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Geometric Properties of Randers and Other Changes of Rizza Manifolds*

H. G. Nagaraja†

Department of Mathematics, Central College, Bangalore University, Bangalore-1, Karnataka, India

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Abstract

In this paper we study the β-change of Rizza manifold associated with the generalized Finsler metric. It is proved that Randers change preserves the C-reducibility. We consider the Randers-change ${}^*L = L + \beta$ and proved the conditions for Rizza manifold to be C-reducible, Berwald and Landsbergian. Also we obtain condition for Rizza manifold to be C-conformal to a locally Minkowski space.

Keywords and Phrases: *Rizza manifold, Kahlerian Finsler, C-conformal, β-change, Randers-change.*

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†E-mail: hgnraj@yahoo.com

1. Introduction

Let (M, L) be a Finsler space, where M is an n-dimensional differentiable manifold associated with the fundamental function L. If (M, L) is endowed with an almost complex structure ${f^i}_j(x)$ on M with ${f^i}_r {f^r}_j = -{\delta^i}_j$ satisfying the condition

$$g_{ij} {f^i}_r y^r y^j = 0 \qquad (1.1)$$

where $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$, $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ then the structure $R_n = (M, L, f)$ is called Rizza manifold and the condition (1.1) is called Rizza condition. In section 2, we consider the Randers change and obtain conditions for Rizza manifold to be C-reducible, Berwald and Landsberg. In section 3, we establish that Rizza manifold is C-conformal to a locally Minkowski space under certain conditions.

2. Randers Change ${}^*L = L + \beta$

Let $R_n = (M, L, f)$ be a Rizza manifold and ${}^*R_n = (M, {}^*L, f)$ be obtained from R_n by Randers change

$${}^*L = L + \beta \qquad (2.1)$$

where $\beta = b_i(x) y^i$ for some covariant vector $b_i(x)$. The Randers change of the metric tensor ${}^*g_{ij}$ and its reciprocal ${}^*g^{ij}$ are given by

$${}^*g_{ij} = \rho\ g_{ij} + (1-\rho) l_i l_j + l_i b_j + l_j b_i + b_i b_j \qquad (2.2)$$

where

$$\rho = \frac{{}^*L}{L} \qquad (2.3)$$

and

$$^*g^{ij} = \rho^{-1}\ g^{ij} + \rho^{-2}[(\rho^{-1}\ b^2 + \frac{\beta}{^*L})l^i l^j - l^i b^j - l^j b^i] \tag{2.4}$$

The Randers changes of normalized vector $^*l_i$ and the angular metric tensor $^*h_{ij}$ are given by

$$^*l_i = l_i + b_i \tag{2.5}$$

$$^*h_{ij} =^* L\ ^*l_{ij} = \rho\ h_{ij} \tag{2.6}$$

where $^*l_{ij} = \dot{\partial}_j\ ^*l_i$.

Transvecting (2.2) by $f^i{}_r y^r y^j$, we get

$$^*g_{ij} f^i{}_r y^r y^j =^* L\ f_r\ y^r \tag{2.7}$$

where

$$f_r = f^i{}_r b_i \tag{2.8}$$

(2.7) shows that

Lemma 2.1. *The Randers change (2.1) preserves the Rizza condition (1.1) if and only if* $f_r y^r = 0$ *holds.*

The Randers change of Cartan's tensor $c_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by

$$^*c_{ijk} = \rho\ c_{ijk} + L^{-1}[m_k h_{ij} + m_j h_{ik} + m_i h_{jk}] \tag{2.9}$$

$$m_i = b_i - \frac{\beta}{L} l_i \tag{2.10}$$

Transvecting (2.9) by $^*g^{ij}$ as given in (2.4), and using $c_j{}^i{}_k = c_{jrk} g^{ir}$,
we obtain

$$^*c_j{}^i{}_k = c_j{}^i{}_k + \rho^{-1} L^{-1}[m_k h^i{}_j + m_j h^i{}_k + m^i h_{jk}] \tag{2.11}$$

where $h^i{}_j = h_{rj} g^{ir}$ and

$$m^2 = m_i m^i. \tag{2.12}$$

Contracting (2.11) with i and k and using $c_i = c^i{}_{ji}$,
we obtain

$$^*c_i = c_i \ + \ (\frac{n+1}{^*L}) m_i$$

or

$$m_i = \frac{(^*c_i - c_i)^* L}{n+1} \tag{2.13}$$

In a Rizza manifold $R_n = (M, L, f)$ if we put $G_{ij} = \frac{1}{2}(g_{ij} + g_{pq} f^p{}_i f^q{}_j)$ then we have $G_{ij} = G_{ji}$, $G_{ij} = G_{pq} f^p{}_i f^q{}_j$ and $y^m f^r{}_m \dot{\partial}_r G_{ij} = 0$, $\dot{\partial}_i G_{pq} y^p y^q = 0$.

This G_{ij} is positively homogeneous of degree 0 in y^i and is a generalized Finsler metric(GFM) [1]. Here afterwards we call R_n the Rizza manifold associated with GFM. The Randers change of G_{ij} is given by

$$\begin{aligned} ^*G_{ij} =& \rho G_{ij} + \frac{1}{2}\left((1-\rho) l_i l_j + l_i b_j + l_j b_i + b_i b_j \right. \\ & \left. +(1-\rho) l_p l_q + l_p b_q + l_q b_p + b_p b_q f^p{}_i f^q{}_j\right) \end{aligned} \tag{2.14}$$

Taking the derivative of (2.14) by y^k, we can derive that the C-tensor of the generalized Finsler metric as

$$^*C_{ijk} = \rho C_{ijk} + L^{-1}\{(m_k h_{ij} + m_j h_{ik} + m_i h_{jk}) + (m_q h_{pk} + m_p h_{qk} - l_p l_q) f^p{}_i f^q{}_j\} \tag{2.15}$$

Substituting (2.13) in (2.15) we get

$$\begin{aligned} ^*C_{ijk} = \rho\{C_{ijk} - \frac{1}{n+1}(C_i H_{jk} + C_j \ H_{ik} + C_k \ H_{ij})\} \\ + \frac{1}{n+1}\{^*C_i \ ^*H_{jk} +^* C_j \ ^*H_{ik} +^* C_k \ ^*H_{ij}\} \end{aligned} \tag{2.16}$$

where H_{ij} and ${}^*H_{ij}$ are the angular metric tensors respectively of R_n and ${}^*R_n$ with respect to generalized Finsler metric G_{ij} .

It is well known that a non Riemannian Finsler space F_n is called C-reducible if the hv-torsion tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{n+1}(C_i H_{jk} + C_j H_{ki} + C_k H_{ij}).$$

Thus from (2.16), we have

Theorem 2.1. *The space ${}^*R_n = (M, {}^*L, f)$ obtained from $R_n = (M, L, f)$ by Randers change is C-reducible if and only if R_n is C-reducible.*

Consider the Cartan connection $C\Gamma = (\Gamma^i{}_{jk}, N^i{}_k, C^i{}_j k)$.
Since $C\Gamma$ is h-metrical, we have $g_{ij|k} = 0$, where $|k$ denotes the h-covariant derivative with respect to $C\Gamma$.
Further we have $g^{ij}{}_{|k} = 0, \quad L_{|k} = 0$ and $y^i{}_{|k} = 0$.
If $f^i{}_{j|k} = 0$, then the Rizza manifold R_n is called Kaehlerian Finsler.

If ${}^*R_n$ is also a Rizza manifold, then by lemma (2.1), $f^i{}_r \; b_i \; y^r = 0$.
Further we have $0 = (f^i{}_j \; b_i \; y^j)_{|k} = f^i{}_{j|k} \; b_i \; y^j + f^i{}_j \; b_{i|k} \; y^j$.
In turn this implies $b_{i|k} = 0$, provided the space R_n is Kaehlerian Finsler.
Now from $b_{i|k} = 0$, we obtain ${}^*L_{|k} = 0$ and $\rho_{|k} = 0$.

In view of the above relations, the covariant derivative of (2.15) yields

$$ {}^*C_{ijk|m} = \rho \; C_{ijk|m}. \tag{2.17}$$

Thus we can state the following theorem

Theorem 2.2. *Let $R_n = (M, L, f)$ be a Kahlerian Finsler manifold associated with generalized Finsler metric G_{ij}. Let ${}^*R_n = (M, {}^*L, f)$ be a Rizza manifold* *obtained from R_n by Randers change (2.1). Then ${}^*R_n$ is Berwald if and only if R_n is Berwald.*

As it is well known that, A Landsberg space is characterized by $P_{ijk} = C_{ijk|0} = 0$.
Transvecting (2.17) by y^m, we obtain

$$ {}^*C_{ijk|0} = \rho\, C_{ijk|0}. $$

Thus we have

Theorem 2.3. *Let $R_n = (M, L, f)$ be a Kahlerian Finsler manifold associated with generalized Finsler metric G_{ij}. Let ${}^*R_n = (M, {}^*L, f)$ be a Rizza manifold obtained from R_n by Randers change (2.1). ${}^*R_n$ is Landsberg if and only if R_n is Landsberg.*

3. C-Conformal Change

Let $R_n = (M, L, f)$ be a Rizza manifold and ${}^*R_n = (M, {}^*L, f)$ be obtained from R_n by conformal change

$$ \overline{L} = e^{\sigma(x)}\, L \tag{3.1} $$

where $\sigma = \sigma(x)$ satisfies $\sigma^h C_j{}^i{}_h = 0$.

Let $F\Gamma = (\Gamma_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$, where

$$ \Gamma^i{}_{jk} = \frac{1}{2} G^{im}(X_j G_{mk} + X_k G_{mj} + X_m G_{jk}) $$

$$ C_j{}^i{}_k = \frac{1}{2} G^{im}(\dot{\partial}_j G_{mk} + \dot{\partial}_k G_{mj} - \dot{\partial}_m G_{jk}) $$

and $X_j = \partial_j - N^m{}_j \dot{\partial}_m$, $\partial_j = \frac{\partial}{\partial x^j}$ $\dot{\partial}_j = \frac{\partial}{\partial y^j}$ be a Finsler conection associated with (G, N) -structure.

Under the C-conformal change (3.1), we have

$$\overline{g}_{ij} = e^{2\sigma} g_{ij}, \ \ \overline{g}^{ij} = e^{-2\sigma} g^{ij}, \ \ \overline{C}_{ijk} = e^{2\sigma} C_{ijk}$$
$$\overline{C_j{}^i{}_k} = C_j{}^i{}_k \ \ \overline{G}_{ij} = e^{2\sigma} G_{ij}$$

and

$$\overline{N}^i{}_j = N^i{}_j + A^i{}_j \tag{3.2}$$

$$\overline{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k + A^i{}_{jk} \tag{3.3}$$

where $A^i{}_j = y^i \sigma_j - f^i{}_h y^h f^r{}_j \sigma_r$ and $A^i{}_{jk} = \delta^i{}_j \sigma_k + \delta^i{}_k \sigma_j - G_{jk}\ G^{im} \sigma_m$.

The h-torsion and h-curvature tensors of $F\Gamma$ are given by

$$R^i{}_{jk} = \{X_k N^i{}_j - j/k\}$$

and

$$R_h{}^i{}_{jk} = X_h \Gamma_h{}^i{}_j + \Gamma_m{}^i{}_k \Gamma_h{}^m{}_j - j/k + C_h{}^i{}_m R^m{}_{jk},$$

where -j/k denote the interchange of indices j and k and substraction.

C-conformal change of $R^i{}_{hjk}$ is given by

$$\overline{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} + B^i{}_{hj|k} + \delta^i{}_k(\sigma^2 G_{hj} - \sigma_j \sigma_h) - G_{hk} \sigma^i \sigma_j,$$

where $B^i{}_{hj} = \delta^i{}_h \sigma_j + \delta^i{}_j \sigma_h - G_{hj} G^{ir} \sigma_r$.

Now $\overline{R}_h{}^i{}_{jk} = 0$ if

$$R_h{}^i{}_{jk} = G_{hk} \sigma^i \sigma_j - \delta^i{}_k(\sigma^2 g_{hj} - \sigma_j \sigma_h) - B^i{}_{hj|k}. \tag{3.4}$$

Taking the covariant derivative of $\overline{C}_{hjk}$ with respect to $\overline{F\Gamma}$, we obtain

$$\overline{C}_{hjk\,\bar{|}\,m} = \overline{X}_m\overline{C}_{hjk} - \overline{C}_{hjr}\overline{\Gamma}_k{}^r{}_m - \overline{C}_{hrk}\overline{\Gamma}_j{}^r{}_m - \overline{C}_{rjk}\overline{\Gamma}_h{}^r{}_m$$

where $\overline{X}_m = \partial_m - \overline{N^s}_m\dot{\partial}_s$.
The above equation may be written in the form

$$\overline{C}_{hjk\bar{|}m} = e^{2\sigma}(C_{hjk|m} - \sigma_m C_{hjk}).$$

Thus $\overline{R}_n$ is C^h-recurrent with the recurrence vector σ_h if and only if $\overline{C}_{hjk\bar{|}m} = 0$. As it is known that a locally Minkowski space is characterized by $R_h{}^i{}_{jk} = 0$ and $C_{ijk|m} = 0$. Hence we have

Theorem 3.1. *Suppose the conditon (3.4) holds in the Rizza manifold R_n. Then R_n is C-conformal to a locally Minkowski space if and only if it is C^h-recurrent with respect to the recurrence vector σ_h.*

References

[1] Eun-Seo Choi, Certain conformally invariant connections of Rizza manifolds, *Commun. Korean math. Soc.,* **17,** no.3(2002), 505-517.

[2] M. Matsumoto, *Foundations of Finsler geometry and Special Finsler spaces,* Kaiseisha press, Otsu, Saikawa 1986.

[3] M. Kitayama, Finsler spaces admtting a parallel vector field, *Balkan Journal of Geometry and its applications*, **3,** no.2(1998), 29-36.

[4] M. Hashiguchi, On generalized Finsler spaces, *An. Stiint. univ. AI. I. Cuza Iasi, Sect.1 a Mat.* **30**(1984), no.1, 69-73.

[5] Y. Ichijyō, Kaehlerian Finsler manifolds, *J. Math. Tokushima Univ.* **25** (1994), 19-27.

[6] Y. Ichijyō and M. Hashiguchi, On locally flat generalized (α, β) metrics and conformally flat Randers metrics, *Rep. Fac. Sci. Kagoshima Univ. Math, Phys. Chem.* **27**(1994), 17-55.

[7] Y. Ichijyō and M. Hashiguchi, On (a, b, f)-metrics, *Rep. Fac. Sci. Kagoshima Univ Math, Phys . Chem.* **28**(1995), 1-9.

[8] Y. Ichijyō and M. Hashiguchi, On (a, b, f)-metrics-II,*Rep. Fac. Sci. Kagoshima Univ. Math . Phys . Chem.* **29**(1996), 1-5.

[9] C. Shibata and M. Azuma, C-conformal invariant tensors of Finsler spaces, *Tensor, N. S.* **52**(1993), 76-81.