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On a New Inequality Applicable to Certain Volterra-Fredholm Type Sum-difference Equations*

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Abstract

The aim of the present paper is to establish a new finite difference inequality with explicit estimate which can be used as a tool in the study of some basic properties of solutions of certain Volterra-Fredholm type sum-difference equations.

Keywords and Phrases: *Finite difference inequality, Volterra-Fredholm type, Sum-difference equations, Numerical methods, Uniqueness of solutions, Estimate on the solution.*

1. Introduction

In [5] it was shown how an equation of the form

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega K(x,t,y,s,u(y,s))\,dyds, \tag{1.1}$$

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arises as a model of the spatio-temporal development of an epidemic and analysed. There are also some other physical and engineering problems which can be described by such equations, see [4, p. 18], [7, Chapter VI] and [6,8]. The numerical methods are often used very effectively in studying the behavior of solutions of equations of the forms (1.1), see [1,2,3,12] and the references cited therein. An important approach on numerical methods for solving such equations is to suitably combine discretization in space and in time in order to generate high order approximations at a reasonable computational cost.

In [3] the equation (1.1) is studied by time-stepping methods and the performance is then compared with that of time-stepping based on cocollection methods. In fact, the study of equations of the forms (1.1) by discretization methods, motivated us for the present work. In this paper we investigate a new finite difference inequality with explicit estimate, which can be used to study the qualitative behavior of solutions of discrete versions of more general equations of the forms (1.1) in ready fashion. Some applications to illustrate the usefulness of our main result are also given.

2. A Basic Finite Difference Inequality

Let N denote the set of natural numbers, $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, ...\}$ be the given subsets of R, the set of real numbers. Let $N_i [\alpha, \beta] = \{\alpha_i, \alpha_i + 1, ..., \beta_i\}$, $\alpha_i \in N_0, \beta_i \in N$, $i = 1, ..., m$ and $G = \prod_{i=1}^{m} N_i [\alpha, \beta]$. For any function $w : G \to R$, we denote the m-fold sum over G with respect to the variable $y = (y_1, ..., y_m) \in G$ by $\sum_G w(y) = \sum_{y_1=\alpha_1}^{\beta_1} ... \sum_{y_m=\alpha_m}^{\beta_m} w(y_1, ..., y_m)$. Clearly, $\sum_G w(y) = \sum_G w(x)$ for $x, y \in G$. Let $E = G \times N_0$ and denote by $D(S_1, S_2)$ the class of discrete functions from the set S_1 to the set S_2. We use the usual convention that empty sums and products are taken to be 0 and 1 respectively and assume that all sums and products involved exist on the respective domains of their definitions and are finite.

Our main result is given in the following theorem.

Theorem 1. *Let $u, a, b, p, q \in D(E, R_+)$ and*

$$u(x,n) \leq a(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, u(y,s)$$

$$+ c(x,n) \sum_{s=0}^{\infty} \sum_{G} q(y,s)\, u(y,s), \tag{2.1}$$

for $(x,n) \in E$. If

$$d = \sum_{s=0}^{\infty} \sum_{G} q(y,s)\, B(y,s) < 1, \tag{2.2}$$

then

$$u(x,n) \leq A(x,n) + DB(x,n), \tag{2.3}$$

for $(x,n) \in E$, where

$$A(x,n) = a(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, a(y,s)$$

$$\times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} p(y,\sigma)\, b(y,\sigma) \right], \tag{2.4}$$

$$B(x,n) = c(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, c(y,s)$$

$$\times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} p(y,\sigma)\, b(y,\sigma) \right], \tag{2.5}$$

for $(x,n) \in E$ and

$$D = \frac{1}{1-d} \sum_{s=0}^{\infty} \sum_{G} q(y,s)\, A(y,s). \tag{2.6}$$

Remark 1. By taking $q = 0$ in (2.1), the estimate obtained in (2.3) reduces to

$$u(x,n) \leq a(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, a(y,s) \times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} p(y,\sigma)\, b(y,\sigma)\right], \tag{2.7}$$

for $(x,n) \in E$. If we choose $p = 0$ in (2.1), then the bound obtained in (2.3) reduces to

$$u(x,n) \leq a(x,n) + c(x,n) \left\{ \frac{1}{1-d_0} \sum_{s=0}^{n-1} \sum_{G} q(y,s)\, a(y,s) \right\}, \tag{2.8}$$

for $(x,n) \in E$, where

$$d_0 = \sum_{s=0}^{\infty} \sum_{G} q(y,s)\, c(y,s) < 1. \tag{2.9}$$

For detailed account on such inequalities, see [9,10].

Proof. Let

$$z(n) = \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, u(y,s), \tag{2.10}$$

$$\lambda = \sum_{s=0}^{\infty} \sum_{G} q(y,s)\, u(y,s). \tag{2.11}$$

Then (2.1) can be restated as

$$u(x,n) \leq a(x,n) + b(x,n)\, z(n) + c(x,n)\, \lambda. \tag{2.12}$$

Introducing the notation

$$e(s) = \sum_{G} p(y,s)\, u(y,s), \tag{2.13}$$

in (2.10), we get

$$z(n) = \sum_{s=0}^{n-1} e(s). \tag{2.14}$$

From (2.14) and (2.12), we have

$$\Delta z(n) = z(n+1) - z(n) = e(n)$$
$$= \sum_G p(y,n)\, u(y,n)$$
$$\leq \sum_G p(y,n)\left[a(y,n) + b(y,n)\, z(n) + c(y,n)\,\lambda\right]$$
$$= z(n) \sum_G p(y,n)\, b(y,n) + \sum_G p(y,n)\left[a(y,n) + c(y,n)\,\lambda\right]. \tag{2.15}$$

Now applying the inequality in Theorem 1.2.1 given in [9, p. 11] with $z(0) = 0$ to (2.15) yields

$$z(n) \leq \sum_{s=0}^{n-1} \sum_G p(y,s)\left[a(y,s) + c(y,s)\,\lambda\right]$$
$$\times \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right]$$
$$= \sum_{s=0}^{n-1} \sum_G p(y,s)\, a(y,s) \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right]$$
$$+\lambda \sum_{s=0}^{n-1} \sum_G p(y,s)\, c(y,s) \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right]. \tag{2.16}$$

From (2.12) and (2.16), we get

$$u(x,n) \leq a(x,n) + b(x,n)\left\{\sum_{s=0}^{n-1} \sum_G p(y,s)\, a(y,s) \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right]\right.$$
$$\left.+\lambda \sum_{s=0}^{n-1} \sum_G p(y,s)\, c(y,s) \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right]\right\} + c(x,n)\,\lambda$$
$$= A(x,n) + \lambda B(x,n). \tag{2.17}$$

From (2.11) and (2.17), we observe that

$$\lambda = \sum_{s=0}^{\infty} \sum_{G} q(y,s)\, u(y,s)$$

$$\leq \sum_{s=0}^{\infty} \sum_{G} q(y,s)\left[A(y,s) + \lambda B(y,s)\right],$$

which implies

$$\lambda \leq D. \tag{2.18}$$

Using (2.18) in (2.17), we get (2.3) and the proof is complete.

3. Some Applications

In this section we apply the inequality established in Theorem 1 to study some basic properties of solutions of Volterra-Fredholm type sum-difference equation

$$u(x,n) = f(x,n) + \sum_{s=0}^{n-1} \sum_{G} F(x,n,y,s,u(y,s))$$

$$+ \sum_{s=0}^{\infty} \sum_{G} H(x,n,y,s,u(y,s)), \tag{3.1}$$

for $(x,n) \in E$, where $f \in D(E,R)$, $F, H \in D(E^2 \times R, R)$. One can formulate existence result for the solution of equation (3.1) by using the idea employed in [8,11].

First we shall give the following theorem which deals with the uniqueness of solutions of equation (3.1).

Theorem 2. *Suppose that the functions F, H in equation (3.1) satisfy the conditions*

$$|F(x,n,y,s,u) - F(x,n,y,s,v)| \leq b(x,n)\, p(y,s)\, |u - v|, \tag{3.2}$$

$$|H(x,n,y,s,u) - H(x,n,y,s,v)| \leq c(x,n)\, q(y,s)\, |u - v|, \tag{3.3}$$

where $b, p, c, q \in D(E, R_+)$. Let $d, D, A(x,n), B(x,n)$ be as in Theorem 1. Then the equation (3.1) has at most one solution on E.

Proof. Let $u(x,n)$ and $v(x,n)$ be two solutions of equation (3.1). Then by using the hypotheses, we have

$$|u(x,n)-v(x,n)| \leq \sum_{s=0}^{n-1}\sum_{G}|F(x,n,y,s,u(y,s))-F(x,n,y,s,v(y,s))|$$

$$+\sum_{s=0}^{\infty}\sum_{G}|H(x,n,y,s,u(y,s))-H(x,n,y,s,v(y,s))|$$

$$\leq b(x,n)\sum_{s=0}^{n-1}\sum_{G}p(y,s)|u(y,s)-v(y,s)|$$

$$+c(x,n)\sum_{s=0}^{\infty}\sum_{G}q(y,s)|u(y,s)-v(y,s)|. \tag{3.4}$$

Here, it is easy to see that $A(x,n)$ and D given by (2.4) and (2.6) gives $A(x,n)=0$ and $D=0$. Now a suitable application of Theorem 1 to (3.4) yields $|u(x,n)-v(x,n)| \leq 0$, and hence $u(x,n)=v(x,n)$. Thus there is at most one solution to equation (3.1) on E.

The following theorem deals with the estimate on the solution of equation (3.1).

Theorem 3. *Suppose that the functions F,H in equation (3.1) satisfy the conditions*

$$|F(x,n,y,s,u)| \leq b(x,n)\,p(y,s)\,|u|, \tag{3.5}$$

$$|H(x,n,y,s,u)| \leq c(x,n)\,q(y,s)\,|u|, \tag{3.6}$$

where $b,p,c,q \in D(E,R_+)$. Let $d, B(x,n)$ be as in (2.2), (2.5) and

$$D_1=\frac{1}{1-d}\sum_{s=0}^{\infty}\sum_{G}q(y,s)\,A_1(y,s), \tag{3.7}$$

where $A_1(x,n)$ is defined by the right hand side of (2.4) by replacing $a(x,n)$ by $|f(x,n)|$. If $u(x,n)$ is any solution of equation (3.1), then

$$|u(x,n)| \leq A_1(x,n)+D_1B(x,n), \tag{3.8}$$

for $(x,n)\in E$.

Proof. Using the fact that $u(x,n)$ is a solution of equation (3.1) and the hypotheses, we have

$$\begin{aligned}|u(x,n)| &\le |f(x,n)| + \sum_{s=0}^{n-1}\sum_{G}|F(x,n,y,s,u(y,s))| \\ &\quad + \sum_{s=0}^{\infty}\sum_{G}|H(x,n,y,s,u(y,s))| \\ &\le |f(x,n)| + b(x,n)\sum_{s=0}^{n-1}\sum_{G}p(y,s)\,|u(y,s)| \\ &\quad + c(x,n)\sum_{s=0}^{\infty}\sum_{G}q(y,s)\,|u(y,s)|. \qquad (3.9)\end{aligned}$$

Now an application of Theorem 1 to (3.9) yields (3.8).

The next theorem gives the estimation on the solution of equation (3.1) assuming that the functions F, H in equation (3.1) satisfy the Lipschitz type conditions.

Theorem 4. *Suppose that the functions F, H in equation (3.1) satisfy the conditions (3.2), (3.3). Let $d, B(x,n)$ be as in (2.2), (2.5) and*

$$\begin{aligned}r(x,n) &= \sum_{s=0}^{n-1}\sum_{G}|F(x,n,y,s,f(y,s))| \\ &\quad + \sum_{s=0}^{\infty}\sum_{G}|H(x,n,y,s,f(y,s))|, \qquad (3.10)\end{aligned}$$

$$D_2 = \frac{1}{1-d}\sum_{s=0}^{\infty}\sum_{G}q(y,s)\,A_2(y,s), \qquad (3.11)$$

where $A_2(x,n)$ is defined by the right hand side of (2.4) by replacing $a(x,n)$ by $r(x,n)$. If $u(x,n)$ is any solution of equation (3.1), then

$$|u(x,n) - f(x,n)| \le A_2(x,n) + D_2 B(x,n), \qquad (3.12)$$

for $(x,n) \in E$.

Proof. Using the fact that $u(x,n)$ is a solution of equation (3.1) and the hypotheses, we have

$$|u(x,n) - f(x,n)| \leq \sum_{s=0}^{n-1} \sum_{G} |F(x,n,y,s,u(y,s)) - F(x,n,y,s,f(y,s))|$$

$$+ \sum_{s=0}^{n-1} \sum_{G} |F(x,n,y,s,f(y,s))|$$

$$+ \sum_{s=0}^{\infty} \sum_{G} |H(x,n,y,s,u(y,s)) - H(x,n,y,s,f(y,s))|$$

$$+ \sum_{s=0}^{\infty} \sum_{G} |H(x,n,y,s,f(y,s))|$$

$$\leq r(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s) |u(y,s) - f(y,s)|$$

$$+ c(x,n) \sum_{s=0}^{\infty} \sum_{G} q(y,s) |u(y,s) - f(y,s)|. \tag{3.13}$$

Now an application of Theorem 1 to (3.13) yields (3.12).

Finally, we obtain the estimate on the difference between the solutions of equation (3.1) and the following Volterra type sum-difference equation

$$v(x,n) = \bar{f}(x,n) + \sum_{s=0}^{n-1} \sum_{G} F(x,n,y,s,v(y,s)), \tag{3.14}$$

for $(x,n) \in E$, where $\bar{f} \in D(E,R), F \in D(E^2 \times R, R)$.

The following theorem holds.

Theorem 5. *Suppose that the functions F, H in equations (3.1), (3.14) satisfy the conditions (3.2), (3.3) and $H(x,n,y,s,0) = 0$. Let $v(x,n)$ be a solution of equation (3.14) such that $|v(x,n)| \leq Q$, for $(x,n) \in E$, where $Q \geq 0$ is a constant. Let $d, B(x,n)$ be as in (2.2), (2.5) and*

$$\bar{a}(x,n) = \left|f(x,n) - \bar{f}(x,n)\right| + Qc(x,n)\sum_{s=0}^{\infty}\sum_{G} q(y,s), \tag{3.15}$$

$$D_3 = \frac{1}{1-d}\sum_{s=0}^{\infty}\sum_{G} q(y,s)\, A_3(y,s), \tag{3.16}$$

where $A_3(x,n)$ is defined by the right hand side of (2.4) by replacing $a(x,n)$ by $\bar{a}(x,n)$. If $u(x,n)$ is any solution of equation (3.1) on E, then

$$|u(x,n) - v(x,n)| \leq A_3(x,n) + D_3 B(x,n), \tag{3.17}$$

for $(x,n) \in E$.

Proof. Using the facts that $u(x,n)$ and $v(x,n)$ are respectively the solutions of equations (3.1) and (3.14) and the hypotheses, we observe that

$$\begin{aligned}
|u(x,n) - v(x,n)| &\leq \left|f(x,n) - \bar{f}(x,n)\right| \\
&+ \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,u(y,s)) - F(x,n,y,s,v(y,s))| \\
&+ \sum_{s=0}^{\infty}\sum_{G} |H(x,n,y,s,u(y,s)) - H(x,n,y,s,v(y,s))| \\
&+ \sum_{s=0}^{\infty}\sum_{G} |H(x,n,y,s,v(y,s)) - H(x,n,y,s,0)| \\
&\leq \left|f(x,n) - \bar{f}(x,n)\right| + b(x,n)\sum_{s=0}^{n-1}\sum_{G} p(y,s)\,|u(y,s) - v(y,s)| \\
&+ c(x,n)\sum_{s=0}^{\infty}\sum_{G} q(y,s)\,|u(y,s) - v(y,s)|
\end{aligned}$$

$$+c\,(x,n)\sum_{s=0}^{\infty}\sum_{G}q\,(y,s)\,|v\,(y,s)|$$

$$\leq \bar{a}\,(x,n)+b\,(x,n)\sum_{s=0}^{n-1}\sum_{G}p\,(y,s)\,|u\,(y,s)-v\,(y,s)|$$

$$+c\,(x,n)\sum_{s=0}^{\infty}\sum_{G}q\,(y,s)\,|u\,(y,s)-v\,(y,s)|. \tag{3.18}$$

Now an application of Theorem 1 to (3.18) yields (3.17).

Remark 2. We note that the generality of the equation (3.1) allow us to include the study of sum-difference equations of the forms

$$u\,(x,n)=f\,(x,n)+\sum_{s=0}^{n-1}\sum_{G}F\,(x,n,y,s,u\,(y,s))\,, \tag{3.19}$$

and

$$u\,(x,n)=f\,(x,n)+\sum_{s=0}^{\infty}\sum_{G}H\,(x,n,y,s,u\,(y,s))\,. \tag{3.20}$$

Moreover, our approach here is different and we believe that the results given here are of independent interest.

References

[1] H. Brunner and P. J. Van der Houwen, The numerical solution of Volterra equations, *CWI Monographs*, **Vol. 3**, North-Holland, Amsterdam, 1986.

[2] H. Brunner, On the numerical solution of nonlinear Volterra-Fredholm integral equations by cocollection methods, *SIAM J. Numer. Anal.* **27**(1990), 987-1000.

[3] H. Brunner and E. Messina, Time-stepping methods for Volterra-Fredholm integral equations, *Rediconti di Mathematica*, **Serie VII, 23**(2003), 329-342.

[4] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.

[5] O. Diekmann, Thresholds and travelling waves for the geographical spread of infection, *J. Math. Biology* **6**(1978), 109-130.

[6] O. Diekmann, Run for your life. A note on the asymptotic speed of propagation of an epidemic, *J. Differential Equations* **33**(1979), 58-73.

[7] V. P. Mikhailov, *Partial Differential Equations*, Mir Publishers, Moscow, 1978.

[8] B. G. Pachpatte, On mixed Volterra-Fredholm type integral equations, *Indian J. Pure and Appl. Math.* **17**(1986), 488-496.

[9] B. G. Pachpatte, *Inequalities for Finite Difference Equations*, Marcel Dekker, Inc., New York, 2002.

[10] B. G. Pachpatte, Integral and Finite Difference Inequalities and Applications, *North-Holland Mathematics Studies*, **Vol. 205**, Elsevier Science B.V., Amsterdam, 2006.

[11] B. G. Pachpatte, On Volterra-Fredholm integral equation in two variables, *Demonstratio Mathematica* **XL**(2007), 839-852.

[12] A. M. Wazwaz, A reliable treatment for mixed Volterra-Fredholm integral equations, *Appl. Math. Comput.* **127**(2002), 405-414.