Tamsui Oxford Journal of Mathematical Sciences **26(2)** (2010) 149-159
Aletheia University

The Hardy-Littlewood Inequality for the Solution to P-Harmonic Type System *

Zhenhua Cao, Gejun Bao†, Ronglu Li, Lifeng Guo
Department of Mathematics, Harbin Institute of Technology,
Harbin, 150001, P.R. China

and

Yi Ling‡
Department of Mathematics, The University of Toledo Toledo,
OH 43606, U. S. A.

Received May 7, 2008, Accepted November 6, 2008.

Abstract

Hardy-Littlewood inequality is instrumental in virtually all analytic aspects of the theory of partial differential equations, linear and nonlinear. And conjugate $A-$harmonic tensors, the solutions to conjugate A-harmonic equation, are generalizations of conjugate harmonic functions to differential forms. In this paper, we shall prove the Hardy-Littlewood inequality for the $p-$harmonic type system which is nonhomogeneous conjugate $A-$harmonic equation.

Keywords and Phrases: *Hardy-Littlewood inequality, p-harmonic type system, A-harmonic equation.*

---

*2000 *This work is supported by the NSF of P.R. China.* No.10671046 and No.10771044.
†Corresponding author. E-mail: baogj@hit.edu.cn
‡E-mail: lingyi2001@hotmail.com

1. Introduction

It is well known that the conjugate harmonic functions play very important role in many areas of mathematics such as harmonic analysis, the theory of H^p-spaces and potential theory. Conjugate harmonic functions have lots of analytical properties in common, among which are global L^p-integrability and Hölder continuity. These discoveries essentially began with the work of Hardy and Littlewood in the 1930's (see [1], [2]). And see [3] for an earlier reference on Hölder continuity.

Conjugate $A-$harmonic equation is an important extension of conjugate $p-$harmonic equation which has various applications in many fields, such as potential theory, quasi-regular mappings, and the theory of elasticity. Many interesting results about conjugate $A-$harmonic tensors have been established recently (see [4-7]). In 2004, L. D'Onofrio and T. Iwaniec introduced $p-$harmonic type system in [7], which is an important extension of conjugate $A-$harmonic equation. Now we mention some notions and definitions to p-harmonic type system.

Let $e_1, e_2, ..., e_n$ denote the standard ordered basis of $\mathbb{R}^n$. For $l = 0, 1, .., n$ we denote by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ the linear space of all $l-$vectors, spanned by the exterior product $e_I = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_l}$ corresponding to all ordered $l-$tuples $I = (i_1, i_2, ..., i_l)$, $1 \leq i_1 < i_2 < ... < i_l \leq n$. The Grassmann algebra $\Lambda = \oplus\Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha_I e_I \in \Lambda$ and $\beta = \sum \beta_I e_I \in \Lambda$, then its inner product is obtained by

$$\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I,$$

where the summation taken all $I = (i_1, i_2, ..., i_l)$ and all integers $l = 0, 1, .., n$. The Hodge star operator $*$:$\Lambda \to \Lambda$ is defined by the rule

$$*1 = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_n}$$

and

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1)$$

for all $\alpha, \beta \in \Lambda$. Hence the norm of $\alpha \in \Lambda$ can be given by

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \Lambda_0 = \mathbb{R}.$$

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is an open subset, for any constant $\sigma > 1$, Q denotes a cube such that $Q \subset \sigma Q \subset \Omega$, where σQ denotes the cube which center is as same as Q and $diam(\sigma Q) = \sigma diam Q$. We say $\alpha = \sum \alpha_I e_I \in \Lambda$ is a differential $l-$form on Ω, if every coefficient α_I of α is Schwartz distribution on Ω. And the space spanned by differential $l-$form on Ω denotes by $D^{'}(\Omega, \Lambda^l)$. We write $\int_\Omega f$ short for $\int_\Omega f dx$ and we shall denote $\|du\|^p_{p,Q,\omega^\alpha}$ by $\int_Q |u|^p \omega^\alpha$. We write $L^p(\Omega, \Lambda^l)$ for the $l-$form $\alpha = \sum \alpha_I dx_I$ on Ω with $\alpha_I \in L^p(\Omega)$ for all ordered $l-$tuple I. Thus $L^p(\Omega, \Lambda^l)$ is a Banach space with the norm

$$\|\alpha\|_{p,\Omega} = (\int_\Omega |\alpha|^p)^{1/p} = (\int_\Omega (\sum_I |\alpha_I|^2)^{p/2})^{1/p}.$$

Similarly $W^{k,p}(\Omega, \Lambda^l)$ denotes those $l-$forms on Ω with all coefficients are belong to $W^{k,p}(\Omega)$. We denote the exterior derivative by

$$d : D^{'}(\Omega, \Lambda^l) \to D^{'}(\Omega, \Lambda^{l+1}),$$

and its formal adjoint (the Hodge co-differential) is the operator

$$d^* : D^{'}(\Omega, \Lambda^l) \to D^{'}(\Omega, \Lambda^{l-1}),$$

where operators d and d^* are given by the formulas

$$d\alpha = \sum_I d\alpha_I \wedge dx_I, \text{ and } d^* = (-1)^{nl+1} * d * .$$

Definition1.1.[7]: *(p−harmonic type system) We say the Hodge system*

$$A(x, a + du) = b + d^* v, \tag{1.1}$$

where $a \in L^p(\Omega, \Lambda^l)$ *and* $b \in L^q(\Omega, \Lambda^l)$, *is a p-harmonic type system if* A *is a mapping from* $\Omega \times \Lambda^l$ *to* Λ^l *satisfying:*

1) $x \to A(x, \xi)$ *is measurable in* $x \in \Omega$ *for every* $\xi \in \Lambda^l$

2) $x \to A(x, \xi)$ *is continuous in* $\xi \in \Lambda^l$ *for almost every* $x \in \Omega$

3) $A(x, t\xi) = t^{p-1} A(x, \xi)$ *for every* $t \geq 0$

4) $K\langle A(x, \xi) - A(x, \zeta), \xi - \zeta\rangle \geq |\xi - \zeta|^2 (|\xi| + |\zeta|)^{p-2}$

5) $|A(x,\xi) - A(x,\zeta)| \leq K|\xi - \zeta|(|\xi| + |\zeta|)^{p-2}$

for almost every $x \in \Omega$ *and all* $\xi, \zeta \in \Lambda^l$*, where* $K \geq 1$ *is a constant. It should be noted that* $A(x,*) : \Omega \times \Lambda^l \to \Lambda^l$ *is invertible and its inverse denoted by* A^{-1} *satisfies similar conditions as* A *but with Hölder conjugate exponent* q *in place of* p*.*

Definition1.2.[7]: *(p−harmonic type equation) If the equation (1.1) is a p-harmonic type system,then we say the equation*

$$d^*A(x, a + du) = d^*b \tag{1.2}$$

is a p- harmonic type equation.

Definition1.3.[5]: *A differential form* u *is a weak solution for the equation (1.2) in* Ω *if* u *satisfies*

$$\int_\Omega \langle A(x, a + du), d\varphi\rangle + \langle d^*b, \varphi\rangle \equiv 0 \tag{1.3}$$

for every $\varphi \in W^{k,p}(\Omega, \Lambda^{l-1})$ *with compact support.*

We can find that if we let $a = 0$ and $b = 0$, then the $p-$harmonic type system

$$A(x, a + du) = b + d^*v$$

becomes

$$A(x, du) = d^*v.$$

It is the conjugate $A-$harmonic equation in which $A : \Omega \times \Lambda^l \to \Lambda^l$ is a mapping and satisfies the following conditions

$$|A(x,\xi)| \leq a|\xi|^{p-1} \qquad , \langle A(x,\xi), \xi\rangle \geq |\xi|^p,$$

and if we let $A(x,\xi) = |\xi|^{p-2}\xi$, then conjugate $A-$harmonic equation becomes the form

$$|du|^{p-2}du = d^*v.$$

It is the conjugate $p-$harmonic equation.

So we can see that conjugate $p-$harmonic equation and conjugate $A-$harmonic equation are the specific $p-$harmonic type system.

Before we prove the Hardy-Littlewood inequality for the solution to p-harmonic type system, let us recall the following theorems.

TheoremA.[1]: *For each $p > 0$, there is a constant C such that*

$$\int_{\mathbb{D}} |u - u(0)|^p dxdy \le C \int_{\mathbb{D}} |v - v(0)|^p dxdy$$

for all analytic functions $u + iv$ in the disk $\mathbb{D}$.

TheoremB.[6]: *Let u and v be conjugate $A-$harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$ and $0 < s, t < \infty$. There exists a constant C, independent of u and v, such that*

$$\|u - u_Q\|_{s,Q} \le C|Q|^{\beta} \|v - c_1\|^{q/p}_{t,\sigma Q}$$

and

$$\|v - v_Q\|_{t,Q} \le C|Q|^{-p\beta/q} \|u - c_2\|^{p/q}_{s,\sigma Q}$$

for all cubes Q with $\sigma Q \subset \Omega$, where c_1 is any form in $W^1_{p,loc}(\Omega, \Lambda)$ with $d^* c_1 = 0$, c_2 is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $dc_2 = 0$ and

$$\beta = 1/s + 1/n - q/pt - q/pn.$$

TheoremC.[6]: *Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate $A-$harmonic tensors. If Ω is $\delta -$ John, $q \le p$, $v - c \in L^t(\Omega, \Lambda^2)$ and $s = \phi(t) = npt/(nq - tq - tp)$, then $u - u_{Q_0} \in L^s(\Omega, \Lambda^0)$ and moreover , there exists a constant C, independent of u and v, such that*

$$\|u - u_{Q_0}\|_{s,\Omega} \le C\|v - c\|^{q/p}_{t,\Omega},$$

where c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $dc = 0$ and Q_0 is the distinguished cube in lemma 4.5 of [6].

2. The Local Norm Comparison on L-domain

First of all, we introduce the operator ω_Q. Given $d\omega \in L^p(Q,\Lambda^l)$, $1 \leq p < \infty$, we can construct the closed $l-$form ω_Q which will be used below by the rule: $\omega_Q = \frac{1}{|Q|}\int_Q \omega dx$ for $l = 0$ and $\omega_Q = d(T_Q)$ for the other. By the definition of the operator T, we easily know $\omega_{\sigma Q}|Q = \omega_Q$, for any $\sigma > 1$. The details of the above constructions and results can be found in [5].

For our results we need the following lemma.

Lemma 2.1.[5] *If $\omega \in D^{'}(Q,\Lambda^l)$ and $d\omega \in L^p(Q,\Lambda^{L+1})$, then $\omega - \omega_Q \in W_p^1(Q,\Lambda^l)$ and*

$$\|\omega - \omega_Q\|_{p,Q} \leq C(n,p) diamQ \|d\omega\|_{p,Q}$$

for $1 < p < \infty$. Moreover,

$$\|\omega_Q\|_{p,Q} \leq C_2(n,p)\|\omega\|_{p,Q}.$$

Lemma 2.2.[7] *Let $\Omega \subset \mathbb{R}^n$ be any $L-$domain. The $A-$harmonic system $A(x, a+du) = b+d^*v$ with given $(a,b) \in L^{\lambda p}(\Omega,\Lambda^l) \times L^{\lambda p}(\Omega,\Lambda^l)$, $1 \leq \lambda < \lambda_A$, admits at least one solution $(u,v) \in W_T^{1,\lambda p}(\Omega,\Lambda^{l-1}) \times W^{1,\lambda q}(\Omega,\Lambda^{l+1})$. This solution is unique if Ω is bounded.*

By the lemma 2.1 and lemma 2.2, we have

Lemma 2.3. *If (u,v) is a pair of solution to the $p-$harmonic type system, then we have*

$$\|u - u_Q\|_{p,Q} \leq C(n,p) diamQ \|du\|_{p,Q}. \tag{2.1}$$

We shall use the Caccoippoli estimate and reverse Hölder inequality when we prove the Hardy-Littlewood inequality.

Lemma 2.4.[8] *Let (u,v) be a solution of the $p-$harmonic type system, $\sigma > 1$ is a constant and $Q \subset \sigma Q \subset \Omega$, then we have a constant C only depending on K, l, p, n, such that*

$$\|du\|_{p,Q} \leq C diamQ^{-1}\|u - c\|_{p,\sigma Q} + C\|a\|_{p,\sigma Q},$$

where c is any closed form (i.e. $dc = 0$). Also we have a constant C' only depending on K, n, l, q, such that

$$\|d^*v\|_{q,Q} \le C' diam Q^{-1} \|v - c'\|_{q,\sigma Q} + C' \|b\|_{q,\sigma Q}, \tag{2.2}$$

where c' is any co-closed form (i.e. $d^*c' = 0$) and q is the conjugate exponent of p given by $p^{-1} + q^{-1} = 1$.

Lemma 2.5.[9] *If (u, v) is a pair of a solution to the $p-$harmonic type system, $\sigma > 1$ is any constant then we have a constant C only depending on K, p, n, l, such that*

$$(\frac{1}{|Q|}\int_Q (|a| + |u|)^s dx)^{1/s} \le C(1-\sigma^{-1})^{-t\chi/p(\chi-1)} (\frac{1}{|\sigma Q|}\int_{\sigma Q} (|a| + |u|)^t dx)^{1/t}$$

for any $0 < s, t < \infty$, $\sigma > 1$ and all cubes Q with $Q \subset \sigma Q \subset \Omega$. Also we have a constant C' only depending on K, q, n, l, such that

$$(\frac{1}{|Q|}\int_Q (|b| + |v|)^s dx)^{1/s} \le C'(1-\sigma^{-1})^{-t\chi/q(\chi-1)} (\frac{1}{|\sigma Q|}\int_{\sigma Q} (|b| + |u|)^t dx)^{1/t},$$

where q is the conjugate exponent of p and χ is the Poincaré constant.

By lemma 2.5, we can obtain lemma 2.6.

Lemma 2.6. *If (u, v) is a pair of a solution to the $p-$harmonic type system, $\sigma > 1$ is any given constant then we have a constant C, independent of u, v, a, b and Q, such that*

$$\|u\|_{s,Q} \le C|Q|^{1/s-1/t} (\|u\|_{t,\sigma Q} + \|a\|_{t,\sigma Q}) \tag{2.3}$$

for any $0 < s, t < \infty$ and all cubes Q with $Q \subset \sigma Q \subset \Omega$. Also we have a constant C', independent of u and v, such that

$$\|v\|_{s,Q} \le C|Q|^{1/s-1/t} (\|v\|_{t,\sigma Q} + \|b\|_{t,\sigma Q}) \tag{2.4}$$

for any $0 < s, t < \infty$ and all cubes Q with $Q \subset \sigma Q \subset \Omega$.

Proof. We only prove (2.3) because the proof of (2.4) is similar to (2.3). If $t \ge 1$, then by Minkowski inequality we obtain

$$\|u\|_{s,Q} \le C_1|Q|^{1/s-1/t} (\|u\|_{t,\sigma Q} + \|a\|_{t,\sigma Q}),$$

where $C_1 = C(1-\sigma^{-1})^{-t\chi/p(\chi-1)}\sigma^{-n/t}$ and C is the constant in lemma 2.5. If $0 \le t < 1$, by the elementary inequality

$$(a+b)^p \le a^p + b^p$$

for all $a, b \ge 0$ and $0 \le p < 1$, then we get

$$\left(\int_{\sigma Q}(|a|+|u|)^t dx\right)^{1/t} \le \left(\int_{\sigma Q}(|a|^t + |u|^t)dx\right)^{1/t},$$

and by the elementary inequality

$$(a+b)^p \le 2^p(a^p + b^p)$$

for all $a, b \ge 0$ and $p \ge 0$, then we obtain

$$\||a|+|u|\|_{t,\sigma Q} \le 2^{1/t}(\|u\|_{t,\sigma Q} + \|a\|_{t,\sigma Q}).$$

So we have

$$\|u\|_{s,Q} \le C_2|Q|^{1/s-1/t}(\|u\|_{t,\sigma Q} + \|a\|_{t,\sigma Q}),$$

where $C_2 = 2^{1/t}C_1$. The lemma 2.6 is proved.

Lemma 2.7. *If (u, v) is a pair of a solution to the $p-$harmonic type system and q is the conjugate exponent of p, then we have*

$$K^{-q}|a+du|^p \le |b+d^*v|^q \le K^q|a+du|^p, \tag{2.5}$$

and

$$K^{-p}|b+d^*v|^q \le |a+du|^p \le K^p|b+d^*v|^q. \tag{2.6}$$

Proof. Let (u, v) be a pair of a solution to the $p-$harmonic type system. By 3) in definition 1.1, we have

$$A(x, 0) = 0.$$

And by 4) in definition 1.1, we obtain

$$K\langle b+d^*v, a+du\rangle \ge |a+du|^p.$$

Thus, we can deduce

$$|b+d^*v| \le K|a+du|^{p-1}.$$

That is,

$$|b + d^*v|^q \leq K^q|a + du|^p.$$

Using similar computation , we can obtain

$$|b + d^*v|^q \geq K^{-q}|a + du|^p.$$

(2.5) is proved. The proof of (2.6) is similar to (2.5).

Theorem 2.1. *Let (u, v) be a pair of a solution to the $p-$harmonic type system with $(a, b) \in L^p(\Omega, \Lambda^l) \times L^q(\Omega, \Lambda^l)$ and q be the conjugate exponent of p. if $\sigma > 1$ and $0 < s, t < \infty$, then there exists a constant C, independent of u, v, a, b and Q, such that*

$$\|u - u_Q\|_{s,Q} \leq C|Q|^{\alpha}(\|v - c_1\|_{t,\sigma Q}^{q/p} + \|a\|_{p,\sigma Q} + \|b\|_{q,\sigma Q}^{q/p} + \|b\|_{t,\sigma Q}^{q/p}), \tag{2.7}$$

and

$$\|v - v_Q\|_{t,Q} \leq C|Q|^{-p\alpha/q}(\|u - c_2\|_{s,\sigma Q}^{p/q} + \|b\|_{q,\sigma Q} + \|a\|_{p,\sigma Q}^{p/q} + \|a\|_{s,\sigma Q}^{p/q}) \tag{2.8}$$

for cubes with $Q \subset \sigma Q \subset \Omega$, where c_1 is any co-closed form, c_2 is any closed form, $\alpha = \max(1/s - 1/p + 1/n, 1/s + 1/n - q/tp - q/np)$ for $|Q| > 1$ and $\alpha = \min(1/s - 1/p, 1/s + 1/n - q/tp - q/np)$ for the others.

Proof. Let $\rho = \sigma^{1/3} > 1$. We only prove the inequality (2.7) because (2.8) is similar to (2.7). First by (2.3) in the lemma 2.6, we get

$$\begin{aligned}\|u - u_Q\|_{s,Q} = \|u - u_{\sigma Q}\|_{s,Q} &\leq C|Q|^{1/s-1/p}(\|u - u_{\sigma Q}\|_{p,\rho Q} + \|a\|_{p,\rho Q}) \\ &= C|Q|^{1/s-1/p}(\|u - u_{\rho Q}\|_{p,\rho Q} + \|a\|_{p,\rho Q}).\end{aligned} \tag{2.9}$$

Then by the Poincaré inequality (2.1), we obtain

$$\|u - u_{\rho Q}\|_{p,\rho Q} \leq C(n, p)diam(\rho Q)\|du\|_{p,\rho Q}. \tag{2.10}$$

Combining (2.9) and (2.10) and using Minkowski inequality, we can deduce

$$\begin{aligned}\|u - u_Q\|_{s,Q} \leq &C|Q|^{1/s-1/p}\|a\|_{p,\rho Q} + C|Q|^{1/s-1/p+1/n}\|du\|_{p,\rho Q} \\ \leq &C|Q|^{1/s-1/p}\|a\|_{p,\rho Q} + C|Q|^{1/s-1/p+1/n}\|a\|_{p,\rho Q} \\ &+ C|Q|^{1/s-1/p+1/n}\|a + du\|_{p,\rho Q} \\ \leq &C|Q|^{\beta_1}\|a\|_{p,\sigma Q} + C|Q|^{1/s-1/p+1/n}\|a + du\|_{p,\rho Q},\end{aligned} \tag{2.11}$$

where $\beta = 1/s + 1/n - 1/p$ when $|Q| > 1$ and $\beta = 1/s - 1/p$ for the others (i.e. $\beta = \max(1/s - 1/p, 1/s + 1/n - 1/p)$ when $|Q| \geq 1$ and $\beta = \min(1/s - 1/p, 1/s + 1/n - 1/p)$ for the others). Now using the inequality (2.5) and Minkowski inequality, (2.11) becomes

$$\begin{aligned} \|u - u_Q\|_{s,Q} \leq & C|Q|^{\beta}\|a\|_{p,\sigma Q} + C|Q|^{1/s-1/p+1/n}\|b + d^*v\|_{q,\rho Q}^{q/p} \\ \leq & C|Q|^{\beta}\|a\|_{p,\sigma Q} + C|Q|^{1/s-1/p+1/n}\|b\|_{q,\rho Q}^{q/p} \\ & + C|Q|^{1/s-1/p+1/n}\|d^*v\|_{q,\rho Q}^{q/p} \\ \leq & C|Q|^{\beta}(\|a\|_{p,\sigma Q} + \|b\|_{q,\sigma Q}^{q/p}) + C|Q|^{1/s-1/p+1/n}\|d^*v\|_{q,\rho Q}^{q/p}. \end{aligned} \tag{2.12}$$

By the Caccoippoli estimate (2.2) and the elementary inequality

$$(a + b)^p \leq 2^p(a^p + b^p)$$

for all $a, b \geq 0$ and $p \geq 0$, we have

$$\begin{aligned} \|d^*v\|_{q,\rho Q}^{q/p} &\leq C_1 diam\rho(Q)^{-q/p}\|v - c_1\|_{q,\rho^2 Q}^{q/p} + C_1\|b\|_{q,\rho^2 Q}^{q/p} \\ &\leq C|Q|^{-q/np}\|v - c_1\|_{q,\rho^2 Q}^{q/p} + C\|b\|_{q,\sigma Q}^{q/p}, \end{aligned} \tag{2.13}$$

where c_1 is a co-closed form (i.e. $d^*c_1 = 0$). Combining (2.12) and (2.13), we have

$$\begin{aligned} \|u - u_Q\|_{s,Q} \leq & C|Q|^{\beta}(\|a\|_{p,\sigma Q} + \|b\|_{q,\sigma Q}^{q/p}) \\ & + C|Q|^{1/s-1/p+1/n-q/np}\|v - c_1\|_{q,\rho^2 Q}^{q/p}. \end{aligned} \tag{2.14}$$

And by the inequality (2.4), we can obtain

$$\begin{aligned} \|u - u_Q\|_{s,Q} \leq & C|Q|^{\beta}(\|a\|_{p,\sigma Q} + \|b\|_{q,\sigma Q}^{q/p}) \\ & + C|Q|^{1/s+1/n-q/np-q/pt}(\|v - c_1\|_{t,\sigma Q}^{q/p} + \|b\|_{t,\sigma Q}^{q/p}). \end{aligned} \tag{2.15}$$

Now let $\alpha = \max(1/s - 1/p + 1/n, 1/s + 1/n - q/tp - q/np)$ for $|Q| > 1$ and $\alpha = \min(1/s - 1/p, 1/s + 1/n - q/tp - q/np)$ for the others, then (2.15) becomes

$$\|u - u_Q\|_{s,Q} \leq C|Q|^{\alpha}(\|v - c_1\|_{t,\sigma Q}^{q/p} + \|a\|_{p,\sigma Q} + \|b\|_{q,\sigma Q}^{q/p} + \|b\|_{t,\sigma Q}^{q/p}).$$

The theorem 2.1 is proved.

Remark 2.1. If the $a = 0$ and $b = 0$, then theorem 2.1 becomes theorem B by (2.15).

References

[1] G. H. Hardy and J. E. Littlewood, Some properties of conjugate functions, *J. für Math.* **167**(1931), 405-423.

[2] G. H. Hardy and J. E. Littlewood, Some properties of fractional integral II, *Math. Z.* **34**(1932), 403-439.

[3] I. I. Privalov, Sur les fonctions conjugées, *Bull. Soc. Math. France* **44**(1916), 100-103.

[4] T. Iwaniec, $p-$harmonic tensors and quasiregular mappings, *Ann. of Math.* **136**(1992), 589-624.

[5] T. Iwaniec and A. Lotoborski, Integral estimates for null Lagrangians, *Arch. Rational Mech. Anal.* 125(1993), 25-79.

[6] C. A. Nolder, Hardy-Littlewoood theorems for$A-$harmonic tensors, *Illinois J. Math.* **43** no. 4 (1999), 613–632.

[7] L. D'Onofrio and T. Iwaniec, The $p-$harmonic transform beyond its natural domains of definition, *Indiana Univ. math. j.* **53** no.3 (2004), 683-718.

[8] Zhenhua Cao, Gejun Bao, Ronglu Li and Haijing Zhu, The reverse Hölder inequality for the solution to p-harmonic type system, *Journal of Inequalities and Applications* **2008** (2008), Art. ID 397340, 15 PP.

[9] Zhenhua Cao, Gejun Bao, Ronglu Li and Xiaopan Ma, The Caccoippoli estimate for the solution to p-harmonic type system, *to appear.*