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Some Applications of the Jung-Kim-Srivastava Integral Operator*

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Abstract

In this paper we derive some interesting properties of certain integral operator I^{σ} which was considered recently by Jung, Kim, and Srivastava [*J. Math. Anal. Appl*. **176**(1993), 138-147].

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1. Introduction

Let $A(p,k)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{m=k}^{\infty}a_{p+m}z^{p+m} \qquad (p,k\in N=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $E=\{z:|z|<1\}$. The Hadamard product or convolution $(f_1*f_2)(z)$ of two functions

$$f_j(z)=z^p+\sum_{m=k}^{\infty}a_{p+m,j}z^{p+m}\in A(p,k) \qquad (j=1,2) \tag{1.2}$$

is given by

$$(f_1*f_2)(z)=z^p+\sum_{m=k}^{\infty}a_{p+m,1}a_{p+m,2}z^{p+m}.$$

Let $f(z)$ and $g(z)$ be analytic in E. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in E such that $|w(z)|<1$ $(z\in E)$ and $g(z)=f(w(z))$. For this relation the symbol $g(z)\prec f(z)$ is used. In case $f(z)$ is univalent in E we have that the subordination $g(z)\prec f(z)$ is equivalent to $g(0)=f(0)$ and $g(E)\subset f(E)$.

Recently, Jung, Kim, and Srivastava [2] introduced the following one-parameter family of integral operator

$$I^\sigma f(z)=\frac{2^\sigma}{z\Gamma(\sigma)}\int_0^z\left(\log\frac{z}{t}\right)^{\sigma-1}f(t)dt \tag{1.3}$$

for $f(z)\in A(1,1)$ and $\sigma>0$. They showed that

$$I^\sigma f(z)=z+\sum_{m=1}^{\infty}\left(\frac{2}{m+2}\right)^\sigma a_{m+1}z^{m+1}. \tag{1.4}$$

The operator I^σ is closely related to the multiplier transformations studied earlier by Flett [1]. It follows from (1.4) that one can define the operator I^{σ} for any real number σ. Certain properties of this operator have been investigated by Jung, Kim and Srivastava [2], Uralegaddi and Somanatha [6], Li [3] and the author [4].

Motivated essentially by some recent works [2,3,4], we now extend the operator I^{σ} to multivalent functions, which is given by the following

$$I^{\sigma} f(z) = z^{p} + \sum_{m=k}^{\infty} \left(\frac{p+1}{m+p+1} \right)^{\sigma} a_{p+m} z^{p+m} \tag{1.5}$$

for $f(z) \in A(p,k)$ and any real number σ. It is easily verified from the definition (1.5) that

$$z(I^{\sigma+1} f(z))' = (p+1) I^{\sigma} f(z) - I^{\sigma+1} f(z). \tag{1.6}$$

In this note, we shall derive some interesting properties of the operator I^{σ}.

2. Main Results

We begin by recalling the following result due to Miller and Mocanu [5], which we shall apply in proving our first theorem.

Lemma. *Let* $h(z)$ *be analytic and convex univalent in* E, $h(0)=1$, *and let* $g(z) = 1 + b_k z^k + b_{k+1} z^{k+1} + \cdots$ *be analytic in* E. *If*

$$g(z) + zg'(z)/c \prec h(z), \tag{2.1}$$

then for $c \neq 0$ *and* $\operatorname{Re} c \geq 0$

$$g(z) \prec \frac{c}{k} z^{-c/k} \int_0^z t^{c/k-1} h(t) dt \quad . \tag{2.2}$$

Theorem 1. *Let* $-1 \leq B < A \leq 1$ *and* $0 < \delta < 1$. *Let* $f(z) = z^{p} + \sum_{m=k}^{\infty} a_{p+m} z^{p+m}$ $\in A(p,k)$. *Suppose that*

$$\sum_{m=k}^{\infty} c_m \, | a_{p+m} | \leq 1, \tag{2.3}$$

where

$$c_m = \frac{1-B}{A-B} \cdot \frac{(p+1)^{\sigma}(p+1+m(1-\delta))}{(m+p+1)^{\sigma+1}}. \tag{2.4}$$

(i) *If* $-1 \le B \le 0$, *then*

$$(1-\delta)\frac{I^{\sigma} f(z)}{z^p} + \delta \frac{I^{\sigma+1} f(z)}{z^p} \prec \frac{1+Az}{1+Bz}. \tag{2.5}$$

(ii) *If* $-1 \le B \le 0$ *and* $\lambda \ge 1$, *then for* $z \in E$

$$\operatorname{Re}\left\{\left(\frac{I^{\sigma+1} f(z)}{z^p}\right)^{1/\lambda}\right\} > \left\{\frac{p+1}{k(1-\delta)}\int_0^1 u^{(p+1)/k(1-\delta)-1}\left(\frac{1-Au}{1-Bu}\right)du\right\}^{1/\lambda}. \tag{2.6}$$

The result is sharp.

Proof. (i) Let

$$J = (1-\delta)\frac{I^{\sigma} f(z)}{z^p} + \delta \frac{I^{\sigma+1} f(z)}{z^p}, \tag{2.7}$$

then

$$J = 1 + \sum_{m=k}^{\infty} \frac{(p+1)^{\sigma}(p+1+m(1-\delta))}{(m+p+1)^{\sigma+1}} a_{p+m} z^{p+m} \quad . \tag{2.8}$$

For $-1 \le B \le 0$ and $z \in E$, it follows from (2.3) that

$$\left|\frac{J-1}{A-BJ}\right| = \left|\frac{\sum_{m=k}^{\infty} \frac{(p+1)^{\sigma}(p+1+m(1-\delta))}{(m+p+1)^{\sigma+1}} a_{p+m} z^m}{A-B-B\sum_{m=k}^{\infty} \frac{(p+1)^{\sigma}(p+1+m(1-\delta))}{(m+p+1)^{\sigma+1}} a_{p+m} z^m}\right|$$

$$\le \frac{\sum_{m=k}^{\infty} c_m \,|\, a_{p+m} \,|}{1-B+B\sum_{m=k}^{\infty} c_m \,|\, a_{p+m} \,|}$$

$$\le 1 \quad ,$$

which show that

$$(1-\delta)\frac{I^{\sigma} f(z)}{z^p} + \delta \frac{I^{\sigma+1} f(z)}{z^p} \prec \frac{1+Az}{1+Bz}.$$

(ii) Put

$$g(z) = I^{\sigma+1} f(z)/z^p . \tag{2.9}$$

Then the function $g(z) = 1 + b_k z^k + b_{k+1} z^{k+1} + \cdots$ is analytic in E. Using (1.6) and (2.9) we obtain

$$\frac{I^{\sigma} f(z)}{z^p} = g(z) + \frac{1}{p+1} zg'(z) \quad . \tag{2.10}$$

Thus

$$(1-\delta)\frac{I^{\sigma} f(z)}{z^p} + \delta \frac{I^{\sigma+1} f(z)}{z^p} = g(z) + \frac{1-\delta}{p+1} zg'(z)$$
$$\prec \frac{1+Az}{1+Bz} .$$

Now an application of the lemma leads to

$$g(z) \prec \frac{p+1}{k(1-\delta)} z^{-(p+1)/k(1-\delta)} \int_0^z t^{(p+1)/k(1-\delta)-1} \left(\frac{1+Az}{1+Bz}\right) dt$$

or

$$\frac{I^{\sigma+1} f(z)}{z^p} = \frac{p+1}{k(1-\delta)} \int_0^1 u^{(p+1)/k(1-\delta)-1} \left(\frac{1+Auw(z)}{1+Buw(z)}\right) du , \tag{2.11}$$

where $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ $(z \in E)$.

In view of $-1 \le B < A \le 1$, it follows from (2.11) that

$$\operatorname{Re}\left\{\frac{I^{\sigma+1} f(z)}{z^p}\right\} > \frac{p+1}{k(1-\delta)} \int_0^1 u^{(p+1)/k(1-\delta)-1} \left(\frac{1-Au}{1-Bu}\right) du > 0 \qquad (z \in E) .$$

Therefore, with the aid of the elementary inequality $\operatorname{Re}\left(w^{1/\lambda}\right) \ge \left(\operatorname{Re} w\right)^{1/\lambda}$ for $\operatorname{Re} w > 0$ and $\lambda \ge 1$, the inequality (2.6) follows immediately.

To show the sharpness of (2.6), we take $f(z) \in A(p,k)$ defined by

$$\frac{I^{\sigma+1} f(z)}{z^p} = \frac{p+1}{k(1-\delta)} \int_0^1 u^{(p+1)/k(1-\delta)-1} \left(\frac{1+Auz^k}{1+Buz^k}\right) du . \tag{2.12}$$

For this function we find that

$$(1-\delta)\frac{I^{\sigma} f(z)}{z^{p}}+\delta\frac{I^{\sigma+1} f(z)}{z^{p}}=\frac{1+Az^{k}}{1+Bz^{k}}$$

and

$$\frac{I^{\sigma+1} f(z)}{z^{p}}\rightarrow\frac{p+1}{k(1-\delta)}\int_{0}^{1}u^{(p+1)/k(1-\delta)-1}\left(\frac{1-Au}{1-Bu}\right)du \quad \text{as} \quad z\rightarrow e^{i\pi/k}.$$

Hence the proof of the theorem is complete.

Theorem 2. *Let* $f(z)=z^{p}+\sum_{m=k}^{\infty}a_{p+m}z^{p+m}\in A(p,k)$, $s_{1}(z)=z^{p}$ *and* $s_{n}(z)=z^{p}+\sum_{m=k}^{k+n-2}a_{p+m}z^{p+m}$ $(n\geq 2)$. *If the sequence* $\{c_{m}\}$ $(m\geq k)$ *is nondecreasing with* $c_{k}>1$*, where* c_{m} *is given by (2.4) and satisfies the condition (2.3), then*

$$\operatorname{Re}\left\{\frac{f(z)}{s_{n}(z)}\right\}>\frac{c_{k+n-1}-1}{c_{k+n-1}} \tag{2.13}$$

and

$$\operatorname{Re}\left\{\frac{s_{n}(z)}{f(z)}\right\}>1-\frac{1}{1+c_{k+n-1}}. \tag{2.14}$$

Each of the bounds in (2.13) and (2.14) is best possible for $n\in N$.

Proof. Under the hypothesis of the theorem, we have

$$\sum_{m=k}^{k+n-2}|a_{p+m}|+c_{k+n-1}\sum_{m=k+n-1}^{\infty}|a_{p+m}|\leq\sum_{m=k}^{\infty}c_{m}|a_{p+m}|\leq 1. \tag{2.15}$$

Let

$$g_{1}(z)=c_{k+n-1}\left\{\frac{f(z)}{s_{n}(z)}-\frac{c_{k+n-1}-1}{c_{k+n-1}}\right\},$$

then

$$g_1(z) = 1 + \frac{c_{k+n-1} \sum_{m=k+n-1}^{\infty} a_{p+m} z^m}{1 + \sum_{m=k}^{k+n-2} a_{p+m} z^m}$$

and it follows from (2.15) that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{c_{k+n-1} \sum_{m=k+n-1}^{\infty} |a_{p+m}|}{2 - 2\sum_{m=k}^{k+n-2} |a_{p+m}| - c_{k+n-1} \sum_{m=k+n-1}^{\infty} |a_{p+m}|}$$
$$\le 1 \quad (z \in E),$$

which readily yields the inequality (2.13).

If we take

$$f(z) = z^p - \frac{z^{p+k+n-1}}{c_{k+n-1}}, \tag{2.16}$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^{k+n-1}}{c_{k+n-1}} \to 1 - \frac{1}{c_{k+n-1}} \quad \text{as} \quad z \to 1^-.$$

This show that the bound in (2.13) is best possible for each n.

Similarly, if we put

$$g_2(z) = (1 + c_{k+n-1})\left\{\frac{s_n(z)}{f(z)} - \left(1 - \frac{1}{1 + c_{k+n-1}}\right)\right\},$$

then we deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1 + c_{k+n-1}) \sum_{m=k+n-1}^{\infty} |a_{p+m}|}{2 - 2\sum_{m=k}^{k+n-2} |a_{p+m}| + (1 - c_{k+n-1}) \sum_{m=k+n-1}^{\infty} |a_{p+m}|}$$
$$\le 1 \quad (z \in E),$$

which yields (2.14). The estimate (2.14) is sharp for each n with the extremal function $f(z)$ given by (2.16). The proof is now complete.

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