Tamsui Oxford Journal of Mathematical Sciences **26(2)** (2010) 221-235
Aletheia University

Non-Archimedean Stability of the Monomial Functional Equations *

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Received November 18, 2008, Accepted August 10, 2009.

Abstract

We study the stability of the functional equation

$$\Delta_x^n f(y) = n! f(x)$$

in non-Archimedean spaces in the spirit of Hyers-Ulam-Rassias-Găvruţa. We will give an example to show that some results in the stability of monomial equations in real normed spaces are not valid in non-Archimedean normed spaces.

Keywords and Phrases: *Monomial functional equation, Fixed point alternative, Hyers–Ulam–Rassias stability, non-Archimedean normed space.*

1. Introduction

The following stability is due to S. M. Ulam [24]:
"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" or in other words: Assume that a mathematical object satisfies a certain

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*2000 *Mathematics Subject Classification.* Primary 39B52; Secondary 39B82, 47S10, 47H10.

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property approximately according to some convention. Is it then possible to find an object near this object satisfying the property accurately?

In 1941, D. H. Hyers [12] gave the first significant partial solution to this question for additive mappings. Hyers' theorem was generalized by T. Aoki [1] in 1950 for additive mappings. In 1978, Th.M. Rassias [23] solved the problem for linear mappings. In 1994, a generalization of Th.M. Rassias' theorem was obtained by Găvruţa [6], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called the Hyers–Ulam–Rassias stability. According to P. M. Gruber [11], this kind of stability problems are of particular interest in probability theory and in the case of functional equations of different types.

Let X and Y be linear spaces and Y^X be the vector space of all functions from X to Y. Following [13], for each $x \in X$, define $\Delta_x : Y^X \to Y^X$ by

$$\Delta_x f(y) = f(x + y) - f(x) \quad (f \in Y^X, y \in X).$$

Inductively, we define

$$\Delta_{x_1,\dots,x_n} f(y) = \Delta_{x_1,\dots,x_{n-1}}(\Delta_{x_n} f(y))$$

for each $y, x_1, \dots, x_n \in X$ and $f \in Y^X$, we write

$$\Delta_x^n f(y) = \Delta_{x_1,\dots,x_n} f(y)$$

if $x_1 = \cdots = x_n = x$. By induction on n, it can be easily verified that

$$\Delta_x^n f(y) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(kx + y) \quad (n \in \mathbb{N}, x, y \in X). \tag{1.1}$$

The functional equation

$$\Delta_x^n f(y) = n! f(x) \tag{1.2}$$

is called the *monomial functional equation* of degree n, since the function $f(x) = cx^n$ is a solution of the functional equation. Every solution of the monomial functional equation of degree n is said to be a *monomial mapping* of degree n. In particular additive, quadratic, cubic and quartic functions are monomials of degree one, two, three and four respectively. The stability of monomial equations was initiated by D. H. Hyers in [13]. The problem has been recently considered by many authors see e. g. [3], [7]-[10], [14, 16] and [25].

In 1897, Hensel [5] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. He indeed introduced a field with a valuation norm, which does not have the Archimedean property. During the last three decades the theory of non-Archimedean spaces has gained the interest of physicists for their research, in particular the problems that emerge in quantum physics, p-adic strings and superstrings (see [15]).

L. M. Arriola and W. A. Beyer in [2] initiated the stability of functional equations in non-Archimedean spaces. In fact they established stability of Cauchy functional equations over p-adic fields. In [17], [18] and [20] the stability of Cauchy, quadratic and quartic functional equations in non-Archimedean normed spaces were investigated. Although different methods are known for establishing the stability of functional equations, almost all proofs depend on the Hyers's method in [12]. In 2003, Radu [21] employed the alternative fixed point theorem, due to Diaz and Margolis [4], to prove the stability of Cauchy additive functional equation. Using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see e. g. [3, 19, 22].

Z. Kaiser in [14] proved the stability of monomial functional equation where the functions map a normed space over a field with valuation to a Banach space over a field with valuation and the control function is of the form $\varepsilon\,(||x||^\alpha + ||y||^\alpha)$.

In this paper, we use the fixed point method to prove the Hyers-Ulam-Rassias stability of monomial functional equation of an arbitrary degree in non-Archimedean normed spaces over a field with valuation. More precisely, we will show that if a function f from a linear space X to a complete non-Archimedean normed space Y for some $n \in \mathbb{N}$ satisfies the inequality

$$||\Delta_x^n f(y) - n!f(x)|| \leq \varphi(x,y) \quad (x, y \in X)$$

for suitable control function φ, then f can be suitably approximated by a unique monomial $M : X \to Y$ of degree n. Moreover, we give a counter example to show that the exact form of some results about the stability of monomial functions in real normed spaces may fail in non-Archimedean normed spaces. Then we give some applications of our results for the stability of the monomial functional equations in non-Archimedean normed spaces over a non-Archimedean field. Finally, we will show that, for each $x \in X$, the continuity of $s \mapsto f(sx)$ and the boundedness of $s \mapsto \varphi(sx, sy)$ imply the continuity of $s \mapsto M(sx)$.

2. Preliminaries

We begin with the definition of a non-Archimedean filed and a non-Archimedean normed linear space. Then we give non-Archimedean version of fixed point alternative principle.

Definition 2.1. *Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $| \ | : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have*

(i) $|a| \geq 0$ *and equality holds if and only if* $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strong triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $| \ |$ is non trivial, i.e.,

(iv) there is an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Definition 2.2. *Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $| \ . \ |$. A function $|| \ . \ || : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely,*

$$||x + y|| \leq \max\{||x||, ||y||\} \quad (x, y \in X).$$

Then $(X, || \ . \ ||)$ is called a non-Archimedean space.
By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Remark 2.3. Thanks to the inequality

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all x and $y > 0$, there exists an integer n such that $x < ny$.

Example 2.4. *Let p be a prime number. For any nonzero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\ |$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\ |$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.*
Note that if $p \geq 3$, then $|2^n| = 1$ in for each integer n.

Definition 2.5. *Let X be a nonempty set and $d : X \times X \to [0, \infty]$ satisfy the following properties:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) = d(y, x)$ *(symmetry),*

(iii) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ *(strong triangle inequality),*

for all $x, y, z \in X$. Then (X, d) is called a generalized non-Archimedean metric space. (X, d) is called complete if every d-Cauchy sequence in X is d-convergent.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Definition 2.6. *Let (X, d) be a generalized complete metric space. A mapping $\Lambda : X \to X$ satisfies a Lipschitz condition with a Lipschitz constant $L \geq 0$ if*

$$d(\Lambda(x), \Lambda(y)) \leq L d(x, y) \quad (x, y \in X).$$

If $L < 1$, then Λ is called a strictly contractive operator.

Theorem 2.7. (Non-Archimedean Alternative Contraction Principle) *If (X, d) is a non-Archimedean generalized complete metric space and $J : X \to X$ a strictly contractive mapping (that is $d(J(x), J(y)) \leq Ld(y, x)$, for all $x, y \in X$ and a Lipschitz constant $L < 1$), then either*

(i) $d(J^n(x), J^{n+1}(x)) = \infty$ *for all* $n \geq 0$, *or*

(ii) there exists some $n_0 \geq 0$ *such that* $d(J^n(x), J^{n+1}(x)) < \infty$ *for all* $n \geq n_0$*;*

the sequence $\{J^n(x)\}$ is convergent to a fixed point x^* of J; x^* is the unique fixed point of J in the set

$$\mathcal{Y} = \{y \in X : d(J^{n_0}(x), y) < \infty\}$$

and $d(y, x^*) \leq d(y, J(y))$ for all y in this set.

Proof. A similar argument as [4] can be applied to show that in case (ii), $J|\mathcal{Y}$ has a unique fixed point $x^* \in \mathcal{Y}$ such that for each $y \in \mathcal{Y}$, $\{J^n(y)\}$ converges to x^*. By the strong triangle inequality for all $y \in \mathcal{Y}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y, J^n(y)) &\leq \max\{d(y, J(y)), \ldots, d(J^{n-1}(y), J^n(y))\} \\ &\leq \max\{d(y, J(y), \ldots, L^{n-1}d(y, J(y))\} \\ &= d(y, J(y)). \end{aligned}$$

¿From this the last inequality of the Theorem follows.

3. Stability of Monomial Functional Equations

Thanks to a result due to A. Giányi [7], Z. Kaiser proved the following:

Lemma 3.1. ([14] Lemma 2, page 1190) *For every mapping $f : X \to Y$ and $m, k \in \mathbb{N}$, there correspond positive integers $\eta_0, \ldots, \eta_{m(k-1)}$ such that*

$$f(kx) - k^m f(x) = \frac{1}{m!} \Big(\eta_0 F_0(x) + \cdots + \eta_{m(k-1)} F_{m(k-1)}(x) - G(x) \Big) \quad (x \in X), \tag{3.1}$$

where $F_i : X \to Y$ and $G : X \to Y$ are defined by

$$F_i(x) = \Delta_x^m f(ix) - m!f(x) \quad (x \in X; i = 0, 1, \ldots, m(k-1))$$

and

$$G(x) = \Delta_{kx}^m f(0) - m!f(kx) \quad (x \in X).$$

Corollary 3.2. *If $f : X \to Y$ is a monomial of degree m, then*

$$f(rx) = r^m f(x) \quad (x \in X) \tag{3.2}$$

for every non negative rational number r.

Proof. If f is a monomial of degree m, by the definition,

$$G \equiv F_i \equiv 0 \qquad (i =, 1, \ldots, m(k-1))$$

for each $k \in \mathbb{N}$. Hence (3.2) holds for every positive integer r. Let $k \in \mathbb{N}$, then for each $x \in X$

$$f(x) = f(k\Big(\frac{1}{k}\Big)x) = k^m f\Big(\frac{x}{k}\Big),$$

i.e.

$$f\Big(\frac{x}{k}\Big) = \frac{1}{k^m} f(x) \quad (x \in X).$$

Therefore for each $i, j \in \mathbb{N}$, we have

$$f(\frac{i}{j}\ x) = f(i\ \frac{1}{j}x) = i^m f(\frac{1}{j}x) = \Big(\frac{i}{j}\Big)^m f(x) \quad (x \in X).$$

Note that

$$f(0) = 0!f(0) = \Delta_0^m f(0) = 0.$$

Throughout the rest of this paper, unless otherwise stated, we will assume that X is a linear space over $\mathbb{Q}$ and Y is a non-Archimedean normed space over a non-Archimedean field $\mathbb{K}$.

In the following theorem, we use the notations of Lemma 3.1.

Theorem 3.3. *Let $\varphi : X \times X \to [0, \infty)$ and $f : X \to Y$ satisfy the inequality*

$$||\Delta_x^m f(y) - m! f(x)|| \leq \varphi(x, y) \quad (x, y \in X) \tag{3.3}$$

for some $m \in \mathbb{N}$. For each integer $k \geq 2$, define

$$\psi_k(x) = \frac{1}{|k|^m m!} \max\Big\{|\eta_0|\ \varphi(x, 0), \ldots, |\eta_{m(k-1)}|\varphi(x, m(k-1)x), \varphi(kx, 0)\Big\}(x \in X). \tag{3.4}$$

If for some $k \geq 2$, there exists some $L < 1$ such that

$$|k|^{-m}\psi_k(kx) \leq L\psi_k(x) \qquad (x \in X) \tag{3.5}$$

and $\lim_{n\to\infty} k^{-nm}\varphi(2^n x, 2^n y) = 0$ *for all x, y in X, then there exists a unique monomial mapping $M : X \to Y$ such that*

$$||M(x) - f(x)|| \leq \psi_k(x) \quad (x \in X). \tag{3.6}$$

Proof. By Lemma 3.1 and the strong triangle inequality

$$||k^{-m} f(kx) - f(x)|| \leq \psi_k(x) \quad (x \in X). \tag{3.7}$$

Let $\mathcal{E} = Y^X$ and define

$$d(g, h) = \inf\{a > 0 : ||g(x) - h(x)|| \leq a\psi_k(x), \forall x \in X\} \quad (g, h \in \mathcal{E}).$$

If $d(f, g) = 0$, then $||g(x) - h(x)|| \leq \alpha\psi_k(x)$ for each $x \in X$ and $\alpha > 0$. Hence $f(x) = g(x)$ for each $x \in X$. Let $d(g_1, g_2) < \alpha_1$ and $d(g_2, g_3) < \alpha_2$. Then for each $x \in X$,

$$||g_1(x) - g_2(x)|| \leq \alpha_1\psi_k(x) \text{ and } ||g_2(x) - g_3(x)|| \leq \alpha_2\psi_k(x).$$

By the strong triangle inequality,

$$||g_1(x) - g_3(x)|| \leq \max\{\alpha_1, \alpha_2\}\psi_k(x) \quad (x \in X).$$

Hence $d(g_1, g_3) \leq \max\{d(g_1, g_2), d(g_2, g_3)\}$. If $\{g_n\}$ a Cauchy sequence in $(\mathcal{E}, d)$, then an easy computation shows that for each $x \in X$, $\{g_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete this sequence converges for each $x \in X$. This proves the completeness of $(\mathcal{E}, d)$. Define $J : \mathcal{E} \to \mathcal{E}$ by $J(g)(x) = k^{-m} g(kx)$ for each $g \in \mathcal{E}$ and $x \in X$. Let $d(g, h) < a$, by the definition,

$$||g(x) - h(x)|| \leq a\psi_k(x) \quad (x \in X).$$

Thanks to (3.5), for each $x \in X$,

$$\begin{aligned} ||J(g)(x) - J(h)(x)|| &= ||k^{-m}g(kx) - k^{-m}h(kx)|| \\ &\leq k^{-m}a\psi_k(kx) \\ &\leq La\psi_k(x). \end{aligned}$$

Hence, by the definition, $d(J(g), J(h)) \leq L\ a$. Therefore

$$d(J(g), J(h)) \leq L \quad d(g, h) \quad (g, h \in \mathcal{E}).$$

This means that J is a contractive mapping with a Lipschitz constant $L < 1$. By the inequality (3.7), $d(f, J(f)) \leq 1$, therefore, by Theorem 2.7, J has a unique fixed point $M : X \to Y$ in the set $\mathcal{F} = \{g \in \mathcal{E} : d(f, g) < \infty\}$, where M is defined by

$$M(x) := \lim_{n\to\infty} J^n(f)(x) = \lim_{n\to\infty} k^{-mn}f(k^n x) \qquad (x \in X). \tag{3.8}$$

Moreover,

$$d(f, M) \leq d(f, J(f)).$$

This means that (3.6) holds. Thanks to (1.1) and (3.8), for each $x, y \in X$, we have

$$\begin{aligned} & ||\Delta_x^m M(y) - m!M(x)|| \\ = \ & ||\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}M(jx + y) - m!M(x)|| \\ = \ & \lim_{n\to\infty} k^{-mn}||\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(k^n jx + k^n y) - m!f(k^n x)|| \\ = \ & \lim_{n\to\infty} k^{-mn}||\Delta_{k^n x}f(k^n y) - m!f(k^n x)|| \\ \leq \ & \lim_{n\to\infty} k^{-mn}\varphi(k^n x, k^n y) = 0. \end{aligned}$$

Hence M is a monomial function.

To prove the uniqueness assertion, let us assume that there exists a monomial function $M' : X \to Y$ which satisfies (3.6). Then by Corollary 2.7, M' is a fixed point of J in $\mathcal{F}$. However, J has a unique fixed point in $\mathcal{E}$, hence $M' \equiv M$.

Theorem 3.4. *With the notation of Theorem 3.3, let $f : X \to Y$ satisfy* (3.3). *If for some $k \geq 2$, there exists some $L < 1$ such that*

$$|k|^m \psi_k(k^{-1}x) \leq L\psi_k(x) \qquad (x \in X) \tag{3.9}$$

and $\lim_{n\to\infty} |k|^{nm}\varphi(k^{-n}x, k^{-n}y) = 0$ for all x, y in X, then there exists a unique monomial mapping $M : X \to Y$ of degree m such that

$$||M(x) - f(x)|| \leq L\psi_k(x) \quad (x \in X). \tag{3.10}$$

In the following result, we investigate continuity of monomial mappings in non-Archimedean spaces. In fact, we will show that under some conditions on f, ψ_k (or ψ), the monomial mapping $s \longmapsto M(sx)$ is continuous.

Theorem 3.5. *Let the conditions of Theorem 3.3 or Theorem 3.4 hold. If for each $x \in X$, the function $s \mapsto f(sx)$ is continuous on $\mathbb{K}$ and $s \mapsto \psi_k(sx)$ is bounded in a neighborhood of s_0 for some point $x \in X$, then $s \longmapsto M(sx)$ is continuous at s_0.*

Proof. We prove our theorem when the hypotheses of Theorem 3.3 hold. A similar argument proves the other case. Fix $x \in X$ and choose positive numbers α and δ' such that

$$|s - s_0| < \delta_1 \Rightarrow \psi_k(sx) < \alpha.$$

Let $\varepsilon > 0$. Since $0 < L < 1$, we may take some n_0 so that

$$L^{n_0}\alpha < \varepsilon.$$

By the continuity of $s \mapsto f(k^{n_0}sx)$, we can find some $0 < \delta < \delta'$ such that

$$|s - s_0| < \delta \Longrightarrow ||f(k^{n_0}sx) - f(k^{n_0}s_0x)|| < \frac{\varepsilon}{|k|^{-n_0m}}.$$

Then $|s - s_0| < \delta$ implies that

$$\begin{aligned}
& ||M(sx) - M(s_0x)|| \\
= \ & |k|^{-n_0m}||M(k^{n_0}sx) - M(k^{n_0}s_0x)|| \\
\leq \ & |k|^{-n_0m} \max\Big\{||M(k^{n_0}sx) - f(k^{n_0}sx)||, ||f(k^{n_0}sx) - f(k^{n_0}s_0x)||, \\
& \quad ||f(k^{n_0}s_0x) - M(k^{n_0}s_0x)||\Big\} \\
\leq \ & |k|^{-n_0m} \max\{\psi_k(k^{n_0}sx), \frac{\varepsilon}{|k|^{-n_0m}}, \psi_k(k^{n_0}s_0x)\} \\
\leq \ & \max\{L^{n_0}\psi_k(sx), \varepsilon, L^{n_0}\psi_k(s_0x)\} \\
< \ & \varepsilon.
\end{aligned}$$

This proves the continuity of $s \mapsto M(sx)$ at s_0.

Remark 3.6. In [14] (Lemma 5), it is shown that if $\mathbb{Q}$ is dense in $\mathbb{K}$ with respect to the valuation $|\cdot|$, then every continuous monomial function $g : \mathbb{K} \to Y$ of degree m is of the form $g(t) = t^m g(1)$. Hence, if $\mathbb{Q}$ is dense in $\mathbb{K}$, then the function $t \to M(tx)$ in Theorem 3.5 is of the from $M(tx) = t^m M(x)$ for each $t \in \mathbb{K}$.

The following result is due to Y.-H. Lee in [16]:

Theorem 3.7. *Let X be a normed space and Y be a Banach space. If $f : X \to Y$ satisfies the inequality*

$$||\Delta_x^m f(y) - m!f(x)||_Y \leq \varepsilon\big(\, ||x||_X^r + ||y||_X^r\big) \quad (x, y \in X)$$

for some $0 \leq r < m$. Then

$$M(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{nm}} \quad (x \in X)$$

defines a unique monomial function of degree m such that

$$||f(x) - M(x)||_Y \leq \eta||x||_X^r \quad (x \in X)$$

where $\eta = \eta_{m,r,\varepsilon}$ is a constat which depends on m, r and ε.

The following example shows that this result is not valid in the setting of non-Archimedean normed spaces.

Example 3.8. *Let $p > 2$ be a prime number. Define $f : \mathbb{Q}_p \to \mathbb{Q}_p$ by $f(x) = x^2 + x$. Then*

$$|\Delta_x^2 f(y) - 2!f(y)| = |2x| = |x| \leq |x| + |y| \quad (x, y \in \mathbb{Q}_p).$$

Hence the conditions of Theorem 3.7 for $m = 2$ and $r = 1$ hold. However for each $n \in \mathbb{N}$ we have

$$\Big|\frac{f(2^n x)}{2^{2n}} - \frac{f(2^{n+1}x)}{2^{2(n+1)}}\Big| = \frac{|x|}{|2^{n+1}|} = |x| \quad (x \in \mathbb{Q}_p).$$

*Hence $\{\frac{f(2^n x)}{2^{2n}}\}$ is not convergent for each nonzero $x \in \mathbb{Q}_p$.*In the next result, which can be compared with Theorem 3.7, we will use the notation of Theorem 3.3.

Corollary 3.9. *Let X and Y be non-Archimedean normed space over a non-Archimedean field $\mathbb{K}$. If Y is complete and $\mathbb{K}$ contains an element k with $|k| < 1$. Then for each function $f : X \to Y$ which satisfies the inequality*

$$||\Delta_x^m f(y) - m!f(x)||_Y \leq \varepsilon\Big(||x||_X^r + ||y||_X^r\Big) \quad (x, y \in X)$$

for some $\varepsilon > 0$ and $m \neq r$. There exists a unique monomial mapping $M : X \to Y$ and a constant

$$\alpha_{k,m} = \frac{\varepsilon}{|k|^m m!} \max\{|\eta_0|, \ldots, |\eta_{m(k-1)}|(1 + |m(k-1)|), |k|^r\}$$

such that for $r > m$,

$$||f(x) - M(x)||_Y \leq \alpha_{k,m}||x||_X^r \quad (x \in X), \tag{3.11}$$

and if $r < m$, then

$$||f(x) - M(x)||_Y \leq |k|^{m-r}\alpha_{k,m}||x||_X^r \quad (x \in X). \tag{3.12}$$

Moreover if $t \mapsto f(tx)$ from $\mathbb{K}$ to Y is continuous for each $x \in X$ and $\mathbb{Q}$ is dense in $\mathbb{K}$ with respect to the valuation $|\cdot|$, then $M(tx) = t^m M(x)$ for each $t \in \mathbb{K}$ and $x \in X$.

Proof. Let

$$\varphi(x,y)=\varepsilon\Big(||x||_X^r+||y||_X^r\Big)\quad (x,y\in X).$$

Then $\psi_k(x)=\alpha_{k,m}||x||_X^r$ for each $x\in X$. Hence if $r>m$, then

$$|k|^{-m}\psi_k(kx)=|k|^{-m+r}\psi_k(x)\quad (x\in X).$$

Therefore the conditions of Theorem 3.3 for $L=|k|^{-m+r}<1$ hold. Hence, we can find a unique monomial $M:X\to Y$ of degree m such that (3.11) holds.

If $r<m$, then

$$|k|^m\psi_k(k^{-1}x)=|k|^{m-r}\psi_k(x)\quad (x\in X).$$

By Theorem 3.4, there is a unique monomial $M:X\to Y$ of degree m, which satisfies the inequality (3.12).

The last assertion of the Corollary follows from Remark 3.6.

Acknowledgement

The author would like to thank the referee for his/her helpful remarks and corrections.

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