right) = \frac{k}{|\xi|^2}\Re\left(\overline{\xi}\right) \\ &= \frac{k}{|\xi|^2}\Re\left(\xi\right)\begin{cases} \geqq \frac{1}{|\xi|^2}\Re(\xi) & (\Re(\xi) > 0), \\ = 0 & (\Re(\xi) = 0), \\ \leqq \frac{1}{|\xi|^2}\Re(\xi) & (\Re(\xi) < 0), \end{cases} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \Im(\varphi(z_0)) &= \Im\left(\frac{\overline{\xi}}{|\xi|^2}\frac{z\phi'(z_0)}{\phi(z_0)}\right) = \frac{k}{|\xi|^2}\Im\left(\overline{\xi}\right) \\ &= -\frac{k}{|\xi|^2}\Im\left(\xi\right)\begin{cases} \leqq -\frac{1}{|\xi|^2}\Im(\xi) & (\Im(\xi) > 0), \\ = 0 & (\Im(\xi) = 0), \\ \geqq -\frac{1}{|\xi|^2}\Im(\xi) & (\Im(\xi) < 0). \end{cases} \end{aligned} \tag{3.10}$$

But the inequalities in (3.9) and (3.10) contradict, respectively, the inequalities in (3.2) and (3.3). Therefore, we can conclude that

$$|\phi(z)| < 1 \qquad (z \in \mathbb{U}),$$

which implies that

$$\left|\left(\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu} - 1\right)^{\xi}\right| = (1-\beta)\,|\phi(z)| < 1-\beta.$$

We thus complete the proof of Theorem 1.

By specializing the parameters $\xi,\ \beta,\ p,\ a$ and c, some interesting corollaries of Theorem 1 are presented as follows.

Corollary 1. *Let $f \in \mathcal{A}_p$, $\xi \in \mathbb{R} \setminus \{0\}$ and $0 \leqq \beta < 1$. Also let the function φ be defined by* (3.1)*. If φ satisfies one of the following conditions:*

$$\Re(\varphi(z)) \begin{cases} < \frac{1}{\xi} & (\xi > 0), \\ > -\frac{1}{\xi} & (\xi < 0), \end{cases} \quad or \quad \Im(\varphi(z)) \neq 0,$$

then

$$\left| \left(\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right) \left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \right)^{\mu} - 1 \right)^{\xi} \right| < 1 - \beta.$$

Corollary 2. *Let $f \in \mathcal{A}_p$, $\xi = 1$ and $0 \leqq \beta < 1$. Also let the function φ be defined by* (3.1)*. If φ satisfies one of the following conditions:*

$$\Re(\varphi(z)) < 1 \quad or \quad \Im(\varphi(z)) \neq 0,$$

then

$$f \in \mathcal{N}_{p,\lambda}^{-1,\mu}(a, c; 1; 1 - 2\beta, -1).$$

Corollary 3. *Let $f \in \mathcal{A}$, $\lambda = \beta = 0$ and $p = a = c = \xi = 1$. Also let the function φ be defined by* (3.1)*. If φ satisfies one of the following conditions:*

$$\Re(\varphi(z)) < 1 \quad or \quad \Im(\varphi(z)) \neq 0,$$

then

$$f \in \mathcal{N}(\mu).$$

Theorem 2. *Let q be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$. Suppose also that q satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \ -\Re\left(\frac{(\lambda+p)\mu}{\alpha}\right)\right\}. \tag{3.11}$$

If $f \in \mathcal{A}_p$ satisfies the following subordination

$$(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu} - \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu} \prec q(z) + \frac{\alpha z q'(z)}{(\lambda+p)\mu}, \tag{3.12}$$

then

$$\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu} \prec q(z),$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \qquad (z \in \mathbb{U}). \tag{3.13}$$

By taking the derivatives in the both sides of equality (3.13), we get

$$(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu - \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu = p(z) + \frac{\alpha z p'(z)}{(\lambda+p)\mu}. \tag{3.14}$$

Combining (3.12) and (3.14), we find that

$$p(z) + \frac{\alpha z p'(z)}{(\lambda+p)\mu} \prec q(z) + \frac{\alpha z q'(z)}{(\lambda+p)\mu}. \tag{3.15}$$

With an application of Lemma 2 to (3.15), we can easily get the assertion of Theorem 2.

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 2, we can get the following result.

Corollary 4. *Let $\alpha \in \mathbb{C}$ and $-1 \leqq B < A \leqq 1$. Suppose also that $\frac{1+Az}{1+Bz}$ satisfies the condition* (3.11)*. If $f \in \mathcal{A}_p$ satisfies the following subordination*

$$(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu - \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \prec \frac{1+Az}{1+Bz} + \frac{\alpha(A-B)z}{(\lambda+p)\mu(1+Bz)^2},$$

then

$$\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result for the class $\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$.

Theorem 3. *Let q be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \in \mathcal{H}[q(0),1] \cap Q$$

and

$$(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}-\alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}$$

be univalent in $\mathbb{U}$*. If*

$$q(z)+\frac{\alpha zq'(z)}{(\lambda+p)\mu}\prec(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}-\alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu},$$

then

$$q(z)\prec\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu},$$

and q *is the best dominant.*

Proof. Let the function $p(z)$ be defined by (3.13). Then

$$\begin{aligned}q(z)+\frac{\alpha zq'(z)}{(\lambda+p)\mu}&\prec(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}-\alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}\\&=p(z)+\frac{\alpha zp'(z)}{(\lambda+p)\mu}.\end{aligned}$$

An application of Lemma 3 yields the assertion of Theorem 3.

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 3, we can get the following corollary.

Corollary 5. *Let* q *be convex univalent in* $\mathbb{U}$ *and* $-1\leqq B<A\leqq 1$, $\alpha\in\mathbb{C}$ *with* $\Re(\alpha)>0$*. Also let*

$$\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\beta}\in\mathcal{H}[q(0),1]\cap Q$$

and

$$(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}-\alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}$$

be univalent in $\mathbb{U}$*. If*

$$\begin{aligned}&\frac{1+Az}{1+Bz}+\frac{\alpha(A-B)z}{(\lambda+p)\mu(1+Bz)^2}\\&\qquad\prec(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu}-\alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^{\lambda}(a,c)f(z)}\right)^{\mu},\end{aligned}$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu,$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Combining the above results of subordination and superordination. we can easily get the following "Sandwich results".

Corollary 6. *Let* q_1 *be convex univalent and let* q_2 *be univalent in* $\mathbb{U}$, $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$. *Let* q_2 *satisfies* (3.11). *If*

$$0 \neq \left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \in \mathcal{H}[q(0),1] \cap Q,$$

and

$$(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu - \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu$$

is univalent in $\mathbb{U}$, *also*

$$\begin{aligned} q_1(z) + \frac{\alpha z q_1'(z)}{(\lambda+p)\mu} \prec (1+\alpha)\left(\frac{z^p}{\mathcal{I}_{p,n}^\lambda(a,c)f(z)}\right)^\mu - \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \\ = q_2(z) + \frac{\alpha z q_2'(z)}{(\lambda+p)\mu}, \end{aligned}$$

then

$$q_1(z) \prec \left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and dominant.*

Theorem 4. *Let* $f \in \mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$ *with* $\Re(\alpha) > 0$. *Then*

$$f(z) = \left(z^p\left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{-\frac{1}{\mu}}\right) * \left(z^p + \sum_{k=1}^{\infty}\frac{k!(a)_k}{(\lambda+p)_k(c)_k}z^{k+p}\right), \qquad (3.16)$$

where ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$ with $\Re(\alpha) > 0$ and $p(z)$ is defined by (3.13). By taking the derivatives in the both sides in equality (3.13) and using (1.7), we get

$$\begin{aligned}(1+\alpha)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu &- \alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \\ &= p(z) + \frac{\alpha z p'(z)}{(\lambda+p)\mu} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).\end{aligned} \tag{3.17}$$

An application of Lemma 4 to (3.17) yields

$$\begin{aligned}\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu &\prec \frac{(\lambda+p)\mu}{\alpha} z^{-\frac{(\lambda+p)\mu}{\alpha}} \int_0^z t^{\frac{(\lambda+p)\mu}{\alpha}-1} \frac{1+At}{1+Bt} dt \\ &= \frac{(\lambda+p)\mu}{\alpha} \int_0^1 t^{\frac{(\lambda+p)\mu}{\alpha}-1} \frac{1+Azu}{1+Bzu} du \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).\end{aligned} \tag{3.18}$$

It now follows from (3.18) that

$$\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{3.19}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

By (3.19), we easily find that

$$\mathcal{I}_p^\lambda(a,c)f(z) = z^p \left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{-\frac{1}{\mu}}. \tag{3.20}$$

Combining (1.3), (1.5) and (3.20), we have

$$\left(z^p + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} z^{k+p}\right) * f(z) = z^p \left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{-\frac{1}{\mu}}. \tag{3.21}$$

The assertion (3.16) of Theorem 4 can now easily be derived from (3.21).

Theorem 5. *Let* $f \in \mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$ *with* $\Re(\alpha) > 0$. *Then*

$$\frac{1}{z}\left[\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\left(z^p+\sum_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{k!(a)_k}z^{k+p}\right)*f(z)-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0 \quad (z\in\mathbb{U};\ 0<\theta<2\pi). \tag{3.22}$$

Proof. Suppose that $f \in \mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B)$ with $\Re(\alpha) > 0$. We know that (3.18) holds true, which implies that

$$\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \qquad (z\in\mathbb{U};\ 0<\theta<2\pi). \tag{3.23}$$

It is easy to see that the condition (3.23) can be written as follows:

$$\frac{1}{z}\left[\mathcal{I}_p^\lambda(a,c)f(z)\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}-z^p\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0 \qquad (z\in\mathbb{U};\ 0<\theta<2\pi). \tag{3.24}$$

Combining (1.3), (1.5) and (3.24), we can easily get the convolution property (3.22) asserted by Theorem 5.

Theorem 6. *Let*

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B). \tag{3.25}$$

Then

$$|a_{p+n}| \leqq \frac{(\lambda+p)n!(a)_n}{(\lambda+p)_n(c)_n}\left|\frac{A-B}{(\lambda+p)\mu+n\alpha}\right|. \tag{3.26}$$

The inequality (3.26) *is sharp.*

Proof. Combining (1.8) and (3.25), we can get

$$\begin{aligned}(1+\alpha)&\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu-\alpha\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)\left(\frac{z^p}{\mathcal{I}_p^\lambda(a,c)f(z)}\right)^\mu\\ &=1-\left(1+\frac{n\alpha}{(\lambda+p)\mu}\right)\frac{(\lambda+p)_n(c)_n}{n!(a)_n}\mu a_{p+n}z^n+\cdots\prec\frac{1+Az}{1+Bz}.\end{aligned} \tag{3.27}$$

An application of Lemma 5 to (3.27) yields

$$\left|\left(1+\frac{n\alpha}{(\lambda+p)\mu}\right)\frac{(\lambda+p)_n(c)_n}{n!(a)_n}\mu a_{p+n}\right| \leqq |A-B|. \tag{3.28}$$

The inequality (3.26) can now easily be derived from (3.28). By noting that

$$f(z)=z^p+\frac{(\lambda+p)n!(a)_n}{(\lambda+p)_n(c)_n}\frac{A-B}{(\lambda+p)\mu+n\alpha}z^{p+n}+\cdots\in\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,B),$$

we obtain that the inequality (3.26) is sharp. Theorem 6 is thus proved.

Theorem 7. *Let* $f\in\mathcal{N}_{p,\lambda}^{\alpha,\mu}(a,c;A,0)$ *with*

$$A>0,\quad \Re(\alpha)>0\quad \textit{and}\quad |\alpha|\left(1+\Re\left(\frac{(\lambda+p)\mu}{\alpha}\right)\right)>A(\lambda+p)\mu.$$

Then

$$\left|\frac{z\left(\mathcal{I}_p^\lambda(a,c)f\right)'(z)}{\mathcal{I}_p^\lambda(a,c)f(z)}-p\right|<\frac{A(\lambda+p)\left[|\alpha|\left(1+\Re\left(\frac{(\lambda+p)\mu}{\alpha}\right)\right)+(\lambda+p)\mu\right]}{|\alpha|\left[|\alpha|\left(1+\Re\left(\frac{(\lambda+p)\mu}{\alpha}\right)\right)-A(\lambda+p)\mu\right]}.$$

Proof. Let $p(z)$ be defined by (3.13). It follows from (3.17) that

$$p(z)+\frac{\alpha zp'(z)}{(\lambda+p)\mu}=1+A\omega(z), \tag{3.29}$$

where

$$\omega(z)=\sum_{k=1}^{\infty}\omega_k z^k$$

is analytic in $\mathbb{U}$ with

$$|\omega(z)|<1 \qquad (z\in\mathbb{U}).$$

From (3.29), we can get

$$p(z)=1+A\frac{(\lambda+p)\mu}{\alpha}\int_0^1 t^{\frac{(\lambda+p)\mu}{\alpha}-1}\omega(tz)dt=1+A\frac{(\lambda+p)\mu}{\alpha}\sum_{k=1}^{\infty}\frac{1}{k+\frac{(\lambda+p)\mu}{\alpha}}\omega_k z^k. \tag{3.30}$$

It follows from (3.30) that

$$
\begin{aligned}
(zp(z))' &= 1 + A\frac{(\lambda+p)\mu}{\alpha}\sum_{k=1}^{\infty}\frac{k+1}{k+\frac{(\lambda+p)\mu}{\alpha}}\omega_k z^k \\
&= 1 + A\frac{(\lambda+p)\mu}{\alpha}\sum_{k=1}^{\infty}\frac{1}{k+\frac{(\lambda+p)\mu}{\alpha}}\omega_k z^k \\
&\quad + A\frac{(\lambda+p)\mu}{\alpha}\left(\omega(z) - \frac{(\lambda+p)\mu}{\alpha}\int_0^1 t^{\frac{(\lambda+p)\mu}{\alpha}-1}\omega(tz)dt\right).
\end{aligned} \tag{3.31}
$$

We now find from (3.30) and (3.31) that

$$
zp'(z) = A\frac{(\lambda+p)\mu}{\alpha}\left(\omega(z) - \frac{(\lambda+p)\mu}{\alpha}\int_0^1 t^{\frac{(\lambda+p)\mu}{\alpha}-1}\omega(tz)dt\right). \tag{3.32}
$$

Combining (3.30) and (3.32), we can get

$$
\left|\frac{zp'(z)}{p(z)}\right| < \frac{A(\lambda+p)\mu\left[|\alpha|\left(1+\Re\left(\frac{(\lambda+p)\mu}{\alpha}\right)\right)+(\lambda+p)\mu\right]}{|\alpha|\left[|\alpha|\left(1+\Re\left(\frac{(\lambda+p)\mu}{\alpha}\right)\right)-A(\lambda+p)\mu\right]}. \tag{3.33}
$$

The assertion of Theorem 7 can now easily be derived from (3.33).

Putting $\lambda = 0$ and $p = a = c = 1$ in Theorem 7, we get the following starlikeness criterion for non-Bazilevič functions.

Corollary 7. *Suppose that*

$$
(1+\alpha)\left(\frac{z}{f(z)}\right)^{\mu} - \alpha f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1 + Az
$$

with

$$
A > 0, \quad \Re(\alpha) > 0 \quad \textit{and} \quad |\alpha|\left(1+\Re\left(\frac{\mu}{\alpha}\right)\right) > A\mu.
$$

Then

$$
\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{A\left[|\alpha|\left(1+\Re\left(\frac{\mu}{\alpha}\right)\right)+\mu\right]}{|\alpha|\left[|\alpha|\left(1+\Re\left(\frac{\mu}{\alpha}\right)\right)-A\mu\right]}.
$$

Acknowledgements

The present investigation was supported by the *Scientific Research Fund of Hunan Provincial Education Department* under Grant 08C118 of the People's Republic of China.

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