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Classes of Analytic Functions of Complex Order Involving a Family of Generalized Differential Operators*

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Abstract

Using Hadamard product, we define a new class differential operator. By applying this operator, we introduce new subclasses of analytic functions of complex order. Apart from deriving a set of coefficient bounds for each of these function classes. We establish several inclusion relationships involving the (n, δ)-neighborhoods of analytic functions with negative coefficients belonging to these subclasses.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}_p(n)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \quad (p < n;\ n, p \in \mathbb{N} = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and $p-$valent in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : | z |< 1\}$. For functions $f \in \mathcal{A}_p(n)$ of the form (1.1) and $g \in \mathcal{A}_p(n)$ given by $g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k$ their Hadamard product (or convolution) is given by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_k b_k z^k. \tag{1.2}$$

For a fixed function $g \in \mathcal{A}_p(n)$ defined by

$$g(z) = z^p + \sum_{n=k}^{\infty} b_k z^k \quad (p < n;\ b_k \geq 0;\ n, p \in \mathbb{N} = \{1, 2, \dots\}) \tag{1.3}$$

we now define the following operator $D_\lambda^\delta(f * g)(z) : \mathcal{A}_p(n) \longrightarrow \mathcal{A}_p(n)$ by

$$D_\lambda^0(f * g)(z) = (f * g)(z),$$

$$D_\lambda^1(f * g)(z) = (1 - \lambda)(f * g)(z) + \frac{\lambda}{p} z((f * g)(z))' \tag{1.4}$$

$$D_\lambda^\delta(f * g)(z) = D_\lambda^1(D_\lambda^{\delta-1}(f * g)(z)). \tag{1.5}$$

If $f \in \mathcal{A}_p(n)$, then from (1.4) and (1.5) we may easily deduce that

$$D_\lambda^\delta(f * g)(z) = z^p + \sum_{k=n}^{\infty} \left[1 + \frac{\lambda(k - p)}{p}\right]^\delta a_k b_k z^k \tag{1.6}$$

where $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$.

Remark 1.1. It is interesting to note that several integral and differential operator follows as a special case of $D_\lambda^\delta(f * g)(z)$, here we list few of them.

1. When $p = 1$, $n = 2$ and $g(z) = z + \sum_{k=2}^{\infty} z^k$, $D_\lambda^\delta(f * g)(z)$ reduces to an operator introduced recently by F. Al-Oboudi [1].

2. Let the coefficients b_k be of the form

$$b_k = \frac{(\alpha_1)_{k-p} \ \ldots \ (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \ \ldots \ (\beta_s)_{k-p} \, (k-p)!} \tag{1.7}$$

   and if $\delta = 0$ then $D_\lambda^\delta(f*g)(z)$ reduces to the well-known Dziok-Srivastava operator (see for details [3, 4, 5]) which contains such well-known operators as the Hohlov linear operator [6], Carlson-Shaffer linear operator [2] and Ruscheweyh derivative[11] .

Apart from these,the operator $D_\lambda^\delta(f * g)(z)$ generalizes the well-known operators like Sălăgean operator [10], Bernardi-Libera-Livingston operator.

By making use of the operator $D_\lambda^\delta(f * g)(z)$, we introduce a new class $\mathcal{S}_p^\delta(g; \lambda, n, b, m)$ of functions belonging to the subclasses of $\mathcal{A}_p(n)$, which consists of functions $f(z)$ of the form (1.1), satisfying the following inequality:

$$\left| \frac{1}{b}\left(\frac{z D_\lambda^\delta (f*g)^{m+1}(z)}{D_\lambda^{\delta+1}(f*g)^m(z)} - (p-m) \right) \right| < \gamma, \tag{1.8}$$

$$\big(z \in \mathcal{U},\ m, \delta \in \mathbb{N}_0,\ 0 \le \lambda \le 1,\ 0 < \gamma \le 1,\ b \in \mathbb{C} \setminus \{0\} \big).$$

Further we define $\mathcal{TS}_p^\delta(g; \lambda, n, b, m) = \mathcal{S}_p^\delta(g; \lambda, n, b, m) \cap T_p(n)$, where $T_p(n)$ is the subclass of $\mathcal{A}_p(n)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k, \qquad a_k \ge 0,\ \forall\, k \ge n \tag{1.9}$$

was introduced and studied by Silverman.

Finally, for a fixed function

$$g(z) = z - \sum_{k=n}^{\infty} b_k z^k \in T_p(n) \quad (p < n;\ b_k \ge 0,\ \forall k \ge n;\ n, p \in \mathbb{N}, \tag{1.10}$$

let $\mathcal{Q}_p^\delta(g, \lambda, n, b, m, \gamma)$ denote the subclass of $T_p n$ consisting of functions $f(z)$ of the form (1.9) which satisfies the following inequality

$$\left| \frac{1}{b}\left(\frac{D_\lambda^\delta (f*g)^m(z)}{z^{p-m}} \right) - (p-m) \right| < \gamma \tag{1.11}$$

$$\big(z \in \mathcal{U},\ m, \delta \in \mathbb{N}_0,\ 0 \le \lambda \le 1,\ 0 < \gamma \le 1,\ b \in \mathbb{C} \setminus \{0\}\big).$$

The purpose of this present paper is to investigate the various properties and characteristics of the functions belonging to the above defined subclasses $\mathcal{TS}_p^{\delta}(g;\lambda,n,b,m)$ and $\mathcal{Q}_p^{\delta}(g,\lambda,n,b,m,\gamma)$ of p-valent analytic functions in $\mathcal{U}$. Apart from deriving coefficient inequality for each of this function classes, we establish several inclusion relationships involving (n,δ)-neighborhoods of the functions belonging to these subclasses.

2. Main Results

We begin with the following

Theorem 2.1. *(coefficient inequality)* $f(z) \in \mathcal{TS}_p^{\delta}(g;\lambda,n,b,m)$ *if and only if*

$$\sum_{k=n}^{\infty}\binom{k}{m}\left[1+\frac{\lambda(k-p)}{p}\right]^{\delta}\left\{(k-m)-\left[1+\frac{\lambda(k-p)}{p}\right]\big[p-m+\gamma|b|\big]\right\}a_kb_k \le \gamma|b|\binom{p}{m} \tag{2.1}$$

Proof. Let $f(z) \in \mathcal{TS}_p^{\delta}(g;\lambda,n,b,m)$. Then in view of the inequality (1.8), we get

$$Re\left(\frac{zD_{\lambda}^{\delta}(f*g)^{m+1}(z)-(p-m)D_{\lambda}^{\delta+1}(f*g)^{m}(z)}{D_{\lambda}^{\delta+1}(f*g)^{m}(z)}\right) > -\gamma|b|. \tag{2.2}$$

A simple computation yields

$$Re\left(\frac{\sum_{k=n}^{\infty}\binom{k}{m}\left[1+\frac{\lambda(k-p)}{p}\right]^{\delta}\left((k-m)-(p-m)\left[1+\frac{\lambda(k-p)}{p}\right]\right)a_k\,b_kz^{k-m}}{\binom{p}{m}z^{p-m}-\sum_{k=n}^{\infty}\left[1+\frac{\lambda(k-p)}{p}\right]^{\delta+1}\binom{k}{m}a_k\,b_kz^{k-m}}\right) < \gamma\mid b\mid . \tag{2.3}$$

Upon choosing the values of z on the positive real axis where $0 \le |z| = r < 1$ and by letting $r \longrightarrow 1^{-}$, through real values the above inequality leads to the desired assertion (2.1) of Theorem 2.1.

Conversely by applying the hypothesis (2.1) of Theorem 2.1 and letting $|z| = 1$, we find that

$$\left|\frac{zD_{\lambda}^{\delta}(f*g)^{m+1}(z)}{D_{\lambda}^{\delta+1}(f*g)^{m}(z)}-(p-m)\right| \le \frac{\gamma\mid b\mid\left\{\binom{p}{m}-\sum_{k=n}^{\infty}\left[1+\frac{\lambda(k-p)}{p}\right]^{\delta+1}\binom{k}{m}a_k\,b_k\right\}}{\binom{p}{m}-\sum_{k=n}^{\infty}\left[1+\frac{\lambda(k-p)}{p}\right]^{\delta+1}\binom{k}{m}a_k\,b_k} =\mid b\mid . \tag{2.4}$$

Hence, by the maximum modulus principle, we infer that $f(z) \in \mathcal{TS}_p^{\delta}(g; \lambda, n, b, m)$, which completes the proof of Theorem 2.1.

The following result concerning the class of functions $\mathcal{Q}_p^{\delta}(g, \lambda, n, b, m, \gamma)$ can be proved on the similar lines as given above for the Theorem 2.1.

Theorem 2.2. *$f(z) \in \mathcal{Q}_p^{\delta}(g, \lambda, n, b, m, \gamma)$ if and only if*

$$\sum_{k=n}^{\infty} \binom{k}{m} \left[1 + \frac{\lambda(k-p)}{p}\right]^{\delta} a_k b_k \leq \left[(p-m) + \gamma|b| - \binom{p}{m}\right] \tag{2.5}$$

In Theorem 2.1, let $m = \delta = 0$, $p = 1$ and b_k be defined as in 1.8, then we have the following result proved recently by Murugusundaramoorthy et.al.[8].

Corollary 2.3. *$f(z) \in \mathcal{TS}(q, s; \lambda, n, b, m, \gamma)$ if and only if*

$$\sum_{k=n}^{\infty} \left[\big(1 + \lambda(k-1)\big)\big(\gamma|b| - 1\big) + k\right] \Gamma_k a_k \leq \gamma|b| \tag{2.6}$$

where $\Gamma_k = \dfrac{(\alpha_1)_{k-p} \ \ldots \ (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \ \ldots \ (\beta_s)_{k-p} \, (k-p)!}$.

Suitably specializing the parameters $\delta, \lambda, \gamma, p, m, n$ and for the choice of the fixed function $g(z)$ in Theorem 2.1 yields the coefficient inequalities for various new and known subclasses of analytic functions.

3. Inclusion Properties

Following Murugusundaramoorthy et. al. [8] and B.A.Frasin [7] (also see [9]), we define the $q - \delta$ neighborhood of a function $f \in T_p(n)$ by

$$N^q_{n\,\delta}(f) = \left\{ h \in T_p(n) \ : \ h(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \quad \text{and} \quad \sum_{k=n}^{\infty} k^{q+1} \mid a_k - c_k \mid \leq \delta \right\}. \tag{3.1}$$

So, for $e(z) = z^p$, we see that

$$N^q_{n\,\delta}(e) = \left\{ h \in T(n) \ : \ h(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \quad \text{and} \quad \sum_{k=n}^{\infty} k^{q+1} \mid c_k \mid \leq \delta \right\}. \tag{3.2}$$

where q is a fixed positive integer. Note that $N^{o}_{n\,\delta}(f) = N_{n\,\delta}(f)$, which was defined by Murugusundaramoorthy et.al.[8].

Theorem 3.1. *If* $b_k \geq b_n$ $(k \geq n)$ *and*

$$\delta = \frac{n\,\gamma|b|\binom{p}{m}}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left\{(n-m)-\left(1+\frac{\lambda(n-p)}{p}\right)[p-m+\gamma|b|]\right\}b_n} \tag{3.3}$$

then $\mathcal{TS}^{\delta}_{p}(g;\lambda,n,b,m) \subset N^{0}_{n\,\delta}(f)$.

Proof. Let $f(z) \in \mathcal{TS}^{\delta}_{p}(g;\lambda,n,b,m)$. Then, in view of the assertion (2.1) of Theorem 2.1 and the given condition $b_k \geq b_n$ $(k \geq n)$, we get

$$\begin{aligned}&\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left\{(n-m)-\left(1+\frac{\lambda(n-p)}{p}\right)[p-m+\gamma|b|]\right\}b_n\sum_{k=n}^{\infty}a_k\\ &\leq\sum_{k=n}^{\infty}\binom{k}{m}\left[1+\frac{\lambda(k-p)}{p}\right]^{\delta}\left\{(k-m)-\left[1+\frac{\lambda(k-p)}{p}\right][p-m+\gamma|b|]\right\}a_kb_k \leq \gamma|b|\binom{p}{m},\end{aligned}$$

which implies that

$$\sum_{k=n}^{\infty}a_k \leq \frac{\gamma|b|\binom{p}{m}}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left\{(n-m)-\left(1+\frac{\lambda(n-p)}{p}\right)[p-m+\gamma|b|]\right\}b_n}. \tag{3.4}$$

Applying the assertion (2.1) of Theorem 2.1 again in conjunction (3.4), we obtain

$$\begin{aligned}&\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}b_n\sum_{k=n}^{\infty}ka_k\\ &\leq\gamma|b|\binom{p}{m}+\left\{m\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}+\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta+1}[p-m+\gamma|b|]\right\}b_n\sum_{k=n}^{\infty}a_k\end{aligned}$$

$$\le \gamma|b|\binom{p}{m} + \left\{ m\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta} + \left[1+\frac{\lambda(n-p)}{p}\right]^{\delta+1}\left[p-m+\gamma|b|\right]\right\} b_n$$

$$\times \frac{\gamma|b|\binom{p}{m}}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left[(n-m)-\left(1+\frac{\lambda(n-p)}{p}\right)\left[p-m+\gamma|b|\right]\right]b_n}$$

$$= \frac{n\,\gamma|b|\binom{p}{m}}{\left[(n-m)-\left(1+\frac{\lambda(n-p)}{p}\right)\left[p-m+\gamma|b|\right]\right]}.$$

Hence,

$$\sum_{k=n}^{\infty} k a_k \le \frac{n\,\gamma|b|\binom{p}{m}}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left[(n-m)-\left(1+\frac{\lambda(n-p)}{p}\right)\left[p-m+\gamma|b|\right]\right]b_n}$$

$$= \delta \ \ (p >| b |), \tag{3.5}$$

which by virtue of (3.2) establishes the inclusion relation of Theorem 3.1.

Theorem 3.2. *If* $b_k \ge b_n$ $(k \ge n)$ *and*

$$\delta = \frac{n\left((p-m)+\gamma|b|-\binom{p}{m}\right)}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta} b_n} \tag{3.6}$$

then $\mathcal{Q}_p^{\delta}(g,\lambda,n,b,m,\gamma) \subset N_{n\,\delta}^{0}(f)$.

4. Neighborhood Properties

In this section we determine the neighborhood properties for each of the function classes $\mathcal{S}_p^{(\delta,\alpha)}(g;\lambda,n,b,m)$ and $\mathcal{Q}_p^{(\delta,\alpha)}(g,\lambda,n,b,m,\gamma)$, which are defined as follows.

A function $f(z) \in T_p(n)$ is said to be in the class $\mathcal{S}_p^{(\delta,\alpha)}(g;\lambda,n,b,m)$ if there exists a function $h(z) \in \mathcal{S}_p^{\delta}(g;\lambda,n,b,m)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right| < p-\alpha \qquad (z \in \mathcal{U};\ 0 \le \alpha < p). \tag{4.1}$$

Analogously, a function $f(z) \in T_p(n)$ is said to be in the class $\mathcal{Q}_p^{(\delta,\alpha)}(g,\lambda,n,b,m,\gamma)$ if there exists a function $h(z) \in \mathcal{Q}_p^{\delta}(g,\lambda,n,b,m,\gamma)$ such that inequality (4.1) holds true.

Theorem 4.1. *If* $g(z) \in \mathcal{S}_p^{\delta}(g;\lambda,n,b,m)$ *and*

$$\alpha = p - \frac{\delta}{n^{q+1}} \frac{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left\{(n-m)-\left[1+\frac{\lambda(n-p)}{p}\right](p-m+\gamma \mid b \mid)\right\}b_n}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left\{(n-m)-\left[1+\frac{\lambda(n-p)}{p}\right](p-m+\gamma \mid b \mid)\right\}b_n - \gamma \mid b \mid \binom{p}{m}} \tag{4.2}$$

then

$$N_{n,\delta}^{q}(h) \subset \mathcal{S}_p^{(\delta,\alpha)}(g;\lambda,n,b,m). \tag{4.3}$$

Proof. Suppose that $f(z) \in N_{n,\delta}^{q}(h)$, we then find from (3.1) that

$$\sum_{k=n}^{\infty} k^{q+1} \mid a_k - c_k \mid \leq \delta, \tag{4.4}$$

which readily implies that

$$\sum_{k=n}^{\infty} \mid a_k - c_k \mid \leq \frac{\delta}{n^{q+1}}, \ (n \in \mathbb{N}) \tag{4.5}$$

Next, since $h(z) \in \mathcal{S}_p^{\delta}(g;\lambda,n,b,m)$, we have in view of (3.4) that

$$\sum_{k=n}^{\infty} c_k \leq \frac{\gamma \mid b \mid \binom{p}{m}}{\binom{n}{m}\left[1+\frac{\lambda(n-p)}{p}\right]^{\delta}\left\{(n-m)-\left[1+\frac{\lambda(n-p)}{p}\right](p-m+\gamma \mid b \mid)\right\}b_n} \tag{4.6}$$

So that $\left|\dfrac{f(z)}{g(z)} - 1\right|$

$$\leq \frac{\sum_{k=n}^{\infty} \mid a_k - c_k \mid}{1 - \sum_{k=n}^{\infty} c_k}$$

$$\leq \frac{\delta}{n^{q+1}} \frac{1}{1 - \gamma \mid b \mid \binom{p}{m} / \binom{n}{m} \left[1 + \frac{\lambda(n-p)}{p}\right]^{\delta} \left\{(n-m) - \left[1 + \frac{\lambda(n-p)}{p}\right](p - m + \gamma \mid b \mid) \right\} b_n}$$

$$\leq \frac{\delta}{n^{q+1}} \frac{\binom{n}{m} \left[1 + \frac{\lambda(n-p)}{p}\right]^{\delta} \left\{(n-m) - \left[1 + \frac{\lambda(n-p)}{p}\right](p - m + \gamma \mid b \mid) \right\} b_n}{\binom{n}{m} \left[1 + \frac{\lambda(n-p)}{p}\right]^{\delta} \left\{(n-m) - \left[1 + \frac{\lambda(n-p)}{p}\right](p - m + \gamma \mid b \mid) \right\} b_n - \gamma \mid b \mid \binom{p}{m}},$$

provided that α is given by (4.3). Thus, by the above definition, $f \in \mathcal{S}_p^{(\delta,\alpha)}(g; \lambda, n, b, m)$ where α is given by (4.3), which proves Theorem 4.1.

Theorem 4.2. *If* $g(z) \in \mathcal{Q}_p^{\delta}(g, \lambda, n, b, m, \gamma)$ *and*

$$\alpha = p - \frac{\delta}{n^{q+1}} \frac{\binom{n}{m} \left[1 + \frac{\lambda(n-p)}{p}\right]^{\delta} b_n}{\binom{n}{m} \left[1 + \frac{\lambda(n-p)}{p}\right]^{\delta} b_n - \left((p-m) + \gamma |b| - \binom{p}{m}\right)} \tag{4.7}$$

then

$$N_{n,\delta}^{q}(h) \subset \mathcal{Q}_p^{(\delta,\alpha)}(g, \lambda, n, b, m, \gamma). \tag{4.8}$$

Remark. By specializing the parameters involved, Theorems (4.1) and (4.2) reduces to various results obtained by several authors. For example, Let $p = 1$, $\delta = 1$, $m = 0$ and b_k be defined as in (1.7) then Theorems (4.1) and (4.2) reduces to the corresponding results obtained by Murugusundaramoorthy et.al. [8].

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