Tamsui Oxford Journal of Mathematical Sciences **26(2)** (2010) 203-219
Aletheia University

On Quasi-Conformally Flat Almost Pseudo Ricci Symmetric Manifolds *

A. A. Shaikh[†], S. K. Hui

Department of Mathematics, University of Burdwan, Burdwan – 713104, West Bengal, India.

and

C. S. Bagewadi

Department of Mathematics, Kuvempu University, Jnana Sahyadri, Shankaraghatta – 577451, Karnataka, India.

Received November 11, 2008, Accepted August 10, 2009.

Abstract

The object of the present paper is to study quasi-conformally flat almost pseudo Ricci symmetric manifolds.

Keywords and Phrases: *Pseudo Ricci symmetric manifold, Almost pseudo Ricci symmetric manifold, Scalar curvature, Conformally flat, Quasi-conformally flat, Special conformally flat, Concircular vector field, Warped product.*

*2000 *Mathematics Subject Classification.* 53B30, 53B50, 53C15, 53C25.

[†]Corresponding author. E-mail: aask2003@yahoo.co.in

1. Introduction

As an extended class of pseudo Ricci symmetric manifolds, very recently M. C. Chaki and T. Kawaguchi [4] introduced the notion of almost pseudo Ricci symmetric manifolds. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X)+B(X)]S(Y,Z)+A(Y)S(X,Z)+A(Z)S(Y,X), \quad (1.1)$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are nowhere vanishing 1-forms such that $g(X,\rho) = A(X)$ and $g(X,\mu) = B(X)$ for all X and ρ, μ are called the basic vector fields of the manifold. The 1-forms A and B are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$.

If, in particular, $B = A$ then (1.1) reduces to

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X),$$

which represents a pseudo Ricci symmetric manifold [3]. In [4] Chaki and Kawaguchi also studied conformally flat $A(PRS)_n$. In 1968 Yano and Sawaki [10] defined and studied a tensor field W of type (1, 3) which includes both the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ as special cases and is called the quasi-conformal curvature tensor. The present paper deals with a study of quasi-conformally flat $A(PRS)_n$.

The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of quasi-conformally flat $A(PRS)_n$ (since the conformal curvature tensor vanishes identically for $n = 3$, we assume the condition $n > 3$ throughout the paper) and proved that such a manifold is of quasi-constant curvature. It is shown that in a quasi-conformally flat $A(PRS)_n$, the vector field λ defined by $g(X,\lambda) = T(X)$ is a unit proper concircular vector field and also it is proved that such a non-Einstein manifold is a subprojective manifold in the sense of Kagan [1]. Again it is proved that a non-Einstein quasi-conformally flat $A(PRS)_n$ can be expressed as the warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n-1)$ dimensional Riemannian manifold.

The notion of special conformally flat manifold which generalizes the notion of subprojective manifold was introduced by Chen and Yano [6]. In [6] the authors also introduced the notion of K-special conformally flat manifold which generalizes the notion of special conformally flat manifold as well as subprojective manifold. In this section it is shown that a non-Einstein quasi-conformally flat $A(PRS)_n$ with non-constant and negative scalar curvature is a K-special conformally flat manifold and also it is proved that such a simply connected manifold with non-constant and negative scalar curvature can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.

2. Preliminaries

In this section we will obtain some formulas for an $A(PRS)_n$ which will be required in the sequel.
Let Q be the symmetric endomorphism of the tangent bundle of the manifold corresponding to the Ricci tensor S, i.e., $S(X,Y) = g(QX,Y)$ for all vector fields X, Y.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then setting $Y = Z = e_i$ in (1.1) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$dr(X) = r[A(X) + B(X)] + 2A(QX), \tag{2.1}$$

where r is the scalar curvature of the manifold.
Again from (1.1) we get

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = B(X)S(Y,Z) - B(Y)S(X,Z). \tag{2.2}$$

Setting $Y = Z = e_i$ in (2.2) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$dr(X) = 2rB(X) - 2B(QX). \tag{2.3}$$

If the scalar curvature r is constant then

$$dr(X) = 0 \quad \text{for all } X. \tag{2.4}$$

By virtue of (2.4), (2.3) yields

$$B(QX) = rB(X), \tag{2.5}$$

i.e.,

$$S(X,\mu) = rg(X,\mu). \tag{2.6}$$

This leads to the following:

Proposition 2.1. *In an $A(PRS)_n$ of constant scalar curvature, r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector μ.*

The quasi-conformal curvature tensor W of type (1, 3) is defined by [10]

$$W(X,Y)Z = -(n-2)bC(X,Y)Z + [a+(n-2)b]\tilde{C}(X,Y)Z, \tag{2.7}$$

where a, b are arbitrary constants not simultaneously zero and C, $\tilde{C}$ are respectively the conformal and concircular curvature tensor. Using the expressions of C and $\tilde{C}$ in (2.7) we get

$$\begin{aligned} & W(X,Y)Z \\ = \; & aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & -g(X,Z)QY] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.8}$$

Differentiating (2.8) covariantly and contracting we obtain

$$\begin{aligned} & (divW)(X,Y)Z \\ = \; & a(divR)(X,Y)Z + b[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ & -\frac{2a-(n-1)(n-4)b}{2n(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)], \end{aligned} \tag{2.9}$$

where ‘div’ denotes the divergence.
Again it is known that in a Riemannian manifold, we have

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z).$$

Consequently by virtue of the above relation, (2.9) takes the form

$$\begin{aligned} & (divW)(X,Y)Z \\ = \; & (a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ & -\frac{2a-(n-1)(n-4)b}{2n(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)]. \end{aligned} \tag{2.10}$$

3. quasi-conformally flat $A(PRS)_n$

Let us consider a quasi-conformally flat $A(PRS)_n$. Then we have

$$(divW)(X,Y)Z = 0$$

and hence (2.10) yields

$$\begin{aligned}&(a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ = \;& \frac{2a-(n-1)(n-4)b}{2n(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)].\end{aligned} \tag{3.1}$$

By virtue of (2.2) and (2.3), it follows from (3.1) that

$$\begin{aligned}B(X)S(Y,Z) - B(Y)S(X,Z) = \frac{2a-(n-1)(n-4)b}{n(n-1)(a+b)}[r\{B(X)g(Y,Z) \\ -B(Y)g(X,Z)\} - \{B(QX)g(Y,Z) - B(QY)g(X,Z)\}],\end{aligned} \tag{3.2}$$

provided that $a+b \neq 0$.
Putting $Z = \mu$ in (3.2) we obtain

$$B(X)B(QY) - B(Y)B(QX) = 0, \tag{3.3}$$

provided $a+b \neq 0$ and $(n+1)a + 2(n-1)b \neq 0$.
Let $B(QX) = g(QX,\mu) = P(X) = g(X,\xi)$ for all X. Then from (3.3) we get

$$B(X)P(Y) = B(Y)P(X), \tag{3.4}$$

which implies that the vector field μ and ξ are co-directional. This leads to the following:

Theorem 3.1. *In a quasi-conformally flat $A(PRS)_n$ with $a+b \neq 0$ and $(n+1)a+2(n-1)b \neq 0$, the vector field μ and ξ are co-directional.* If $a+b=0$ and $(n+1)a+2(n-1)b \neq 0$, then using (2.3) in (3.1), it can be easily shown that the relation (3.4) holds. Hence we can state the following:

Corollary 3.1. *In a quasi-conformally flat $A(PRS)_n$ with $a+b=0$ and $(n+1)a+2(n-1)b \neq 0$, the vector field μ and ξ are co-directional.*

Again, if $(n+1)a+2(n-1)b=0$ and $a+b\neq 0$, then using (2.2) in (3.1), it follows that (3.4) holds. Hence we can state the following:

Corollary 3.2. *In a quasi-conformally flat* $A(PRS)_n$ *with* $a+b\neq 0$ *and* $(n+1)a+2(n-1)b=0$*, the vector field* μ *and* ξ *are co-directional.*

It may be noted that in a quasi-conformally flat $A(PRS)_n$, the relations $a+b=0$ and $(n+1)a+2(n-1)b=0$ can not hold simultaneously as a and b are not simultaneously zero.

Again setting $Y=Z=e_i$ in (3.2) and then taking summation over i, $1\leq i\leq n$, we obtain

$$B(QX)=rB(X), \quad \text{provided that } a+(n-2)b\neq 0, \tag{3.5}$$

i.e.,

$$S(X,\mu)=rg(X,\mu). \tag{3.6}$$

Hence we can state the following:

Theorem 3.2. *In a quasi-conformally flat* $A(PRS)_n$ *with* $a+b\neq 0$ *and* $a+(n-2)b\neq 0$*,* r *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* μ*.* If $a+b=0$, then it follows from (3.2) that (3.6) holds provided that $2a-(n-1)(n-4)b\neq 0$. Hence we can state the following:

Corollary 3.3. *In a quasi-conformally flat* $A(PRS)_n$ *with* $a+b=0$ *and* $2a-(n-1)(n-4)b\neq 0$*,* r *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* μ*.*

Also for $2a-(n-1)(n-4)b=0$ and $a+b\neq 0$, we can state the following:

Corollary 3.4. *In a quasi-conformally flat* $A(PRS)_n$ *with* $a+b\neq 0$ *and* $2a-(n-1)(n-4)b=0$*,* r *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* μ*.*

In view of (3.5), (3.2) yields

$$B(X)S(Y,Z)-B(Y)S(X,Z)=0. \tag{3.7}$$

Setting $X = \mu$ in (3.7) we get

$$S(Y,Z) = \frac{1}{B(\mu)} B(Y)B(QZ). \tag{3.8}$$

In view of (3.5), (3.8) yields

$$S(Y,Z) = rT(Y)T(Z), \tag{3.9}$$

where $T(X) = g(X,\lambda) = \frac{1}{\sqrt{B(\mu)}} B(X)$, λ being a unit vector field associated with the nowhere vanishing 1-form T.

From (3.9), it follows that if $r = 0$, then $S(Y,Z) = 0$, which is inadmissible by the definition of $A(PRS)_n$. Hence we can state the following:

Theorem 3.3. *In a quasi-conformally flat $A(PRS)_n$ with $a + b \neq 0$ and $a+(n-2)b \neq 0$, the scalar curvature can not vanish and the Ricci tensor is of the form (3.9).* As a generalization of the manifold of constant curvature, the notion of the manifold of quasi-constant curvature arose during the study of conformally flat hypersurfaces by Chen and Yano [5]. A Riemannian manifold $(M^n, g)(n > 3)$ is said to be the manifold of quasi-constant curvature [5] if it is conformally flat and its curvature tensor R of type (0, 4) is of the form:

$$\begin{aligned} R(X,Y,Z,U) \;=\;& a_1[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\ &+a_2[g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\ &+g(X,U)A(Y)A(Z) - g(Y,U)A(X)A(Z)], \end{aligned} \tag{3.10}$$

where A is a nowhere vanishing 1-form and a_1, a_2 are scalars of which $a_2 \neq 0$. Now from (2.8) it follows that in a quasi-conformally flat $A(PRS)_n$, the curvature tensor R of type (0, 4) is of the following form:

$$\begin{aligned} R(X,Y,Z,U) \;=\;& -\frac{b}{a}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+S(X,U)g(Y,Z) - S(Y,U)g(X,Z)] \\ &+\frac{r}{na}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \end{aligned} \tag{3.11}$$

for $a \neq 0$.

Using (3.9) in (3.11) we obtain

$$\begin{aligned} R(X,Y,Z,U) &= \tilde{a}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\ &\quad +\tilde{b}[g(X,U)T(Y)T(Z) - g(Y,U)T(X)T(Z) \\ &\quad +g(Y,Z)T(X)T(U) - g(X,Z)T(Y)T(U)] \end{aligned} \tag{3.12}$$

for $a \neq 0$, where $\tilde{a} = \frac{r}{na}(\frac{a}{n-1} + 2b)$ and $\tilde{b} = -\frac{br}{a}$ are non-zero scalars.
By virtue of (3.10), it follows from (3.12) that a quasi-conformally flat $A(PRS)_n$ is a manifold of quasi-constant curvature. This leads to the following:

Theorem 3.4. *Every quasi-conformally flat $A(PRS)_n$ with $a \neq 0$, $a+b \neq 0$ and $a+(n-2)b \neq 0$ is a manifold of quasi-constant curvature.* Again from (3.9) we have

$$(\nabla_X S)(Y,Z) = dr(X)T(Y)T(Z) + r[(\nabla_X T)(Y)T(Z) + T(Y)(\nabla_X T)(Z)]. \tag{3.13}$$

Using (3.13) in (3.1) we obtain

$$\begin{aligned} &(a+b)[dr(X)T(Y)T(Z) - dr(Y)T(X)T(Z)] \\ &+r[(\nabla_X T)(Y)T(Z) \\ &+T(Y)(\nabla_X T)(Z) - (\nabla_Y T)(X)T(Z) - T(X)(\nabla_Y T)(Z)] \\ = &\frac{2a-(n-1)(n-4)b}{2n(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)]. \end{aligned} \tag{3.14}$$

Setting $Y = Z = e_i$ in (3.14) and taking summation over $i, 1 \leq i \leq n$, we get

$$\begin{aligned} &(a+b)dr(\lambda)T(X) + r\{(\nabla_\lambda T)(X) + T(X)\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i)\} \\ = &\frac{2(n-1)a + (n^2-3n+4)b}{2n}dr(X). \end{aligned} \tag{3.15}$$

Again putting $Y = Z = \lambda$ in (3.14) we obtain

$$r(a+b)(\nabla_\lambda T)(X) = \frac{2(n^2-n-1)a + (n-1)(3n-4)b}{2n(n-1)}[dr(X) - dr(\lambda)T(X)]. \tag{3.16}$$

Using (3.16) in (3.15) we get

$$r(a+b)T(X)\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i) + (n-2)Edr(X) + Edr(\lambda)T(X) = 0, \tag{3.17}$$

where $E = \frac{2a-(n-1)(n-4)b}{2n(n-1)}$.
Substituting $X = \lambda$ in (3.17) we get

$$r(a+b)\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i) = -(n-1)Edr(\lambda). \tag{3.18}$$

From (3.17) and (3.18) it follows that

$$dr(X) = dr(\lambda)T(X), \text{ provided } 2a-(n-1)(n-4)b \neq 0. \tag{3.19}$$

Again plugging $Z = \lambda$ in (3.14) and then using (3.19) we obtain

$$r(a+b)\{(\nabla_X T)(Y) - (\nabla_Y T)(X)\} = 0,$$

which implies that

$$(\nabla_X T)(Y) - (\nabla_Y T)(X) = 0, \quad \text{since } r \neq 0 \text{ and } \quad a+b \neq 0. \tag{3.20}$$

The relation (3.20) implies that the 1-form T is closed.
In view of (3.19) it follows from (3.16) that

$$(\nabla_\lambda T)(X) = 0, \quad \text{provided } a+b \neq 0. \tag{3.21}$$

which implies that $\nabla_\lambda \lambda = 0$ and hence we can state the following:

Theorem 3.5. *In a quasi-conformally flat $A(PRS)_n$ with $a+b \neq 0$, $a+(n-2)b \neq 0$ and $2a-(n-1)(n-4)b \neq 0$, the integral curves of the generator λ are geodesics.* Also setting $Y = \lambda$ in (3.14) we obtain by virtue of (3.19) and (3.21) that

$$(\nabla_X T)(Z) = \frac{E}{r(a+b)}dr(\lambda)[T(X)T(Z) - g(X,Z)] \quad \text{for } a+b \neq 0. \tag{3.22}$$

Let us now consider a non-zero scalar function $f = \frac{E}{r(a+b)}dr(\lambda)$, where the scalar curvature r is non-constant. Then we have

$$\nabla_X f = -\frac{E}{r^2(a+b)}dr(\lambda)dr(X) + \frac{E}{r(a+b)}d^2r(\lambda, X). \tag{3.23}$$

From (3.19) it follows that

$$d^2r(X,Y) = d^2r(\lambda,Y)T(X) + dr(\lambda)(\nabla_Y T)(X). \tag{3.24}$$

Again in a Riemannian manifold the second covariant differential of any function $h \in C^\infty(M)$ is defined by

$$d^2h(X,Y) = X(Yh) - (\nabla_X Y)h \quad \text{for } X, Y \in \chi(M),$$

which implies that

$$d^2h(X,Y) = d^2h(Y,X) \quad \text{for all } X, Y \in \chi(M)$$

and hence (3.24) implies that

$$d^2r(\lambda, Y)T(X) = d^2r(\lambda, X)T(Y). \tag{3.25}$$

Replacing Y by λ in (3.25) we have

$$d^2r(\lambda, X) = d^2r(\lambda, \lambda)T(X) = -\psi T(X), \tag{3.26}$$

where $\psi = -d^2r(\lambda, \lambda)$ is a scalar function.
Using (3.19) and (3.26) in (3.23) we obtain

$$\nabla_X f = \sigma T(X), \tag{3.27}$$

where

$$\sigma = -\frac{E}{r^2(a+b)}[r\psi + \{dr(\lambda)\}^2] \quad \text{is a non-zero scalar.}$$

We now consider an 1-form ω given by

$$\omega(X) = \frac{E}{r(a+b)} dr(\lambda)T(X) = fT(X).$$

Then by virtue of (3.20) and (3.27) we have

$$d\omega(X,Y) = 0.$$

Hence the 1-form ω is closed. Therefore (3.22) can be rewritten as

$$(\nabla_X T)(Z) = -fg(X,Z) + \omega(X)T(Z), \tag{3.28}$$

which implies that the vector field λ corresponding to the 1-form T defined by $g(X, \lambda) = T(X)$ is a proper concircular vector field ([8],[9]). Hence we can state the following:

Theorem 3.6. *In a quasi-conformally flat $A(PRS)_n$ of non-constant scalar curvature with $a+b \neq 0$, $a+(n-2)b \neq 0$ and $2a-(n-1)(n-4)b \neq 0$, the vector field λ defined* by $g(X,\lambda) = T(X)$ is a unit proper concircular vector field. If, in particular, $a+b=0$ then from (3.1) we get

$$dr(X)g(Y,Z) - dr(Y)g(X,Z) = 0,$$

which yields

$$dr(X) = 0 \quad \text{for all } X, \tag{3.29}$$

provided $2a-(n-1)(n-4)b \neq 0$. This means that the scalar curvature of the quasi-conformally flat $A(PRS)_n$ is constant.
Putting $Y = Z = e_i$ in (3.13) and taking summation over i, $1 \leq i \leq n$, we obtain by virtue of (3.29) that

$$(\nabla_\lambda T)(X) + T(X)\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i) = 0, \tag{3.30}$$

which implies for $X = \rho$ that

$$\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i) = 0.$$

Using this relation in (3.30) we get

$$(\nabla_\lambda T)(X) = 0. \tag{3.31}$$

Setting $Y = \lambda$ in (3.13) and then using (3.30) and (3.31) we obtain

$$(\nabla_\lambda S)(X,Z) = 0 \quad \text{for all } X, Z,$$

which implies that the manifold under consideration is Ricci symmetric along the direction of the generator λ. This leads to the following:

Corollary 3.5. *A quasi-conformally flat $A(PRS)_n$ with $2a-(n-1)(n-4)b \neq 0$ and $a+b=0$ is of constant scalar curvature and Ricci symmetric along the direction of the generator λ.*

Again if $2a-(n-1)(n-4)b=0$ and $a+b\neq 0$, then (3.13) implies that

$$(\nabla_X S)(Y,Z)=(\nabla_Y S)(X,Z) \quad \text{for all } X,Y,Z, \tag{3.32}$$

which means that the Ricci tensor S is of Codazzi type [7] and hence

$$dr(X)=0 \quad \text{for all } X. \tag{3.33}$$

Hence (3.14) takes the form

$$(\nabla_X T)(Y)T(Z)+T(Y)(\nabla_X T)(Z)-(\nabla_Y T)(X)T(Z)-T(X)(\nabla_Y T)(Z)=0. \tag{3.34}$$

Putting $Z=\lambda$ in (3.34) we get the relation (3.20).
Also for $Y=\lambda$, (3.20) implies that

$$(\nabla_\lambda T)(X)=0 \quad \text{for all } X. \tag{3.35}$$

Using this relation we obtain from (3.34) (for $Y=\lambda$) that

$$(\nabla_X T)(Z)=0 \quad \text{for all } X,Z.$$

This implies that

$$g(Z,\nabla_X\lambda)=0 \quad \text{for all } X,Z.$$

Since g is non-degenerate, the last relation yields

$$(\nabla_X\lambda)=0 \quad \text{for all } X,$$

which means that λ is a parallel vector field. Thus we can state the following:

Corollary 3.6. *In a quasi-conformally flat $A(PRS)_n$ with $2a-(n-1)(n-4)b=0$ and $a+b\neq 0$, the Ricci tensor is of Codazzi type and the vector field λ defined by $g(X,\lambda)=T(X)$ is a unit parallel vector field.*

Remark: It may be noted that in a quasi-conformally flat $A(PRS)_n$, the cases $2a-(n-1)(n-4)b=0$ and $a+b=0$ can not occur simultaneously. For, if they occur simultaneously then we have

$$a=-b \quad \text{and } a=\frac{(n-1)(n-4)}{2}b.$$

This together implies that (since $n > 3$) $a = 0 = b$, which is inadmissible, by the definition of quasi-conformal curvature tensor.

In [2] Amur and Maralabhavi proved that a quasi-conformally flat Riemannian manifold is either conformally flat or Einstein. We now consider a quasi-conformally flat $A(PRS)_n$, which is non-Einstein. Then by [2] such an $A(PRS)_n$ is conformally flat. Again it is known that [1] if a conformally flat Riemannian manifold $(M^n, g)(n > 3)$ admits a proper concircular vector field then the manifold is a subprojective manifold in the sense of Kagan. Hence by virtue of Theorem 3.6, we can state the following:

Theorem 3.7. *A non-Einstein quasi-conformally flat $A(PRS)_n$ of non-constant scalar curvature with $2a - (n-1)(n-4)b \neq 0$, $a + (n-2)b \neq 0$ and $a + b \neq 0$ is a subprojective manifold in the sense of Kagan.*

In[9] K. Yano proved that in order that a Riemannian manifold admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form can be written as

$$ds^2 = dx'^2 + e^p \overset{*}{g}_{ij}\, dx^i dx^j,$$

where $\overset{*}{g}_{ij} = g_{ij}[x^k]$ are the functions of x^k only $(i, j, k = 2, 3, \cdots, n)$ and $p = p(x')$ is a non-constant function of x' only. Hence if a $A(PRS)_n$ is conformally flat, then it is a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n-1)$ dimensional Riemannian manifold. Hence we can state the following:

Theorem 3.8. *A non-Einstein quasi-conformally flat $A(PRS)_n$ of non-constant scalar curvature with $a + b \neq 0$, $a + (n-2)b \neq 0$ and $2a - (n-1)(n-4)b \neq 0$ can be expressed as a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n-1)$ dimensional Riemannian manifold.*

Let us consider a quasi-conformally flat $A(PRS)_n$. In [6] Chen and Yano introduced the notion of special conformally flat manifold which generalizes the notion of subprojective manifold. According to them a conformally flat Riemannian manifold is said to be a special conformally flat manifold if the tensor field H of type (0, 2) defined by

$$H(X,Y) = -\frac{1}{n-2}S(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y) \tag{3.36}$$

is expressible in the form

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha), \tag{3.37}$$

where α, β are two scalars such that α is positive.
In view of (3.9), (3.36) can be written as

$$H(X,Y) = -\frac{r}{n-2}T(X)T(Y) + \frac{r}{2(n-1)(n-2)}g(X,Y). \tag{3.38}$$

We now put

$$\alpha^2 = -\frac{r}{(n-1)(n-2)} > 0, \quad \text{provided } r < 0. \tag{3.39}$$

Then

$$2\alpha(X\alpha) = -\frac{dr(X)}{(n-1)(n-2)},$$

which implies by virtue of (3.19) that

$$2\alpha(X\alpha) = -\frac{dr(\lambda)T(X)}{(n-1)(n-2)}.$$

Hence

$$T(X)T(Y) = -\frac{4(n-1)(n-2)r}{\delta^2}(X\alpha)(Y\alpha),$$

where $\delta = dr(\lambda)$. Therefore, by virtue of (3.39), (3.38) can be expressed as

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha),$$

where $\beta = \frac{4(n-1)r^2}{\delta^2}$. Hence the manifold under consideration is a special conformally flat manifold. Since a non-Einstein quasi-conformally flat manifold is conformally flat [2], we can state the following:

Theorem 3.9. *A non-Einstein quasi-conformally flat $A(PRS)_n$ with non-constant negative scalar curvature, $a+b \neq 0$, $a+(n-2)b \neq 0$ and $2a-(n-1)(n-4)b \neq 0$ is a special conformally flat manifold.*

Also in [6] Chen and Yano proved that every simply-connected special conformally flat manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface. Therefore by virtue of Theorem 3.9, we can state the following:

Theorem 3.10. *Every simply-connected non-Einstein quasi-conformally flat $A(PRS)_n$ of non-constant negative scalar curvature with $a+b \neq 0$, $a+(n-2)b \neq 0$ and $2a-(n-1)(n-4)b \neq 0$ can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.*

The notion of K-special conformally flat manifold which generalizes the notion of special conformally flat manifold as well as subprojective manifold was introduced by Chen and Yano [6]. According to them a conformally flat manifold is said to be K-special conformally flat manifold if the tensor field H of type (0, 2) defined in (3.23) is expressible in the form

$$H(X,Y) = -\frac{K+\alpha^2}{2} g(X,Y) + \beta\gamma\pi(X)\pi(Y), \tag{3.40}$$

where $(X\alpha) = \beta\pi(X)$ on G, G is an open set of M^n defined by

$$G = \{p \in M^n : \beta \neq 0\}$$

and π is an 1-form on G, and α, β, γ are scalar functions and K is a constant.

We consider a non-Einstein quasi-conformally flat $A(PRS)_n$. Then such a manifold is conformally flat. Using (3.9) in (3.36) we obtain (3.38). Let us now put

$$K + \alpha^2 = -\frac{r}{(n-1)(n-2)} > 0, \quad \text{provided } r < 0,$$

where K is a constant. Then proceeding similarly as before it can be easily shown that

$$H(X,Y) = -\frac{K+\alpha^2}{2} g(X,Y) + \beta\gamma\pi(X)\pi(Y),$$

where $\beta = \frac{4(n-1)r^2}{\delta^2}$, $\gamma = \frac{16r^3(n-1)^2\{r+K(n-1)(n-2)\}}{\delta^4}$. Thus we can state the following:

Theorem 3.11. *A non-Einstein quasi-conformally flat $A(PRS)_n$ with non-constant negative scalar curvature, $a+b \neq 0$, $a+(n-2)b \neq 0$ and $2a-(n-1)(n-4)b \neq 0$ is a K-special conformally flat manifold.*

References

[1] T. Adati, On subprojective spaces III, *Tohoku Math. J.,* **3** (1951), 343–358.

[2] K. Amur and Y. B. Maralabhavi, On quasi-conformally flat spaces, *Tensor N. S.,* **31**(1977), 194–198.

[3] M. C. Chaki, On pseudo Ricci symmetric manifolds, *Bulg. J. Phys.,* **15**(1988), 526–531.

[4] M. C. Chaki and T. Kawaguchi, On almost pseudo Ricci symmetric manifolds, T*ensor N. S.* **68**(2007), 10–14.

[5] B. Y. Chen and K. Yano, Hypersurfaces of conformally flat spaces, *Tensor N. S.* **26**(1972), 318–322.

[6] B. Y. Chen and K. Yano, Special conformally flat spaces and canal hypersurfaces, *Tohoku Math. J.* **25**(1973), 177–184.

[7] D. Ferus, *A remark on Codazzi tensors on constant curvature space*, Lecture Notes Math., **838**, Global Differential Geometry and Global Analysis, Springer-Verlag, New York, 1981.

[8] K. Yano, Concircular geometry, I, *Proc. Imp. Acad. Tokyo,* **16** (1940), 195–200.

[9] K. Yano, On the torseforming direction in Riemannian spaces,*Proc. Imp. Acad. Tokyo,* **20** (1944), 340–345.

[10] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Diff. Geom.* **2**(1968), 161–184.