Some Applications of the q-Mellin Transform

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Abstract

In this paper we show that the q-integral transforms can be used to solve some q-differential difference equation.

1. Introduction

It is well known that one of the purposes of integral transforms like Fourier, Laplace and Mellin is to solve differential equations like the Heat equation the wave equation and some EDDs.

Even though in [1] the author solves some q-differential equations using the Mellin transform, it seems more appropriate to use the q-Mellin transform, that is why in this paper the q-Mellin transform studied in [3] is used to solve some analogous Heat and Wave equations.

This paper is organized as follows :

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- In section 2 we present some preliminary results and notations.

- in sections 3, 4 and 5 we solve respectively the q-diffution, the q-wave and the ∂_q-diffusion equations using the q-Mellin transform.

2. Notations and Preliminaries

We take q in $]0,1[$ throughout this paper .

In this section we provide a summary of the mathematical notations and definitions used in this papaer (see [4] and [6]).

Let's take $a \in \mathbb{C}$, we have

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-derivative $D_q f$, of a function f is given by

$$\begin{cases} (D_q f)(x) = \frac{f(q^{-1}x)-f(x)}{(1-q)x}, & \text{for} \quad x \neq 0 \\ (D_q f)(0) = q^{-1} f'(0), & \text{when } f'(0) \text{ exists.} \end{cases} \tag{1}$$

The q-Jackson integrals are defined by (see [5])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2}$$

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} f(q^n) q^n, \tag{3}$$

$$\int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} \left\{ f(q^n) + f(-q^n) \right\} q^n. \tag{4}$$

Using these q-integrals, we set for $p > 0$,

$$L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty. \right\} \tag{5}$$

A q-analogue of the exponential function is given by [4]

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{[n]_q!}. \tag{6}$$

The q-sine and the q-cosine associated to the q-exponential E_q^z, are given by

$$Sin_q(x) = \frac{E_q^{ix} - E_q^{-ix}}{2i}, \quad Cos_q(x) = \frac{E_q^{ix} + E_q^{-ix}}{2}. \tag{7}$$

Jackson [5] defined the q-analogue of the Gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0, -1, -2, \ldots. \tag{8}$$

which have the following properties :

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x), \ \Re(x) > 0. \tag{9}$$

It has the q-integral representations [2]

$$\Gamma_q(s) = \int_0^{\frac{1}{1-q}} t^{s-1}E_q^{-qt}d_qt. \tag{10}$$

When $\dfrac{\log(1-q)}{\log(q)} \in \mathbb{Z}$, we have

$$\Gamma_q(s) = \int_0^\infty t^{s-1}E_q^{-qt}d_qt. \tag{11}$$

The q-Mellin transform of a suitable function f on $\mathbb{R}_{q,+}$ is given by [3]

$$M_q(f)(s) = \int_0^{+\infty} t^{s-1}f(t)d_qt. \tag{12}$$

The inversion formula for the q-Mellin transform is given by [3]

$$\forall x \in \mathbb{R}_{q,+}, \quad f(x) = \frac{logq}{2i\pi(1-q)} \int_{c-\frac{i\pi}{logq}}^{c+\frac{i\pi}{logq}} M_q(f)(s)x^{-s}ds. \tag{13}$$

In [7, 8] Rubin defined a q-analogue of differential operators by

$$\partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}. \tag{14}$$

We notice that if f is differentiable at z, then

$$\lim_{q \to 1} \partial_q(f)(z) = f'(z).$$

The operator ∂_q is closely related to the classical q-derivative operators studied in [4, 6].
The q-trigonometric functions q-cosine and q-sine are defined by (see [7, 8]):

$$\cos(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!} \tag{15}$$

and

$$\sin(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}. \tag{16}$$

These functions induce a ∂_q-adapted q^2-analogue exponential function by

$$e(z; q^2) = \cos(-iz; q^2) + i \sin(-iz; q^2). \tag{17}$$

In [8], R. L. Rubin defines the q^2-analogue Fourier transform by

$$\widehat{f}(x; q^2) = \mathcal{F}_q(f)(x) = K \int_{-\infty}^{\infty} f(t) e(-itx; q^2) d_q t, \quad x \in \widetilde{\mathbb{R}}_q, \tag{18}$$

where

$$K = \frac{(q; q^2)_\infty}{2(q^2; q^2)_\infty (1-q)^{1/2}}.$$

The following conditions

$$\begin{cases} q \to 1 \\ \frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z} \end{cases} \tag{19}$$

gives, at least formally, the classical Fourier transform (see [6]).
In this paper, we assume that $\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}$.
It has been shown in ([7] and [8]) that the q^2-analogue Fourier transform $\mathcal{F}_q$ verifies the following properties:

(1) If $f(u)$ and $uf(u) \in L_q^1(\mathbb{R}_q)$, then

$$\partial_q(\mathcal{F}_q f)(x) = \mathcal{F}_q(-iuf(u))(x).$$

(2) If f and $\partial_q f \in L_q^1(\mathbb{R}_q)$, then

$$\mathcal{F}_q(\partial_q f)(x) = ix\mathcal{F}_q(f)(x). \tag{20}$$

(3) For $f \in L_q^2(\mathbb{R}_q)$, we have

$$f(t) = K \int_{-\infty}^{\infty} \mathcal{F}_q(f)(x)e(itx; q^2)d_q x \ , \ \forall t \in \mathbb{R}_q. \tag{21}$$

3. q-diffusion Equation

Let's consider the following q-diffusion equation

$$D_{q,t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) \ , \ x \in]-\infty, +\infty[\quad \text{and} \quad t \in \mathbb{R}_{q,+} \tag{22}$$

with the initial condition

$$u(x,0) = f(x). \tag{23}$$

We assume that $f \in L^1$, $\hat{f} \in L^1$, where

$$\hat{h}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x)e^{iyx}dx.$$

By taking a Fourier transform in x and a q-Mellin transform in t, the equation (22) becomes

$$[s-1]_q U(\xi, s-1) = \xi^2 U(\xi, s). \tag{24}$$

A solution of the equation (24) is given by [1]

$$U(\xi, s) = A(\xi)\xi^{-2s}\Gamma_q(s), \tag{25}$$

where $A(\xi)$ is a function of ξ only.

According to the relation (11), the inversion q-Mellin transform of $\xi^{-2s}\Gamma_q(s)$ is

$$\frac{logq}{2i\pi(1-q)} \int_{c-i\frac{\pi}{logq}}^{c+i\frac{\pi}{logq}} \xi^{-2s}\Gamma_q(s)t^{-s}ds = \frac{logq}{2i\pi(1-q)} \int_{c-i\frac{\pi}{logq}}^{c+i\frac{\pi}{logq}} \Gamma_q(s)(\xi^2 t)^{-s}ds = E_q^{-q\xi^2 t} \tag{26}$$

it follows that

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(\xi) E_q^{-q\xi^2 t} e^{-i\xi x} d\xi. \tag{27}$$

For $t = 0$, we get

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(\xi) e^{-i\xi x} d\xi = f(x) \tag{28}$$

then

$$A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{i\xi x} dx = \hat{f}(\xi). \tag{29}$$

Therefore, a solution of (22) is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\xi) E_q^{-q\xi^2 t} e^{-i\xi x} d\xi. \tag{30}$$

Conversely, from the relation $D_{q,t} E_q^{\lambda t} = \frac{\lambda}{q} E_q^{\lambda t}$, we conclude that (30) satisfies (22).

Remark. The equation (28) is well justified as $\hat{f} \in L^1$.

4. q-wave Equation

We consider now the following q-wave Equation

$$(D_{q,t})^2 u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \ , \ x \in]-\infty,+\infty[\text{ and } t \in \mathbb{R}_{q,+} \tag{31}$$

with the initial conditions

$$u(x,0) = f(x), \quad D_{q,t} u(x,0) = g(x).$$

We assume the following conditions

$$\begin{cases} f,g \in L^1 \\ \hat{f} \text{ and } \hat{g} \text{ have compact supports.} \end{cases} \tag{32}$$

By applying the Fourier and the q-Mellin transform, we obtain

$$[s-1]_q[s-2]_q U(\xi, s-2) = -\xi^2 U(\xi, s). \tag{33}$$

This is a solution of (33) [1]

$$U(\xi, s) = \left[A(\xi)(-i\xi)^{-s} + B(\xi)(i\xi)^{-s} \right] \Gamma_q(s) \qquad (34)$$

where $A(\xi)$ and $B(\xi)$ are functions of ξ only.

¿From the q-Mellin inversion formula, we get

$$\hat{u}(\xi, t) = \left[A(\xi) E_q^{iq\xi t} + B(\xi) E_q^{-i\xi t} \right], \qquad (35)$$

where $\hat{u}(\xi, t)$ is the Fourier transform of $u(x, t)$ with respect to x.

It follows from the relations (7) that

$$\hat{u}(\xi, t) = C(\xi) Cos_q(q\xi t) + D(\xi) Sin_q(q\xi t) \qquad (36)$$

where $C(\xi)$ and $D(\xi)$ are functions of ξ.

Now, the inverse-Fourier transform of (36) gives

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(C(\xi) Cos_q(q\xi t) + D(\xi) Sin_q(q\xi t) \right) e^{-i\xi x} d\xi. \qquad (37)$$

By taking $t = 0$ in (37), we get $C(\xi) = \hat{f}(\xi)$.

On the other hand, by using the relations

$$D_{q,t} Sin_q(\lambda t) = \frac{\lambda}{q} Cos_q(\lambda t),$$

and

$$D_{q,t} Cos_q(\lambda t) = -\frac{\lambda}{q} Sin_q(\lambda t)$$

we get

$$\hat{g}(\xi) = D(\xi)\xi. \qquad (38)$$

Therefore the final solution of (31) is

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\hat{f}(\xi) Cos_q(q\xi t) + \frac{\hat{g}(\xi)}{\xi} Sin_q(q\xi t) \right) e^{-i\xi x} d\xi. \qquad (39)$$

Remark. The same as in section 2, when the conditions (32) are true, we can interchange the limit $(t \to 0)$ and the integral by applying the dominated convergence theorem.

5. ∂_q-diffusion Equation

In this section, we study the following q-equation by using the ∂_q-operator introduced by Rubin [8, 7].

$$D_{q,t}u(x,t) = (\partial_{q,x})^2 u(x,t) \ , \ x \in \mathbb{R}_q \ , \ t \in \mathbb{R}_{q,+} \tag{40}$$

with the initial condition

$$u(x,0) = f(x), f \in L_q^2(\mathbb{R}_q). \tag{41}$$

By taking a q^2-Fourier transform respecting to x and a q-Mellin transform respecting to t, we get

$$[s-1]_q U(\xi, s-1) = \xi^2 U(\xi, s), \tag{42}$$

thus

$$U(\xi, s) = A(\xi)\xi^{-2s}\Gamma_q(s), \tag{43}$$

where $A(\xi)$ is a function of ξ only.

According to the equation (26) the inversion q-Mellin transform of $\xi^{-2s}\Gamma_q(s)$ is $E_q^{-q\xi^2 t}$.

Then

$$u(x,t) = K \int_{-\infty}^{+\infty} A(\xi)E_q^{-q\xi^2 t}e(-i\xi x, q^2)d_q\xi. \tag{44}$$

For $t = 0$, we obtain

$$u(x,0) = K \int_{-\infty}^{+\infty} A(\xi)e(-i\xi x, q^2)d_q\xi = f(x), \tag{45}$$

so

$$A(\xi) = K \int_{-\infty}^{+\infty} f(x)e(-i\xi x, q^2)d_q x = \hat{f}(\xi, q^2), \tag{46}$$

and the final solution of (40) is

$$u(x,t) = K \int_{-\infty}^{+\infty} \hat{f}(\xi, q^2)E_q^{-q\xi^2 t}e(-i\xi x, q^2)d_q\xi. \tag{47}$$

Conversely, from the relation $D_{q,t}E_q^{\lambda t} = \frac{\lambda}{q}E_q^{\lambda t}$, we conclude that (47) satisfies (40).

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