Tamsui Oxford Journal of Mathematical Sciences **26(2)** (2010) 237-254
Aletheia University

Roe's Theorem Associated with a Dunkl Type Differential-Difference Operator on the Real Line *

Hatem Mejjaoli†
(Dedicated to Khalifa Trimèche)‡
Department of Mathematics, Faculty of sciences of Tunis- CAMPUS-1060, Tunis, Tunisia.

Received December 5, 2008, Accepted July 22, 2009.

Abstract

In this paper we characterize the eigenfunctions for the linear differential-difference operators with constant coefficients.

Keywords and Phrases: *Differential-difference operator, Generalized Fourier transform, Roe's theorem.*

1. Introduction

If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period 2π. This is Roe's theorem [14].

---

*2000 *Mathematics Subject Classification.* Primary 33C80, 43A15, 43A32, 44A15.

†E-mail: hatem.mejjaoli@ipest.rnu.tn

‡Thanks to Professors Khalifa Trimèche and Mohamed Ali Mourou for their helps and encouragements. Thanks is also due to the referee for his suggestions and comments.

The purpose of this work is to generalize Roe's theorem. In place of derivatives and antiderivatives, we shall extended this to all formally self-adjoint constant-coefficient differential-difference operators. Indeed, we consider the first-order singular differential-difference operator Λ_A on $\mathbb{R}$ introduced by Mourou and Trimèche in [12] defined for a function f of class C^1 on $\mathbb{R}$ by

$$\Lambda_A f = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right),$$

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. We suppose in addition that

(i) A is increasing on $[0,\infty[$;

(ii) There exists a constant $\delta > 0$ such that the function $e^{\delta x}B'(x)/B(x)$ is bounded for large $x \in [0,\infty[$ together with its derivatives.

This operator extend the usual partial derivatives by additional reflection terms and give generalizations of many analytic structures like the exponential function, the Fourier transform and the convolution product (cf. [12]).

For $A(x) = |x|^{2\alpha+1}$, $\alpha \geq -\frac{1}{2}$, we regain the differential-difference operator

$$D_\alpha f = \frac{df}{dx} + \left(\alpha + \frac{1}{2}\right)\frac{f(x)-f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ (cf. [5]).

For $A(x) = \left(\frac{\sinh|x|}{\cosh x}\right)^{2\alpha+1}$, $\alpha \geq -\frac{1}{2}$, we regain the differential-difference operator

$$l_\alpha f = \frac{df}{dx} + \Big[(2\alpha+1)(\coth x - \tanh x)\Big]\frac{f(x)-f(-x)}{x},$$

which is referred to as the special case of Jacobi-Dunkl operator on $\mathbb{R}$ (cf. [4]).

For an even function f of class C^2 on $\mathbb{R}$, we have

$$\Lambda_A^2 f = \mathcal{L}_A f,$$

with

$$\mathcal{L}_A = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}. \tag{1.1}$$

(See [18]).

We note $\Lambda_A = \mathcal{L}_A^{\frac{1}{2}}$ if and only if, for an even function f of class C^2 on $\mathbb{R}$, we have

$$\Lambda_A^2 f = \mathcal{L}_A f.$$

We now state our main result.

Theorem 1. *Suppose $P(\xi) = \sum_n a_n \xi^n$ is a polynomial in ξ with real coefficients and $L_A = P(-i\Lambda_A)$. Let $a \geq 0$ and let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$L_A f_j = f_{j+1}$$

and

$$|f_j(x)| \leq M_j(1+|x|)^a, \tag{1.2}$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{1.3}$$

Then $f_0 = f_+ + f_-$ where $L_A f_+ = f_+$ and $L_A f_- = -f_-$. If 1 (or -1) is not in the range of P then $f_+ = 0$ (or $f_- = 0$).

2. Preliminaries

In this section we recall some facts about harmonic analysis related to the differential-difference operator Λ_A. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [12].

For each $\lambda \in \mathbb{C}$ the differential-difference equation

$$\Lambda_A u = \lambda u, \quad u(0) = 1,$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_{i\lambda}(x) + \frac{1}{\lambda}\frac{d}{dx}\varphi_{i\lambda}(x) & \text{if } \lambda \neq 0, \\ \\ 1 & \text{if } \lambda = 0, \end{cases}$$

where for each $z \in \mathbb{C}$, φ_z designates the solution of the differential equation

$$\mathcal{L}_A u = -z^2 u, \quad u(0) = 1$$

$\mathcal{L}_A$ being the second-order singular differential operator on $\mathbb{R}$ defined by (1.1). Moreover, $\Phi_\lambda(x)$ is entire in λ. Recently Mourou in [13] have proved for all λ, $x \in \mathbb{R}$,

$$|\Phi_{i\lambda}(x)| \leq 1. \tag{2.4}$$

Notations. We denote by

- $C_b^p(\mathbb{R})$ the space of bounded functions of class C^p on $\mathbb{R}$.
- $\mathcal{E}(\mathbb{R})$ The space of infinitely differentiable functions on $\mathbb{R}$.
- $D(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ which are of compact support.
- $S(\mathbb{R})$ the space of C^∞-functions g on $\mathbb{R}$ which are rapidly decreasing together with their derivatives.
- $D'(\mathbb{R})$ the space of distributions on $\mathbb{R}$. It is the topological dual of $D(\mathbb{R})$.
- $S'(\mathbb{R})$ the space of temperate distributions on $\mathbb{R}$. It is the topological dual of $S(\mathbb{R})$.

  We provide these spaces with the classical topology.
- For a Borel positive measure μ on $\mathbb{R}$, and $1 \leq p \leq \infty$, we write $L_\mu^p(\mathbb{R})$ for the Lebesgue space equipped with the norm $\|\cdot\|_{p,\mu}$ defined by

$$\|f\|_{p,\mu} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu(x)\right)^{1/p}, \quad \text{if } p < \infty,$$

  and

$$\|f\|_{\infty,\mu} = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)|.$$

  When $\mu = w(x)dx$, with w a nonnegative function on $\mathbb{R}$, we replace the μ in the norms by w.

Definition 1. *The generalized Fourier transform of a function* $f \in L^1_A(\mathbb{R})$ *is defined by*

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-i\lambda}(x)A(x)dx, \quad \lambda \in \mathbb{R}.$$

Many of the important properties of Fourier transforms on locally compact abelian groups are proved to hold true for $\mathcal{F}$.

Theorem 2. *The generalized Fourier transform* $\mathcal{F}$ *is a bijection from* $\mathcal{S}(\mathbb{R})$ *onto itself.*

Theorem 3. i*) Plancherel formula: There is an even positive measure* σ *on* $\mathbb{R}$ *such that for all* $f \in \mathcal{S}(\mathbb{R})$*,*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2 d\sigma(\lambda).$$

ii*) Plancherel theorem: The generalized Fourier transform* $\mathcal{F}$ *extends uniquely to a unitary isomorphism from* $L^2_A(\mathbb{R})$ *onto* $L^2_\sigma(\mathbb{R})$*.*

iii*) Inversion formula: Let* f *be a function in* $L^1_A(\mathbb{R})$*, such that* $\mathcal{F}(f) \in L^1_\sigma(\mathbb{R})$*. Then the inverse transform of* f *is given by*

$$\mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}} f(\lambda)\Phi_{i\lambda}(x)d\sigma(\lambda), \quad a.e.\ x \in \mathbb{R}.$$

The measure σ is called the spectral measure associated with the differential-difference operator Λ_A. Under our assumptions on the function A, it is known that the spectral measure σ takes the form

$$d\sigma(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2}, \quad \lambda \in \mathbb{R},$$

where $c(s)$ is a continuous function on $]0, \infty[$ such that

$$c(s)^{-1} \sim k_1\, s^{\alpha+\frac{1}{2}}, \quad \text{as } s \to \infty,$$

$$c(s)^{-1} \sim k_2\, s^{\alpha+\frac{1}{2}}, \quad \text{as } s \to 0,$$

for some $k_1, k_2 \in \mathbb{C}$.

In the Dunkl operator case corresponding to $A(x) = |x|^{2\alpha+1}$, $\alpha \geq -\frac{1}{2}$, the spectral measure σ is given by

$$d\sigma(\lambda) = \frac{|\lambda|^{2\alpha+1} d\lambda}{2^{2\alpha+2}\,(\Gamma(\alpha+1))^2}.$$

In the special case of Jacobi-Dunkl operator corresponding to $A(x) = \left(\frac{\sinh|x|}{\cosh x}\right)^{2\alpha+1}$, $\alpha \geq -\frac{1}{2}$, the spectral measure σ is given by

$$d\sigma(\lambda) = \frac{d\lambda}{8\pi\,|c(|\lambda|)|^2},$$

where

$$c(\mu) = \frac{\Gamma(\alpha+1)\Gamma(i\mu)}{2^{i\mu}\Gamma(\frac{i\mu}{2})\Gamma(\alpha+1+\frac{i\mu}{2})},\ \mu \in \mathbb{C}\backslash\{i\mathbb{N}\}.$$

We shall need the following properties.

Proposition 1. i) *Let $f \in C_b^1(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \Lambda_A f(x) g(x) A(x) dx = -\int_{\mathbb{R}} f(x) \Lambda_A g(x) A(x) dx. \tag{2.5}$$

ii) *For $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\mathcal{F}\left(\Lambda_A f\right)(\lambda) = i\lambda \mathcal{F} f(\lambda), \quad \lambda \in \mathbb{R}. \tag{2.6}$$

Proof. An integration by parts yields assertion (i). Assertion (ii) follows by substituting in (2.5) g by $\Phi_{-i\lambda}$.

2.1 The Generalized Convolution

Definition 2. *Let y be in $\mathbb{R}$. The generalized translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}(\mathbb{R})$ by*

$$\mathcal{F}(\tau_y f)(x) = \Phi_{ix}(y)\mathcal{F}(f)(x), \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.7}$$

At the moment an explicit formula for the generalized translation operator is known only in the following three cases.

1st case: In the Dunkl operator case corresponding to $A(x) = |x|^{2\alpha+1}$, $\alpha \geq -\frac{1}{2}$.
2nd case: In the special case of Jacobi-Dunkl operator corresponding to

$$A(x) = \Big(\frac{\sinh|x|}{\cosh x}\Big)^{2\alpha+1}, \ \alpha \geq -\frac{1}{2}.$$

3rd case: In the Chébli-Trimèche hypergroups.

In all these cases the generalized translation operator can be written as follow:

Definition 3. *For $x, y \in \mathbb{R}$, the generalized translation operator is defined on $C_b(\mathbb{R})$ by*

$$\tau_x f(y) = \int_{\mathbb{R}} f(u)\, d\nu^A_{x,y}(u), \tag{2.8}$$

where the measure $d\nu^A_{x,y}$ is of the form

$$d\nu^A_{x,y}(u) = \begin{cases} \mathcal{W}_A(x,y,u)A(u)du & \text{if} \quad xy \neq 0, \\ \delta_x & \text{if} \quad y = 0, \\ \delta_y & \text{if} \quad x = 0, \end{cases} \tag{2.9}$$

$\mathcal{W}_A$ is a kernel which is not necessarily positive.

In the following we shall give some properties of the measure $\nu^A_{x,y}$ (cf. [1, 15, 16, 18]).

Proposition 2. *For all $x, y \in \mathbb{R}$, we have*

i) $\nu^A_{x,y}(\mathbb{R}) = 1$.

ii) $||\nu^A_{x,y}|| \leq 4$.

iii) *supp* $(\nu^A_{x,y}) \subset I_{x,y}$ *where*

$$I_{x,y} = [-|x| - |y|, -||x| - |y||] \cup [||x| - |y||, |x| + |y|]. \tag{2.10}$$

Using the generalized translation operator, we define the generalized convolution product of functions as follows.

Definition 4. *The generalized convolution product of f and g in $\mathcal{S}(\mathbb{R})$ is the function $f * g$ defined by*

$$f * g(x) = \int_{\mathbb{R}} \tau_x f(-y) g(y) A(y) dy, \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.11}$$

This convolution is commutative and associative and satisfies the following properties. (See [12]).

Proposition 3. i*) For f,g in $D(\mathbb{R})$(resp. $\mathcal{S}(\mathbb{R})$) the function $f * g$ belongs to $D(\mathbb{R})$(resp. $\mathcal{S}(\mathbb{R})$) and we have*

$$\mathcal{F}(f * g)(y) = \mathcal{F}(f)(y)\mathcal{F}(g)(y), \quad \text{for all} \quad y \in \mathbb{R}. \tag{2.12}$$

ii*) We assume that $\Lambda_A \in \left\{D_\alpha, l_\alpha, \mathcal{L}_A^{\frac{1}{2}}\right\}$. Let $1 \leq p, q, r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If f is in $L^p_A(\mathbb{R})$ and g is an element of $L^q_A(\mathbb{R})$, then $f * g$ belongs to in $L^r_A(\mathbb{R})$ and we have*

$$\|f * g\|_{L^r_A(\mathbb{R})} \leq 4 \|f\|_{L^p_A(\mathbb{R})} \|g\|_{L^q_A(\mathbb{R})}. \tag{2.13}$$

For f in $L^p_A(\mathbb{R})$, we define the tempered distribution $\mathcal{T}_f$ associated with f by

$$\langle \mathcal{T}_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)A(x)dx, \ \phi \in \mathcal{S}(\mathbb{R}). \tag{2.14}$$

We denote by $\langle f, \phi \rangle_A$ the second member.

Definition 5. i*) The generalized Fourier transform of a distribution τ in $\mathcal{S}'(\mathbb{R})$ is defined by*

$$\langle \mathcal{F}(\tau), \phi \rangle = \langle \tau, \mathcal{F}^{-1}(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}). \tag{2.15}$$

ii*) The generalized Fourier transform of f in $L^p_A(\mathbb{R})$ denoted also by $\mathcal{F}(f)$, is defined by*

$$\langle \mathcal{F}(f), \phi \rangle = \langle \mathcal{F}(\mathcal{T}_f), \phi \rangle = \langle \mathcal{T}_f, \mathcal{F}^{-1}(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

Thus from (2.14) we have

$$\langle \mathcal{F}(f), \phi \rangle = \int_{\mathbb{R}} f(x) \mathcal{F}^{-1}(\phi)(x) A(x) dx.$$

Theorem 4. *The generalized Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}'(\mathbb{R})$ onto itself.*

Definition 6. *The generalized convolution product of a distribution S in $\mathcal{S}'(\mathbb{R})$ and a function ϕ in $\mathcal{S}(\mathbb{R})$ (or S in $D'(\mathbb{R})$ and a function ϕ in $D(\mathbb{R})$) is the function $S * \phi$ defined by*

$$S * \phi(x) = \langle S_y, \tau_x \phi(-y) \rangle, \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.16}$$

Let u be in $D'(\mathbb{R})$. We define the distribution $\Lambda_A u$ by

$$\langle \Lambda_A u, \psi \rangle \;\; = \;\; -\langle u, \Lambda_A \psi \rangle, \;\; \text{for all} \; \psi \in \; D(\mathbb{R}).$$

Remark 1. As in the clasical case we have for S in $\mathcal{S}'(\mathbb{R})$ and ϕ in $\mathcal{S}(\mathbb{R})$ (or S in $D'(\mathbb{R})$ and ϕ in $D(\mathbb{R})$)

$$\Lambda_A(S * \phi) = (\Lambda_A S) * \phi. \tag{2.17}$$

Proposition 4. *We assume that $\Lambda_A \in \left\{ D_\alpha, l_\alpha, \mathcal{L}_A^{\frac{1}{2}} \right\}$.*

i*) Let f be in $L^p_A(\mathbb{R})$, $p \in [1, \infty]$ and ϕ in $\mathcal{S}(\mathbb{R})$. Then the distribution $\mathcal{T}_f * \phi$ is given by the function $f * \phi$ which belongs to $L^p_A(\mathbb{R})$.*

ii*) Let f be in $L^p_A(\mathbb{R})$, $p \in [1, \infty]$ and ϕ_1, ϕ_2 in $\mathcal{S}(\mathbb{R})$. Then we have*

$$\langle \mathcal{T}_f * \phi_1, \phi_2 \rangle = \langle \check{f}, \phi_1 * \check{\phi}_2 \rangle_A, \tag{2.18}$$

where $\check{h}(x) = h(-x)$.

iii*) Let f be in $L^p_A(\mathbb{R})$, $p \in [1, \infty]$ and ϕ in $\mathcal{S}(\mathbb{R})$. Then we have*

$$\mathcal{F}(\mathcal{T}_f * \phi) = \mathcal{F}(\mathcal{T}_f) \mathcal{F}(\phi). \tag{2.19}$$

Proof. i) Let f be in $L^p_A(\mathbb{R})$, $p \in [1, \infty]$ and ϕ in $\mathcal{S}(\mathbb{R})$. From the relations (2.16), (2.14) and (2.11) we have

$$\begin{aligned} \forall\, x \in \mathbb{R}, \; \mathcal{T}_f * \phi(x) &= \langle (\mathcal{T}_f)_y, \tau_x \phi(-y) \rangle, \\ &= \langle f, \tau_x \phi(-y) \rangle_A, \\ &= f * \phi(x). \end{aligned}$$

By applying the Proposition 3 ii) we see that $\mathcal{T}_f * \phi$ belongs to $L^p_A(\mathbb{R})$.

ii) Let f be in $L^p_A(\mathbb{R})$, $p \in [1,\infty]$ and ϕ_1, ϕ_2 in $\mathcal{S}(\mathbb{R})$. Then from Fubini-Tonelli's theorem the function $(x,y) \mapsto f(-y)\tau_x\phi_1(y)\phi_2(x)$ is integrable on $\mathbb{R}\times\mathbb{R}$ with respect to the measure $A(y)dy\,A(x)dx$. Thus from Fubini's theorem we obtain

$$\begin{aligned}\langle \mathcal{T}_f * \phi_1, \phi_2\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(-y)\tau_x\phi_1(y)\phi_2(x)A(y)dyA(x)dx,\\ &= \int_{\mathbb{R}} f(-y)\Big(\int_{\mathbb{R}} \tau_y\phi_1(x)\phi_2(x)A(x)dx\Big)A(y)dy\\ &= \int_{\mathbb{R}} f(-y)\phi_1 * \check{\phi}_2(y)A(y)dy,\\ &= \langle \check{f}, \phi_1 * \check{\phi}_2\rangle_A.\end{aligned}$$

iii) Let f be in $L^p_A(\mathbb{R})$, $p \in [1,\infty]$ and ϕ in $\mathcal{S}(\mathbb{R})$. Then from i) and the relations (2.12) and (2.18) we have for any φ in $\mathcal{S}(\mathbb{R})$

$$\begin{aligned}\langle \mathcal{F}(\mathcal{T}_f * \phi), \varphi\rangle &= \langle \mathcal{T}_f * \phi, \mathcal{F}^{-1}(\varphi)\rangle,\\ &= \langle \check{f}, \phi * \mathcal{F}^{-1}(\varphi)\rangle_A\\ &= \langle f, \mathcal{F}^{-1}(\mathcal{F}(\phi)\varphi)\rangle_A\\ &= \langle \mathcal{F}(\mathcal{T}_f)\mathcal{F}(\phi), \varphi\rangle.\end{aligned}$$

Thus we have the result.

In the following $\mathcal{T}_f$ will be denoted by f.

3. Proof of Theorem 1

We break the proof up into three steps. In the first step we consider the generalized Fourier transform $\mathcal{F}(f_0)$ of f_0, which exists as a distribution.

Lemma 1. *Let $(f_j)_{j\in\mathbb{Z}}$ is a sequence of functions on $\mathbb{R}$ satisfying*

$$L_A f_j = f_{j+1}, \tag{3.20}$$

$$|f_j(x)| \leq M_j(1+|x|)^a, \tag{3.21}$$

and

$$\lim_{j\to\infty} \frac{M_j}{(1+\varepsilon)^j} = 0, \tag{3.22}$$

for all $\varepsilon > 0$, *then*

$$\text{support}(\mathcal{F}(f_0)) \subset S = \Big\{\xi,\ |P(\xi)| = 1\Big\}.$$

Proof. First we show that $\mathcal{F}(f_0)$ is supported in $\Big\{\xi,\ |P(\xi)| \le 1\Big\}$. To do this we need to show that $\langle \mathcal{F}(f_0), \phi\rangle = 0$ if $\phi \in D(\mathbb{R})$ and $\text{support}(\phi) \cap \Big\{\xi,\ |P(\xi)| \le 1\Big\} = \emptyset$. Since support($\phi$) is compact, there is some $r < 1$ so that $\frac{1}{|P(\xi)|} \le r$, for all $\xi \in \text{support}(\phi)$. Then

$$\begin{aligned} \langle \mathcal{F}(f_0), \phi\rangle &= \langle P^j \mathcal{F}(f_0), \tfrac{\phi}{P^j}\rangle \\ &= \langle \mathcal{F}(L_A^j f_0), \tfrac{\phi}{P^j}\rangle \\ &= \langle L_A^j f_0, \mathcal{F}^{-1}(\tfrac{\phi}{P^j})\rangle. \end{aligned}$$

Choose an integer m with $2m \ge 2a+2\alpha+2$. A calculation, using the hypothesis of the lemma and Cauchy-Schwartz inequality, implies

$$\begin{aligned} |\langle \mathcal{F}(f_0), \phi\rangle| &\le \int_{\mathbb{R}} |L^j f_0(x)||\mathcal{F}^{-1}(\frac{\phi}{P^j})(x)|A(x)dx \\ &\le M_j\Big(\int_{\mathbb{R}} \frac{(1+|x|)^{2a}}{(1+|x|^2)^m} A(x)dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}} (1+|x|^2)^m |\mathcal{F}^{-1}(\frac{\phi}{P^j})(x)|^2 A(x)dx\Big)^{\frac{1}{2}} \\ &\le CM_j\Big(\int_{\mathbb{R}} (1+|x|^2)^m |\mathcal{F}^{-1}(\frac{\phi}{P^j})(x)|^2 A(x)dx\Big)^{\frac{1}{2}}. \end{aligned}$$

Since ϕ is supported in $\Big\{\xi,\ |P(\xi)| \ge 1+\varepsilon\Big\}$ for some fixed $\varepsilon > 0$, it is not hard to prove that the right-hand side of this goes to zero as $j \to \infty$ and so $\langle \mathcal{F}(f_0), \phi\rangle = 0$. To complete the proof we need to show that $\mathcal{F}(f_0)$ is also supported in $\Big\{\xi,\ |P(\xi)| \ge 1\Big\}$, which means $\langle \mathcal{F}(f_0), \phi\rangle = 0$ if ϕ is supported in $\Big\{\xi,\ |P(\xi)| \le 1\Big\}$. Here we use (3.20) to obtain

$$\langle \mathcal{F}(f_0), \phi\rangle = \langle \mathcal{F}(f_{-j}), P^j \phi\rangle$$

and the argument proceeds as before.

The next step in the proof we assume firstly that -1 is not a value of $P(\xi)$, and show that $L_A f_0 = f_0$.

Lemma 2. *There exists an integer N such that*

$$(P-1)^{N+1}\mathcal{F}(f_0) = 0. \tag{3.23}$$

Proof. From the growth conditions on the sequence $(f_j)_{j\in\mathbb{Z}}$, Lemma 1, and the assumption that $P(\xi) \neq -1$, we obtain

$$\text{support}(\mathcal{F}(f_0)) \subset \Big\{\xi,\ P(\xi) = 1\Big\}.$$

As $\mathcal{F}(f_0)$ is a continuous linear functional on $\mathcal{S}(\mathbb{R})$, there is a constant C and integers m and N so that

$$|\langle \mathcal{F}(f_0), \phi\rangle| \leq C||\phi||_{N,m} \tag{3.24}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$ when the topology on the space $\mathcal{S}(\mathbb{R})$ is defined by the seminorms

$$||\phi||_{N,m} = \sup_{x\in\mathbb{R}} \sum_{n\leq N} (1+|x|)^m |\Lambda_A^n \phi(x)|.$$

Thus a distribution $\mathcal{F}(f_0)$ is of order $\leq N$. For this N we want to prove that

$$(P-1)^{N+1}\mathcal{F}(f_0) = 0.$$

To simplify notation set $Q := (P-1)$. Then we need to show, for any compactly supported C^∞ function ϕ, that

$$\langle Q^{N+1}\mathcal{F}(f_0), \phi\rangle = \langle \mathcal{F}(f_0), Q^{N+1}\phi\rangle = 0.$$

Let $g : \mathbb{R} \to [0,1]$ be a C^∞ function with $g = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $g = 0$ outside $(-1,1)$.
Set $g_r(t) := g(\frac{t}{r})$, $Q_r = g_r(Q)Q^{N+1}\phi$. Then $Q_r = Q^{N+1}\phi$ in a neighborhood of

$$\text{support}\,\mathcal{F}(f_0) \subset \Big\{\xi : Q(\xi) = 0\Big\} = \Big\{\xi : P(\xi) = 1\Big\}.$$

Thus by (3.24) we have

$$|\langle \mathcal{F}(f_0), Q^{N+1}\phi\rangle| = |\langle \mathcal{F}(f_0), Q_r\rangle| \leq C||Q_r||_{N,m}.$$

We proceed as in [8] to prove that $||Q_r||_{N,m} \to 0$ as $r \to 0$. Thus (3.23) is proved.

Conclusion of the proof of Theorem 1

Inverting the generalized Fourier transform in (3.23) yields that

$$(L_A - 1)^{N+1} f_0 = 0. \tag{3.25}$$

This equation implies

$$\mathrm{span}\Big\{ f_0, f_1, f_2, ... \Big\} = \mathrm{span}\Big\{ f_0, L_A f_0, L_A^2 f_0, ... \Big\} = \mathrm{span}\Big\{ f_0, L_A f_0, L_A^2 f_0, ... L_A^N f_0 \Big\}$$

We shall now show that we can take $N = 0$ in (3.25). If not then $(L_A - 1) f_0 \neq 0$. Let p be the largest positive integer so that $(L_A - 1)^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := (L_A - 1)^{p-1} f_0 \in \mathrm{span}\Big\{ f_0, f_1, ..., f_N \Big\}$$

will satisfy

$$(L_A - 1)^2 f = 0 \quad \text{and} \quad (L_A - 1) f \neq 0. \tag{3.26}$$

Write

$$f = a_0 f_0 + ... + a_N f_N,$$

for constants $a_0, ..., a_N$. Then

$$L_A^j f = a_0 f_j + ... + a_N f_{N+j}.$$

If

$$C_j = |a_0| M_j + ... + |a_N| M_{j+N},$$

then this and (1.2) imply

$$|L_A^j f(x)| \leq C_j (1 + |x|)^a. \tag{3.27}$$

By (1.3) these satisfy the sublinear growth condition

$$\lim_{j \to \infty} \frac{C_j}{j} = 0. \tag{3.28}$$

An induction using (3.26) implies for $j \geq 2$ that

$$L_A^j f = j L_A f - (j-1) f = j(L_A - 1) f + f.$$

Thus

$$|(L_A - 1) f(x)| \leq \frac{1}{j} |L_A^j f(x)| + \frac{|f(x)|}{j} \leq \frac{C_j}{j} (1 + |x|)^a + \frac{|f(x)|}{j}.$$

Letting $j \to \infty$ and using (3.28) implies $(L_A - 1)f = 0$. But this contradicts (3.26). Consequently, $N = 0$ in (3.25). This completes the proof in the case that -1 is not in the range of P.

In the case that 1 is not in the range of P we apply the same argument to $-L_A$ to conclude $L_A f_0 = -f_0$. In the general case, let $\mathfrak{L}_A = L_A^2$. Then $\mathcal{F}(\mathfrak{L}_A f)(\xi) = P(\xi)^2 \mathcal{F}(f)(\xi)$. $\mathfrak{L}_A f_{2p} = f_{2(p+1)}$ and $P(\xi)^2 \neq -1$. Thus we can (as before) conclude, for the sequence $(f_{2p})_{p\in\mathbb{Z}}$ that

$$\mathfrak{L}_A f_0 = L_A^2 f_0 = f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + L_A f_0)$ and $f_- = \frac{1}{2}(f_0 - L_A f_0)$. Then $f_0 = f_+ + f_-$, $L_A f_+ = f_+$ and $L_A f_- = -f_-$. This completes the proof of Theorem 1.

Remark 2. If we take $P(y) = -|y|^2$, then $L_A = \triangle_A$ and Theorem 1 give $\triangle_A f_0 = -f_0$. This characterizes eigenfunctions f of generalized Laplace operator $\triangle_A$ with polynomial growth in terms of the size of the powers $\triangle_A^j f$, $-\infty < j < \infty$. It also generalizes results of Roe [14] (where $\frac{A'}{A} = 0$, $a = 0$, $M_j = M$, and $d = 1$) and Strichartz [17](where $\frac{A'}{A} = 0$, $a = 0$, $M_j = M$, for $d = 1$).

As an application of the above theorem we have the following corollary.

Corollary 1. *We assume that* $\Lambda_A \in \left\{D_\alpha, l_\alpha, \mathcal{L}_A^{\frac{1}{2}}\right\}$. *If in Theorem 1 , we replace (1.2) by*

$$||f_j||_{L^p_A(\mathbb{R})} \leq M_j, \tag{3.29}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0.$$

Then $f_0 = f_+ + f_-$ *where* $L_A f_+ = f_+$ *and* $L f_- = -f_-$. *If 1 (or -1) is not in the range of* P *then* $f_+ = 0$ *(or* $f_- = 0$*).*

Proof. Let $t > 0$. Consider the functions $F_{j,t}(x) = (f_j * E_t)(x)$ where E_t the heat kernel given by :

$$E_t(y) = \mathcal{F}^{-1}\Big(\exp(-\frac{|.|^2}{4t})\Big)(y), \quad \text{for all} \quad y \in \mathbb{R}. \tag{3.30}$$

¿From the relation (2.13) we deduce that

$$|F_{j,t}(x)| \leq 4||f_j||_{L^p_A(\mathbb{R})}||E_t||_{L^{p'}_A(\mathbb{R})}, \quad \text{for all} \quad x \in \mathbb{R}$$

where p' is the conjugate exponent of p. On the other hand

$$L_A F_{j,t} = F_{j+1,t}, \quad j \in \mathbb{Z}.$$

Thus $\{F_{j,t}\}_{j\in\mathbb{Z}}$ verifies the relations (1.2) and (1.3) of Theorem 1 and then the result is immediately.

In the space of distributions $D'(\mathbb{R})$, we use the regularization of distributions to obtain the analogue of Theorem 1.

Theorem 5. *Let $P(\xi) = \sum_\nu a_\nu \xi^\nu$ be a polynomial in ξ with real coefficients and let*

$$L_A = P(-i\Lambda_A) = \sum_\nu (-i)^\nu a_\nu \Lambda_A^\nu.$$

Let $u_j \in D'(\mathbb{R})$, $j \in \mathbb{Z}$. Suppose that for every compact subset K of $\mathbb{R}$, there exist a nonnegative integer N and a positive constants $M_j := M_j(K, N)$ such that

i*)* $L_A u_j = u_{j+1}$.

ii*)* $||u_j * \varphi||_{L^\infty_A(\mathbb{R})} \leq M_j \sum_{n\leq N} \sup_{x\in K} |\Lambda_A^n \varphi(x)|$ *for all $j \in \mathbb{Z}$ and all $\varphi \in D(\mathbb{R})$,*

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0.$$

Then $u_0 = u_+ + u_-$ where $L_A u_+ = u_+$ and $L_A u_- = -u_-$. If 1 (or -1) is not in the range of P then $u_+ = 0$ (or $u_- = 0$).

Proof. Let $\chi \in D(\mathbb{R})$ such that $\int_\mathbb{R} \chi(x)A(x)dx = 1$, and set $\chi_n(x) = \frac{A(nx)}{nA(x)}\chi(nx)$, $n \in \mathbb{N}^*$. ¿From the relation (2.14) we have

$$\langle \chi_n, \varphi\rangle_A = \int_\mathbb{R} \chi(x)\varphi(\frac{x}{n})A(x)dx \underset{n\to\infty}{\longrightarrow} \Big(\int_\mathbb{R} \chi(x)A(x)dx\Big)\varphi(0) = \langle \delta, \varphi\rangle_A.$$

Then $\chi_n \to \delta$ in $D'(\mathbb{R})$, and support $\chi_n \subset$ support χ for all n. For each $j \in \mathbb{Z}$, $u_j * \chi_n \in \mathcal{E}(\mathbb{R})$ which is a regularization of u_j and $u_j * \chi_n \to u_j$ in $D'(\mathbb{R})$ as $n \to \infty$. Let $h_{j,n} := u_j * \chi_n$. Using the relation (2.17) we obtain

$$L_A h_{j,n} = (L_A u_j) * \chi_n = u_{j+1} * \chi_n.$$

Then for $K :=$ support χ and all $j \in \mathbb{Z}$, it follows from the hypothesis (i) and (ii) that

$$\begin{array}{lcl} L_A h_{j,n} & = & u_{j+1} * \chi_n = h_{j+1,n} \\ ||h_{j,n}||_{L^\infty_A(\mathbb{R})} & \leq & \widetilde{M_j}, \end{array} \tag{3.31}$$

where

$$\widetilde{M_j} := M_j \sum_{m \leq N} \sup_{x \in K} |\Lambda_A^m \chi_n(x)|$$

is a positive constant. It then follows from (3.31) and Theorem 1 that $u_n = u_{n,+} + u_{n,-}$ where $L_A u_{n,+} = u_{n,+}$ and $L_A u_{n,-} = -u_{n,-}$. If 1 (or -1) is not in the range of P then $u_{n,+} = 0$ (or $u_{n,-} = 0$). Letting $n \to \infty$, we obtain the result.

Theorem 6. *If, in Theorem 1, we replace (1.3) with*

$$\lim_{j \to \infty} \frac{M_{|j|}}{(1+\varepsilon)^j} = 0, \tag{3.32}$$

for all $j > 0$, then the span of $(f_j)_j$ is finite dimensional. Moreover, $f_0 = f_+ + f_-$, where, for some integer N, $(L_A - 1)^N f_+ = 0$ and $(L_A + 1)^N f_- = 0$. Thus f_+ (or f_-) is a generalized eigenfunction of L_A with eigenvalue 1 *(or* -1*).*

The proof will be based on the following result from linear algebra (cf. [2], Chapter 10).

Lemma 3. *Let X be a finite dimensional complex vector space, and let $T : X \to X$ be a linear map with eigenvalues $\lambda_1, ..., \lambda_p$. Then $X = X_1 \oplus ... \oplus X_p$, where $X_j = ker((T - \lambda_j)^N)$ and $dim X = N$.*

We first prove Theorem 3 under the assumption that $P(\xi) \neq -1$. Using the growth condition (3.32) and the Lemma 3, we may still conclude that

$$\text{support}(\mathcal{F}(f_0)) \subset S = \Big\{\xi : P(\xi) = 1\Big\}.$$

But then, as before, we can conclude that (3.25) holds. But this is enough to complete the proof in this case. A similar argument shows that if $P(\xi) \neq 1$, then $(L_A + 1)^N f_0 = 0$.

In the general case we again let $\mathfrak{L}_A = L_A^2$ and $P_0 = P^2$. Then $P_0(\xi) \neq -1$ and the span of $(f_{2j})_j$ is finite dimensional. The map L_A takes the span of $(f_{2j})_j$ onto the span of $(f_{2j+1})_j$. Thus X is finite dimensional. Any $f \in X$ will have support$(\mathcal{F}(f))$ inside the set defined by $P(\xi) = \pm 1$. From this it is not hard to show the only possible eigenvalues of L_A restricted to X are 1 and -1. The result now follows from the last lemma.

References

[1] N. Ben Salem and A. Oueld Ahmed Salem, A convolution structure associated with the generalized operator on $\mathbb{R}$, *The Ramanuan J.* **12** no. 3 (2006), 359-378.

[2] G. Birkoff and S. MacLane, *A Survey of Modern Algebra*, MacMillan, New York, 1965.

[3] H. Burkill, Sequences characterizing the sine function, *Math. Proc. Camb. Phil. Soc.* **89**(1981), 71-77.

[4] F. Chouchane, M. Mili, and K. Trimèche, Positivity of the intertwining operator and harmonic analysis associated with the generalized operator on $\mathbb{R}$, *J. Anal. Appl.* **1** no. 4 (2003), 387-412.

[5] C. F. Dunkl, Differential-difference operators associated to reflection group, *Trans. Am. Math. Soc.* **311** (1989), 167-183.

[6] C. F. Dunkl, Integral kernels with reflection group invariant, *Can. J. Math.* **43** (1991), 1213-1227.

[7] C. F. Dunkl, Hankel transforms associated to finite reflection groups, *Contemp. Math.* **138** (1992), 123-138.

[8] J.-P. Gabardo, Tempered distributions with spectral gaps, *Math. Proc. Camb. Phil. Soc.* **106** (1989), 143-162.

[9] M. F. E. DE Jeu, The Dunkl transform, *Invent.Math.* **113** (1993), 147-162.

[10] L. Hormander, *Linear Partial Differential Operators*, Springer-Verlag, New York, 1969.

[11] R. Howard, A note on Roe's characterization of the sine function, *Proc. Amer. Math. Soc.* **105** (1989), 658-663.

[12] M. A. Mourou and K. Trimèche, Transmutation operators and Paley-Wiener theorem associated with a singular differential-difference operator on the real line, *Analysis and Applications.* **1** no. 1 (2003), 43-70.

[13] M. A. Mourou, Hardy, and Cowling-Price, Theorems associated with a Dunkl type differential-difference operator on the real line, *Communications in Mathematical Analysis Proceedings of the 16th Symposium of the Tunisian Mathematical Society Conf.* **02** (2008), 56-62.

[14] J. Roe, A characterization of the sine function, *Math. Proc. Comb. Phil. Soc.* **87** (1980), 69-73.

[15] M. Rösler, *Bessel-type signed hypergroups on* $\mathbb{R}$. Pr*obability Measures on Groups and related structures,* XI, Proceeding, Oberwalfach, 1994. (H.Heyer and A. Mukherjea. Eds). Word Scientific Singafore 1995.

[16] S. Thangavelu and Y. Xu, Convolution operator and maximal functions for Dunkl transform, *J. d'Analyse Mathematique* **97** (2005), 25-56.

[17] R. S. Strichartz, Characterization of eigenfunctions of the Laplacian by boundedness conditions, *Trans. Amer. Math. Soc.* **338** (1993), 971-979.

[18] K. Trimèche, Inversion of the Lions transmutation operators using generalized wavelets, *App. Comput. Harm. Anal.* **4** (1997), 97-112.