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# Convex Functions and Functions with Bounded Turning*

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**Abstract**

Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ that are normalized with $f(0) = f'(0) - 1 = 0$ and let $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$. In this paper the following generalizations of the class of convex functions and of the class of functions with bounded turn are studied

$$K[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$R_k[C,D] = \left\{ f \in \mathcal{A} : \sqrt[k]{f'(z)} \prec \frac{1+Cz}{1+Dz} \right\},$$

$k \geq 1$. Conditions when $K[A,B] \subset R_k[C,D]$ are given together with several corollaries for different choices of $A$, $B$, $C$, $D$ and $k$.

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# 1. Introduction and Preliminaries

A region $\Omega$ from the complex plane $\mathbb{C}$ is called convex if for every two points $\omega_1, \omega_2 \in \Omega$ the closed line segment $[\omega_1, \omega_2] = \{(1-t)\omega_1 + t\omega_2 : 0 \le t \le 1\}$ lies in $\Omega$. Fixing $\omega_1 = 0$ brings the definition of starlike region. If $\mathcal{A}$ denotes the class of functions $f(z)$ that are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, then a function $f \in \mathcal{A}$ is called *convex* or *starlike* if it maps $\mathcal{U}$ into a convex or starlike region, respectively. Corresponding classes are denoted by $K$ and $S^*$. It is well known that $K \subset S^*$, that both are subclasses of the class of univalent functions and have the following analytical representations

$$f \in K \quad \Leftrightarrow \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\; z \in \mathcal{U},$$

and

$$f \in S^* \quad \Leftrightarrow \quad \operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \;\; z \in \mathcal{U}.$$

More on these classes can be found in [1].

Further, let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk $\mathcal{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$ than $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

In the terms of subordination we have

$$S^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$$

and

$$K = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \right\}.$$

Now, let $A$ and $B$ be real numbers such that $-1 \le B < A \le 1$. Then, the function $\frac{1+Az}{1+Bz}$ maps the unit disc univalently onto an open disk that lies in the right half of the complex plane and is centered on the real axis with diameter end points $(1-A)/(1-B)$ and $(1+A)/(1+B)$. Thus, classes $S^*$ and $K$ can be generalized with

$$S^*[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$K[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

Similarly, the class of *functions with bounded turning*

$$R = \{ f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathcal{U} \} = \left\{ f \in \mathcal{A} : f'(z) \prec \frac{1+z}{1-z}, z \in \mathcal{U} \right\}$$

can be generalized by

$$R_k[A,B] = \left\{ f \in \mathcal{A} : \sqrt[k]{f'(z)} \prec \frac{1+Az}{1+Bz} \right\},$$

where $k \geq 1$ and the root is taken by its principal value. The name of the class comes from the fact that $\operatorname{Re} f'(z) > 0$ is equivalent with $|\arg f'(z)| < \pi/2$ and $\arg f'(z)$ is the angle of rotation of the image of a line segment from $z$ under the mapping $f$.

It is well known sharp result due to A. Marx ([5]) that $K \subset R_2[0,-1]$, i.e., $\operatorname{Re} \sqrt{f'(z)} > 1/2$, $z \in \mathcal{U}$. This paper aims to obtain conditions when $K[A,B] \subset R_k[C,D]$, i.e., we will look for a conditions over $k$ so that for fixed $A$, $B$, $C$ and $D$, the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}$ implies $f'(z) \prec \left(\frac{1+Cz}{1+Dz}\right)^k$. Also, some corollaries and examples for different choices of $A$, $B$, $C$, $D$ and $k$ will be given having in mind that

- $K(\alpha) \equiv K[1-2\alpha,-1]$, $0 \leq \alpha < 1$, is the class of *convex functions of order* $\alpha$, with analytical representation $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$, $z \in \mathcal{U}$;
- $K \equiv K[1,-1] = K(0)$ is the class of *convex functions*; and
- $R(\alpha,k) \equiv R_k[1-2\alpha,-1]$, $0 \leq \alpha < 1$, $k \geq 1$, with analytical representation $\operatorname{Re} \sqrt[k]{f'(z)} > \alpha$, $z \in \mathcal{U}$.

For obtaining the main result we will use the method of differential subordinations. Valuable reference on this topic is [2]. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [3] and [4]. Namely, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ is analytic in a domain $D$, if $h(z)$ is univalent in $\mathcal{U}$, and if $p(z)$ is analytic in $\mathcal{U}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathcal{U}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1.1}$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (1.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). If $\widetilde{q}(z)$ is a dominant of (1.1) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (1.1), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (1.1).

From the theory of first-order differential subordinations we will make use of the following lemma.

**Lemma 1** ([4]). *Let $q(z)$ be univalent in the unit disk $\mathcal{U}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{U})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

*i)* $Q(z) \in S^*$; *and*

*ii)* $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0,\ z \in \mathcal{U}.$

*If $p(z)$ is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq \mathcal{D}$ and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{1.2}$$

*then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.2).*

# 2. Main Results and Consequences

In the beginning, using Lemma 1 we will prove the following useful result.

**Lemma 2.** *Let $f \in \mathcal{A}$, $k \geq 1$ and let $C$ and $D$ be real numbers such that $-1 \leq D < C \leq 1$. If*

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{kz(C-D)}{(1+Cz)(1+Dz)} \equiv h(z) \tag{2.1}$$

*then $\sqrt[k]{f'(z)} \prec \frac{1+Cz}{1+Dz} \equiv q(z)$, i.e., $f \in R_k[C, D]$, where function $q(z)$ is the best dominant of (2.1).*

**Proof.** We choose $p(z) = \sqrt[k]{f'(z)}$, $q(z) = \frac{1+Cz}{1+Dz}$, $\theta(\omega) = 1$ and $\phi(\omega) = k/\omega$. Then $q(z)$ is convex, thus univalent, because $1 + zq''(z)/q'(z) = (1-Dz)/(1+Dz)$; $\theta(\omega)$ and $\phi(\omega)$ are analytic with domain $\mathcal{D} = \mathbb{C} \setminus \{0\}$ which contains $q(\mathcal{U})$ and $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Further,

$$Q(z) = zq'(z)\phi(q(z)) = \frac{k(C-D)z}{(1+Cz)(1+Dz)}$$

and for $z = e^{i\lambda}$, $\lambda \in [-\pi, \pi]$ we have

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \frac{1 - CDz^2}{(1+Cz)(1+Dz)} = \frac{(1-CD)[1 + CD + (C+D)\cos\lambda]}{|1+Ce^{i\lambda}|^2 \cdot |1+De^{i\lambda}|^2}.$$

If $CD \geq 0$ then $1 + CD + (C+D)\cos\lambda \geq (1-|C|)(1-|D|)$ and if $CD < 0$, i.e., $-1 \leq D < 0 < C \leq 1$, then

$$1+CD+(C+D)\cos\lambda \geq 1-C|D|-|C-|D|| = \begin{cases} (1-C)(1+|D|) \geq 0, & C \geq |D| \\ (1+C)(1-|D|) \geq 0, & C < |D| \end{cases}.$$

Thus $Q(z)$ is a starlike function.

Also, $h'(z) = Q'(z)$, $p(z)$ is analytic in $\mathcal{U}$, $p(0) = q(0) = 1$, $p(\mathcal{U}) \subseteq \mathcal{D}$, and so, the conditions of Lemma 1 are satisfied. Finally, concerning that subordinations (1.2) and (2.1) are equivalent we receive the conclusion of Lemma 2.

The next theorem gives conditions when $K[A,B] \subset R_k[C,D]$.

**Theorem 1.** *Let $A$, $B$, $C$, $D$ and $k$ be real numbers such that $-1 \leq B < A \leq 1$, $-1 \leq D < C \leq 1$ and $k \geq 1$. Also let $a = CD(A-B)$, $b = (A-B)(C+D) - kB(C-D)$, $c = A - B$ and*

$$\Delta = \begin{cases} (c-a) \cdot \sqrt{1 - \frac{b^2}{4ac}} \equiv \Delta_1, & ac < 0 \text{ and } |b|(a+c) \leq -4ac \\ a + c + |b| \equiv \Delta_2, & \text{otherwise} \end{cases}.$$

*If $\Delta \leq k(C-D)$ then $K[A,B] \subset R_k[C,D]$, i.e.,*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \tag{2.2}$$

*implies the subordination $\sqrt[k]{f'(z)} \prec \frac{1+Cz}{1+Dz} \equiv q(z)$, meaning that $q^k(\mathcal{U})$ contains $f'(\mathcal{U})$ for all $f \in K[A,B]$.*

**Proof.** According to Lemma 2 in order to prove this theorem it is enough to show that $g(z) \equiv \frac{1+Az}{1+Bz} \prec h(z)$. By the definition of subordination and the fact that $h(z)$ is starlike univalent function (shown in the proof of lemma 2 since $zQ'(z)/Q(z) = zh'(z)/h(z)$), subordination $g \prec h$ is equivalent to

$$\left(\frac{A-B}{g(z)-1} - B\right)^{-1} \prec \left(\frac{A-B}{h(z)-1} - B\right)^{-1},$$

i.e.,

$$z \prec \frac{k(C-D)z}{(A-B)(1+Cz)(1+Dz) - kB(C-D)z} \equiv g_1(z). \tag{2.3}$$

Further, function $g_1(z)$ is univalent because $h(z)$ is univalent and so (2.3) can be rewritten as $\mathcal{U} \subset g_1(\mathcal{U})$ or as

$$|g_1(z)| \geq 1 \tag{2.4}$$

for all $|z| = 1$. Now, putting $z = e^{i\lambda}$ and $t = \cos\lambda$ and using notations for $a$, $b$ and $c$ given in the statement of the theorem we obtain that inequality (2.4) is equivalent with

$$\Sigma \equiv \max\{\psi(t) : -1 \leq t \leq 1\} \leq k^2(C-D)^2,$$

where

$$\begin{aligned}\psi(t) &= |(A-B)(1+Cz)(1+Dz) - kB(C-D)z|^2 = |az^2 + bz + c|^2 = \\ &= 4act^2 + 2b(a+c)t + b^2 + (a-c)^2.\end{aligned}$$

Simple calculus shows that the parabola $\psi(t)$ has vertex $(t_*, \psi(t_*)$, where $t_* = -\frac{b(a+c)}{4ac}$ and $\psi(t_*) = (c-a)^2\left(1 - \frac{b^2}{4ac}\right)$. Therefore if $ac < 0$ and $\left|\frac{b(a+c)}{4ac}\right| \leq 1$ then $\Sigma = \psi(t_*0$. Otherwise, $\Sigma = \max\{\psi(1), \psi(-1)\} = (a+c+|b|)^2$. $\square$

Combining all this we receive the statement of the theorem.

In order to obtain more explicit conditions over $k$ the last theorem can be rewritten in the following way.

**Theorem 2.** *Let $A$, $B$, $C$, $D$ and $k$ be real numbers such that $-1 \leq B < A \leq 1$, $-1 \leq D < C \leq 1$ and $k \geq 1$. Also, let*

$$k_1 \equiv \frac{A-B}{C-D}\left[C(1+D)^2 + D(1+C)^2\right], \quad k_2 \equiv \frac{A-B}{C-D}\left[C(1-D)^2 + D(1-C)^2\right],$$

$$\overline{a} = 4CD + B^2(1-CD)^2, \quad \overline{b} = -2B\frac{A-B}{C-D}(C+D)(1-CD)^2$$

*and* $\overline{c} = (A-B)^2(1-CD)^2$. *If one of the following two sets of conditions is satisfied*

*i)* $CD < 0$, $k_1 \le kB(1+CD) \le k_2$ *and* $\overline{a}k^2 + \overline{b}k + \overline{c} \le 0$;

*ii)* $CD \ge 0$ *or* $k_1 \ge kB(1+CD)$ *or* $k_2 \le kB(1+CD)$, *together with*

$$k \ge \begin{cases} \frac{(1+A)(1-C)}{1+C} \equiv \widehat{k_3}, & B = D = -1 \\ \frac{A-B}{C-D} \cdot \max\left\{\frac{(1-C)(1-D)}{1-B}, \frac{(1+C)(1+D)}{1+B}\right\} \equiv \overline{k_3}, & B \ne -1 \end{cases}; \tag{2.5}$$

*then* $K[A,B] \subset R_k[C,D]$, *i.e.,* $1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}$ *implies* $\sqrt[k]{f'(z)} \prec \frac{1+Cz}{1+Dz} \equiv q(z)$, *meaning that* $q^k(\mathcal{U})$ *contains* $f'(\mathcal{U})$ *for all* $f \in K[A,B]$. *If none of the conditions (i) and (ii) is satisfied or in (ii)* $B = -1$ *and* $D \ne -1$ *then* $K[A,B] \not\subseteq R_k[C,D]$.

**Proof.** Considering notations defined in Theorem 1 it can be easily verified that inequalities $ac < 0$, $|b|(a+c) \le -4ac$ and $\Delta_1 \le k(C-D)$ are equivalent to $CD < 0$, $k_1 \le kB(1+CD) \le k_2$ and $\overline{a}k^2 + \overline{b}k + \overline{c} \le 0$, respectively. Equivalency between $|b|(a+c) \le -4ac$ and $k_1 \le kB(1+CD) \le k_2$ is not so obvious and it follows from

$$|b|(a+c) \le -4ac \quad \Leftrightarrow \quad \frac{4ac}{a+c} \le b \le -\frac{4ac}{a+c}.$$

This, according to Theorem 1, shows that condition (i) implies $K[A,B] \subset R_k[C,D]$.

Now we will show that $\Delta_2 \le k(C-D)$ is equivalent to (2.5). First let note that $\Delta_2 \le k(C-D)$ if and only if

$$-k(C-D) + (A-B)(1+CD) \le (A-B)(C+D) - kB(C-D) \le \\ \le k(C-D) - (A-B)(1+CD),$$

i.e., if and only if both

$$(A-B)(1-C)(1-D) \le k(C-D)(1-B) \tag{2.6}$$

and

$$(A-B)(1+C)(1+D) \le k(C-D)(1+B) \tag{2.7}$$

hold. In the case when $B \neq -1$ we can divide both sides of (2.7) and (2.6) with $(C-D)(1-B)$ and $(C-D)(1+B)$, respectively, and obtain that $\Delta_2 \leq k(C-D)$ is equivalent to $k \geq \overline{k_3}$. If $B = -1$ then in order (2.6) to hold it is necessary and sufficient $D = -1$ and then (2.7) is equivalent to $k \geq \widehat{k_3}$.

If none of the conditions (i) and (ii) is satisfied or in (ii) $B = -1$ and $D \neq -1$ then $\Delta > k(C-D)$ that by the proof of Theorem 1 implies $g(z) \not\prec h(z)$. Therefore, for a function $f(z) \in K[A, B]$ defined by

$$1 + \frac{zf''(z)}{f'(z)} = g(z)$$

we have $f(z) \notin R_k[C, D]$ since $q(z)$ is the best dominant of the subordination (2.1).

Now we will study the problem of finding the smallest $k \geq 1$, if any, such that for given $A$, $B$, $C$ and $D$, $q^k(\mathcal{U})$ is the smallest region contains $f'(\mathcal{U})$ for all $f \in K[A, B]$. Such $k$ will be denoted by $\widetilde{k}$. The idea is to see under which conditions over $k$ Theorem 2 can be applied. We will consider the following cases:

case 1 : $B = 0$

case 2 : $B \neq 0$, $B \neq -1$ and $CD = -1$

case 3 : $B = -1$

case 4 : $B \neq 0$, $B \neq -1$ and $CD \neq -1$

The following theorem covers the cases 1 and 2.

**Theorem 3.** *Let $A$, $B$, $C$ and $D$ be real numbers such that $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$. Then $\widetilde{k} = \max\{1, \widehat{k}\}$, where*

$$\widehat{k} = \begin{cases} \frac{A(1-CD)}{2\sqrt{-CD}}, & \textit{if} \quad B = 0, CD < 0, k_1 \leq 0 \leq k_2 \\ \frac{A}{C-D}(1+|C|)(1+|D|), & \textit{if} \quad B = 0 \textit{ and } (CD \geq 0 \textit{ or } k_2 \leq 0 \textit{ or } k_1 \geq 0) \\ \frac{A-B}{\sqrt{1-B^2}}, & \textit{if} \quad B \neq -1, CD = -1 \end{cases} .$$

**Proof.** Let $B = 0$. Then $\overline{a} = 4CD$, $\overline{b} = 0$ and $\overline{c} = A^2(1-CD)^2 > 0$.

If $CD < 0$ and $k_1 \leq 0 \leq k_2$ then, by Theorem 2(i), $\widetilde{k}$ is the smallest $k \geq 1$ such that $\overline{a}k^2 + \overline{c} \leq 0$. Since $\overline{a} < 0$, we have $\widetilde{k} = \max\{1, \widehat{k}\}$, where $\widehat{k} = \sqrt{-\frac{\overline{c}}{\overline{a}}} = \frac{A(1-CD)}{2\sqrt{-CD}}$ is the positive root of the equation $\overline{a}k^2 + \overline{c} = 0$.

If $CD \geq 0$ or $k_2 \leq 0$ or $k_1 \geq 0$ then by Theorem 2(ii) we have $\widetilde{k} = \max\{1, \widehat{k}\}$, where $\widehat{k} = \overline{k_3} = \frac{A}{C-D}(1+|C|)(1+|D|)$.

Now let $B \neq -1$ and $CD = -1$, i.e., $B \neq -1$, $C = 1$ and $D = -1$. Then, $k_1 = -2(A-B)$, $k_2 = 2(A-B)$, $\overline{a} = 4(B^2-1) < 0$, $\overline{b} = 0$ and $\overline{c} = 4(A-B)^2 > 0$. So, according to Theorem 2(i), $\widetilde{k} = \max\{1, \widehat{k}\}$, where $\widehat{k} = \sqrt{-\frac{\overline{c}}{\overline{a}}} = \frac{A-B}{\sqrt{1-B^2}}$ is the positive root of the equation $\overline{a}k^2 + \overline{c} = 0$. □

Now we will explore case 3 when $B = -1$.

**Theorem 4.** *Let $C$ and $D$ be real numbers such that $-1 \leq D < C \leq 1$ and let $0 \leq \alpha < 1$. If $CD = -1$ or $D \neq -1$ then $\widetilde{k}$ does not exist. Otherwise, i.e., if $D = -1$ and $C \neq 1$ then $\widetilde{k} = \max\{1, \widehat{k}\}$, where $\widehat{k} = \widehat{k_3} = (1+A)(1-C)/(1+C)$.*

**Proof.** First we will prove the cases when $\widetilde{k}$ does not exist.

If $CD = -1$, i.e., $C = 1$ and $D = -1$, then $\overline{a} = \overline{b} = 0$, $\overline{c} = 4(1+A)^2 > 0$, i.e., for any real $k$ condition (i) of Theorem 2 can not be satisfied.

The same holds if $D \neq -1$. First note that (ii) from Theorem 2 can not be applied because $B = -1$ and $D \neq -1$. Further, for $CD < 0$ and $C \neq 1$ we have

$$\overline{b}^2 - 4 \cdot \overline{a} \cdot \overline{c} = \frac{16CD(A-B)^2(1-CD)^2(1-D)(1+D)(1-C)(1+C)}{(C-D)^2} < 0$$

and $\overline{a} = (1+CD)^2 > 0$, and so $\overline{a}k^2 + \overline{b}k + \overline{c} > 0$ for any $k$. Or, if $CD < 0$ and $C = 1$ then $\overline{a} > 0$, $\overline{b}^2 - 4 \cdot \overline{a} \cdot \overline{c} = 0$ and so $\overline{a}k^2 + \overline{b}k + \overline{c} \leq 0$ only if $k = -\overline{b}/(2\overline{a}) = -(1+A)\frac{1-D}{1+D} < 0$ and (i) can not be satisfied because we are looking for $k \geq 1$.

The case $CD \geq 0$ and $D \neq -1$ is excluded since expression (2.5) can not be true.

In all the remaining cases $\widetilde{k}$ is the lowest $k \geq 1$ that satisfies (i) or (ii) from Theorem 2, i.e., $\widetilde{k} = \max\{1, \widehat{k}\}$, where $\widehat{k}$ is the lowest positive $k$ that satisfies (i) or (ii) from Theorem 2.

So, let $D = -1$ and $CD \neq -1$, i.e., $D = -1$ and $C \neq 1$. Then $\overline{a} = (1-C)^2 > 0$ and $\overline{b}^2 - 4\overline{a}\overline{c} = 0$. Thus, $\overline{a}k^2 + \overline{b}k + \overline{c} \leq 0$ only for $k = -\overline{b}/(2\overline{a}) = (1+A)(1+C)/(1-C)$. For such $k$ we have

$$k_1 \leq kB(1+CD) \quad \Leftrightarrow \quad C \leq 0$$

and
$$kB(1+CD) \le k_2 \quad \Leftrightarrow \quad C \ge 0.$$
Therefore, part (i) from Theorem 2 can not be applied due to the condition $CD < 0$ ($\Rightarrow C \neq 0$). Also, either $k_1 \ge kB(1+CD)$ or $k_2 \le kB(1+CD)$ and by Theorem 2(ii) we obtain $\widehat{k} = \widehat{k_3}$. □

Now we will study a part of case 4: $B \neq 0$, $CD \neq -1$ and $B \neq -1$. We will use the notations
$$\overline{k_1} = \min\left\{\frac{k_1}{B(1+CD)}, \frac{k_2}{B(1+CD)}\right\}, \quad \overline{k_2} = \max\left\{\frac{k_1}{B(1+CD)}, \frac{k_2}{B(1+CD)}\right\}.$$

**Theorem 5.** *Let $A$, $B$, $C$ and $D$ be real numbers such that $-1 \le B < A \le 1$ and $-1 \le D < C \le 1$. Also let $B \neq 0$, $B \neq -1$ and $CD \neq -1$.*

*i) If $CD < 0$ then $\widetilde{k} = \min\{(I_1 \cup I_2) \cap [1, +\infty)\}\}$, where $I_1 = \left[\overline{k_1}, \overline{k_2}\right] \cap \overline{I_1}$ with*
$$\overline{I_1} = \begin{cases} \emptyset, & \text{if } |B| > \frac{C-D}{1-CD} \text{ or } \left(|B| = \frac{2\sqrt{-CD}}{1-CD} \text{ and } B(C+D) \le 0\right) \\ [k_4, k_5], & \text{if } \frac{2\cdot\sqrt{-CD}}{1-CD} < |B| \le \frac{C-D}{1-CD} \\ ((-\infty, k_4] \cup [k_5, +\infty)), & \text{if } |B| < \frac{2\cdot\sqrt{-CD}}{1-CD} \\ [-\overline{c}/\overline{b}, +\infty), & \text{if } |B| = \frac{2\cdot\sqrt{-CD}}{1-CD} \text{ and } B(C+D) > 0 \end{cases}$$
*and $k_4 \le k_5$ are the real roots of the equation $\overline{a}k^2 + \overline{b}k + \overline{c} = 0$. The second interval is*
$$I_2 = ((-\infty, \overline{k_1}] \cup [\overline{k_2}, +\infty)) \cap [\overline{k_3}, +\infty).$$

*ii) If $CD \ge 0$ then $\widetilde{k} = \max\{1, \overline{k_3}\}$.*

**Proof.** (i) Let $CD < 0$. Then it is enough to show that $I_1$ and $I_2$ are the sets of all $k$ satisfying conditions (i) and (ii) of Theorem 2, respectively. Expressions for $I_1$ follow after simple calculus from the fact that
$$I_1 = \left[\overline{k_1}, \overline{k_2}\right] \cap \left\{k : \overline{a}k^2 + \overline{b}k + \overline{c} \le 0\right\}$$

and

$$\overline{b}^2 - 4 \cdot \overline{a} \cdot \overline{c} = \frac{16CD(A-B)^2(1-CD)^2[B^2(1-CD)^2 - (C-D)^2]}{(C-D)^2}.$$

Namely,

$$\overline{b}^2 - 4 \cdot \overline{a} \cdot \overline{c} < 0 \quad \Leftrightarrow \quad B^2(1-CD)^2 - (C-D)^2 > 0 \quad \Leftrightarrow \quad |B| > \frac{C-D}{1-CD}$$

and also

$$B^2(1-CD)^2 - (C-D)^2 > 0 \quad \Rightarrow \quad \overline{a} = 4CD + B^2(1-CD)^2 > 0.$$

Further,

$$|B| > \frac{2\sqrt{-CD}}{1-CD} \quad \Leftrightarrow \quad \overline{a} > 0$$

and

$$B(C+D) > 0 \quad \Leftrightarrow \quad \overline{b} < 0.$$

Set $I_2$ is obvious from Theorem 2(ii).

(ii) If $CD \geq 0$ from Theorem 2(ii) directly follows that $\widetilde{k} = \max\{1, \overline{k_3}\}$.

**Remark 1.** Theorem 5 does not close the problem. For example there may be values of $A$, $B$, $C$ and $D$ when $(I_1 \cup I_2) \cap [1, +\infty) = \emptyset$ but $\widetilde{k}$ can be obtained from part (ii) of Theorem 2. This was too robust to solve and we leave it for further investigation.

**Remark 2.** It can be verified that

$$i_2 \equiv \min\{I_2 \cap [1, \infty)\} = \begin{cases} \overline{k_2}, & \text{if } \overline{k_1} \leq \max\{1, \overline{k_3}\} \leq \overline{k_2} \\ \max\{1, \overline{k_3}\}, & \text{otherwise} \end{cases}.$$

Thus in Theorem 5(i) we can put $\widetilde{k} = \min\{(I_1 \cap [1, +\infty)) \cup i_2\}\}$.

Theorems 3, 4 and 5 yield to a more simple ones for special choices of $A$, $B$, $C$ and $D$. Next is the case $B = D = -1$, $A = 1 - 2\alpha$, $0 \leq \alpha < 1$, and $C = 1 - 2\beta$, $0 \leq \beta < 1$.

**Corollary 1.** *Let* $0 \leq \alpha < 1$ *and* $0 \leq \beta < 1$. *If* $\beta = 0$ *then* $\widetilde{k}$ *does not exist. Otherwise* $\widetilde{k} = \max\{1, \widehat{k}\}$, *where*

$$\widehat{k} = \begin{cases} \frac{2(1-\alpha)(1-\beta)}{\beta}, & 0 < \beta < 1/2 \\ \frac{2\beta(1-\alpha)}{1-\beta}, & 1/2 \leq \beta < 1 \end{cases}.$$

**Remark 3.** For $\alpha = 0$ and $\beta = 1/2$ in Corollary 1 we receive $\widetilde{k} = 2$. This is the classical result obtained by A. Marx, [5].

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