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Paley-Wiener Theorem for the q^2-Fourier-Rubin Transform*

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Abstract

In this paper we deal with the q-translation related with the q^2-derivative and we show that it can be expressed as a series of Taylor's type and proceeding as in [6, 7] and we caracterize the correspondent Paley-Wiener space.

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1. Introduction

R.L.Rubin in [4, 5] introduced a q^2-derivative for which he established a constructive q^2-Fourier transform. The aim of this is to complete the q-Fourier analysis elaborated by the previous authors in studying the q-analogues of some basic theorems with the same technic that those used in [6, 7]. More precisely we state some new properties of the q^2-Fourier Rubin transform and show that its associated q^2-translation. We prove that if f run in the q- analogue of the Scharwtz space then its q-translation can be expand in series involving the powers of the q-Rubin derivative of f. This last result plays a central role for the study of the related q-Paley-Wiener space and the afferent theorem.

2. Notations and Preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. For this purpose, we fix $q \in]0,1[$ and we refer to the book by G. Gasper and M. Rahman [1], for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.
Note
$\mathbb{R}_q = \{\pm q^n \quad : \quad n \in \mathbb{Z}\}$, $\widetilde{\mathbb{R}}_q = \{\pm q^n \quad : \quad n \in \mathbb{Z}\} \cup \{0\}$.
For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2,\ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1}$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C} \;\; ; \;\; [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}. \tag{2}$$

The q^2-analogue differential operator is (see [4, 5])

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & if \quad z \neq 0 \\ \lim\limits_{x \to 0} \partial_q(f)(x) \qquad (\text{in } \; \mathbb{R}_q) & if \quad z = 0. \end{cases} \tag{3}$$

Remark that if f is differentiable at z, then $\lim_{q\to 1} \partial_q(f)(z) = f'(z)$.
A repeated application of the q^2-analogue differential operator is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

The following lemma lists some useful computational properties of ∂_q.

Lemma 1.
1) For all function f on $\mathbb{R}_q$, $\partial_q f(z) = \dfrac{f_e(q^{-1}z) - f_e(z)}{(1-q)z} + \dfrac{f_o(z) - f_o(qz)}{(1-q)z}$.
2) For two functions f and g, we have

- *if f is even and g is odd then*

$$\partial_q(fg)(z) = qg(z)\partial_q(f)(qz) + f(qz)\partial_q(g)(z) = f(z)\partial_q(g)(z) + qg(qz)\partial_q(f)(qz);$$

- *if f and g are even then*

$$\partial_q(fg)(z) = g(q^{-1}z)\partial_q(f)(z) + f(z)\partial_q(g)(z).$$

- *if f and g are odd then*

$$\partial_q(fg)(z) = q^{-1}g(q^{-1}z)\partial_q(q^{-1}f)(z) + q^{-1}f(z)\partial_q(g)(q^{-1}z),$$

where, for a function f defined on $\mathbb{R}_q$, f_e and f_o are, respectively, its even and odd parts.

The q-trigonometric functions q-cosine and q-sine are defined by (see [4, 5]):

$$\cos(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!} \tag{4}$$

and

$$\sin(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}. \tag{5}$$

These functions induce a ∂_q-adapted q^2-analogue exponential function as

$$e(z; q^2) = \cos(-iz; q^2) + i \sin(-iz; q^2). \tag{6}$$

Remark that $e(z;q^2)$ is absolutely convergent for all z in the complex plane since both of its component functions are. Moreover, $\lim_{q\to 1^-} e(z;q^2) = e^z$ (exponential function) pointwise and uniformly on compacts.
Using the same technique as in [4], one can prove that for all $x \in \mathbb{R}_q$, we have

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty} \quad \text{and} \quad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty},$$

so,

$$\forall x \in \mathbb{R}_q, \quad |e(-ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{7}$$

The q-Jackson integrals are defined by (see [3])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty} q^n \left[bf(bq^n) - af(aq^n)\right],$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n), \quad \int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n \left[f(q^n) + f(-q^n)\right],$$

provided the sums converge absolutely.
Using this q-integrals, we note for $p > 0$,

- $L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}$.
- $L_q^p([-a,a]) = \left\{ f : \|f\|_{p,q} = \left(\int_{-a}^{a} |f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}$.
- $L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}$.

By the use of the q^2-analogue differential operator ∂_q, we note
• $\mathcal{S}_q(\mathbb{R}_q)$ the space of infinitely q-differentiable and fast decreasing functions and all its q-derivatives on $\mathbb{R}_q$ **i.e.**

$$\forall n, m \in \mathbb{N}, \qquad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q; 0\leq k\leq n} |\ (1+|x|)^m \partial_q^k f(x)\ | < +\infty.$$

$\mathcal{S}_q(\mathbb{R}_q)$ is equipped with the induced topology defined by the semi-norms $P_{n,m,q}$.
• $\mathcal{D}_q(\mathbb{R}_q)$ the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ composed of functions with compact support in $\mathbb{R}_q$ and for $A \subset \mathbb{R}$, $\mathcal{D}_q(A)$ is the subspace of $\mathcal{D}_q(\mathbb{R}_q)$ constituted of functions

with supports in A.

The following result can be verified by direct computation.

Lemma 2.

1) If $\int_{-\infty}^{\infty} f(t)d_qt$ *exists, then*

for all $a \in \mathbb{R}_q$, $\int_{-\infty}^{\infty} f(at)d_qt = |a|^{-1} \int_{-\infty}^{\infty} f(t)d_qt$;

2) For $a > 0$, *if* $\int_{-a}^{a} (\partial_q f)(x)g(x)d_qx$ *exists, then*

$$\int_{-a}^{a} (\partial_q f)(x)g(x)d_qx = 2\left[f_e(q^{-1}a)g_o(a) + f_o(a)g_e(q^{-1}a)\right] - \int_{-a}^{a} f(x)(\partial_q g)(x)d_qx. \tag{8}$$

3) If $\int_{-\infty}^{\infty} (\partial_q f)(x)g(x)d_qx$ *exists, then*

$$\int_{-\infty}^{\infty} (\partial_q f)(x)g(x)d_qx = -\int_{-\infty}^{\infty} f(x)(\partial_q g)(x)d_qx. \tag{9}$$

3. The q^2-analogue Fourier Transform and the q-translation Operator

In [5], R. L. Rubin defined the q^2-analogue Fourier transform as

$$\widehat{f}(x; q^2) = \mathcal{F}_q(f)(x) = K \int_{-\infty}^{\infty} f(t)e(-itx; q^2)d_qt, \tag{10}$$

where

$$K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \tag{11}$$

and

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$$

be the q-Gamma function.

Letting $q \uparrow 1$ subject to the condition

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \tag{12}$$

gives, at least formally, the classical Fourier transform (see [4] and [5]). In the remainder of this paper, we assume that the condition (12) holds.

It was shown in ([4] and [5]) that the q^2-analogue Fourier transform $\mathcal{F}_q$ verifies the following properties:

1) If $f(u),\ \ uf(u) \in L^1_q(\mathbb{R}_q)$, then

$$\partial_q\left(\mathcal{F}_q(f)\ \right)(x) = \mathcal{F}_q(-iuf(u))(x). \tag{13}$$

2) If $f,\ \ \partial_q f \in L^1_q(\mathbb{R}_q)$, then

$$\mathcal{F}_q(\partial_q f)(x) = ix\mathcal{F}_q(f)\ (x). \tag{14}$$

We have the following theorem.

Theorem 1. *$\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ (resp $L^2_q(\mathbb{R}_q)$) onto itself. For $f \in L^2_q(\mathbb{R}_q)$,*

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q} \tag{15}$$

and

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx. \tag{16}$$

Let us state the following result

Theorem 2. *For $f \in L^1_q(\mathbb{R}_q)$, we have $\mathcal{F}_q(f) \in L^\infty_q(\mathbb{R}_q)$ and*

$$\|\mathcal{F}_q(f)\|_{\infty,q} \leq \frac{2K}{(q;q)_\infty}\|f\|_{1,q}, \tag{17}$$

$$\lim_{\substack{|x|\to+\infty \\ x\in\mathbb{R}_q}} \mathcal{F}_q(f)(x) = 0, \tag{18}$$

$$\lim_{\substack{|x|\to 0 \\ x\in\widetilde{\mathbb{R}}_q}} \mathcal{F}_q(f)(x) = \mathcal{F}_q(f)(0). \tag{19}$$

Proof. Using the relation (7), we have for $f \in L^1_q(\mathbb{R}_q)$ and $x \in \mathbb{R}_q$,

$$|f(t)||e(-itx;q^2)| \leq \frac{2}{(q;q)_\infty}|f(t)|, \quad \forall t \in \mathbb{R}_q.$$

Then by q-integration, we obtain the inequality (17) and by the Lebesgue theorem we obtain the two limits. □

The q-translation operator $T_{q,x},\ \ x\in\widetilde{\mathbb{R}}_q$ is defined (see [4]) on $L^1_q(\mathbb{R}_q)$ by

$$T_{q,x}(f)(y)=K\int_{-\infty}^{\infty}\mathcal{F}_q(f)(t)e(itx;q^2)e(ity;q^2)d_qt,\ y\in\mathbb{R}_q, \tag{20}$$

$$T_{q,0}(f)(y)=f(y). \tag{21}$$

In the following result, we will give some of its properties.

Proposition 1. *For $f,g\in L^1_q(\mathbb{R}_q)$, we have*
i) For all $x,y\in\mathbb{R}_q$,

$$T_{q,x}(f)(y)=T_{q,y}(f)(x).$$

ii) For all $x\in\widetilde{\mathbb{R}}_q$,

$$\int_{-\infty}^{\infty}T_{q,x}(f)(-y)g(y)d_qy=\int_{-\infty}^{\infty}f(y)T_{q,x}(g)(-y)d_qy. \tag{22}$$

iii) For all $x,t\in\mathbb{R}_q,\ \ y\in\widetilde{\mathbb{R}}_q$

$$T_{q,y}(e(it.;q^2))(x)=e(itx;q^2)e(ity;q^2). \tag{23}$$

iv) For all $x\in\widetilde{\mathbb{R}}_q$

$$\partial_q\left(T_{q,x}f\right)=T_{q,x}\left(\partial_qf\right). \tag{24}$$

Proof. i) The definition of $T_{q,x}f$ gives the result.
ii) Let $f,g\in L^1_q(\mathbb{R}_q)$, we have $\forall t,x,y\in\mathbb{R}_q$
$|\mathcal{F}_q(f)(t)e(itx;q^2)e(-ity;q^2)g(y)|\leq\dfrac{2}{(q,q)_\infty}\|\mathcal{F}_q(f)\|_{\infty,q}|e(itx;q^2)||g(y)|$,
since $e(ix.\ \ ;q^2)$ and g are in $L^1_q(\mathbb{R}_q)$ so, by the Fubini's theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}T_{q,x}(f)(-y)g(y)d_qy &= K\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\mathcal{F}_q(f)(t)e(itx;q^2)e(-ity;q^2)d_qt\right]g(y)d_qy\\ &= K\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e(-ity;q^2)g(y)d_qy\right]\mathcal{F}_q(f)(t)e(itx;q^2)d_qt\\ &= \int_{-\infty}^{\infty}\mathcal{F}_q(g)(t)\mathcal{F}_q(f)(t)e(itx;q^2)d_qt\\ &= K\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(y)e(-ity;q^2)d_qy\right]\mathcal{F}_q(g)(t)e(itx;q^2)d_qt\\ &= \int_{-\infty}^{\infty}f(y)T_{q,x}(g)(-y)d_qy.\end{aligned}$$

iii) Using [[4], Theorem 3, e)] and [[4], property 2, c)], one can prove the following orthogonality relation:

$$\int_{-\infty}^{\infty} e(-i\lambda x;q^2)e(i\lambda y;q^2)d_q\lambda = \frac{1}{K^2(1-q)|xy|^{1/2}}\delta_{x,y}, \qquad x,y\in\mathbb{R}_q,$$

which together with the properties of the q-Jackson integral give the result.
iv) The result follows from the relation (14) and the properties of ∂_q. □

Proposition 2. *(see [4]) Let $f\in L^2_q(\mathbb{R}_q)$ then*
i) $T_{q,x}f\in L^2_q(\mathbb{R}_q)$ and

$$\|T_{q,x}f\|_{q,2}\leq \frac{2}{(q;q)_\infty}\|f\|_{q,2}, \qquad x\in\widetilde{\mathbb{R}}_q \tag{25}$$

ii) For all $x\in\widetilde{\mathbb{R}}_q$, $\lambda\in\mathbb{R}_q$,

$$\mathcal{F}_q(T_{q,x}f)(\lambda) = e(i\lambda x;q^2)\mathcal{F}_q(f)(\lambda). \tag{26}$$

The following result gives a Taylor formula for the q-translation operator $T_{q,.}$.

Proposition 3. *Let $f\in\mathcal{S}_q(\mathbb{R}_q)$ satisfying:*

$$\exists C>0, \quad \exists R>0 \qquad \textit{such that} \qquad \forall n\in\mathbb{N}, \|\partial_q^n f\|_{1,q}\leq CR^n.$$

Then,

$$\forall x,y\in\mathbb{R}_q, \qquad T_{q,y}(f)(x)=\sum_{n=0}^{+\infty}a_{n,q}\partial_q^n(f)(y)x^n=\sum_{n=0}^{+\infty}a_{n,q}\partial_q^n(f)(x)y^n, \tag{27}$$

where

$$\begin{cases} a_{2n,q} &= \dfrac{q^{n(n+1)}}{[2n]_q!} \\ a_{2n+1,q} &= \dfrac{q^{n(n+1)}}{[2n+1]_q!}. \end{cases} \tag{28}$$

Proof. Let $f \in \mathcal{S}_q(\mathbb{R}_q)$, satisfying the condition of the proposition and fix $x, y \in \mathbb{R}_q$. On the one hand, from (6),(5) and (4), we have

$$e(i\lambda x; q^2) = \sum_{n=0}^{+\infty} a_{n,q}(i\lambda x)^n.$$

On the other hand, from the Plancheral theorem, we have

$$\sum_{n=0}^{+\infty} a_{n,q}\partial_q^n(f)(y)x^n = K\sum_{n=0}^{+\infty} a_{n,q}x^n \int_{-\infty}^{\infty} \mathcal{F}_q(\partial_q^n f)(\lambda)e(i\lambda x; q^2)d_q\lambda.$$

Now, using the fact that the function $\lambda \longmapsto e(i\lambda y; q^2)$ is in $L_q^1(\mathbb{R}_q)$ and for all $n \in \mathbb{N}$,

$$\|\mathcal{F}_q(\partial_q^n f)\|_{\infty,q} \leq \frac{2K}{(q;q)_\infty}\|\partial_q^n f\|_{1,q} \leq \frac{2KC}{(q;q)_\infty}R^n,$$

we deduce that

$$\sum_{n\geq 0}\int_{-\infty}^{\infty} |a_{n,q}\mathcal{F}_q(\partial_q^n f)(\lambda)e(i\lambda y; q^2)x^n|d_q\lambda$$

converges. Then, the Fubini's theorem implies that we can exchange the order of the sum and the q-integral signs, and we obtain

$$\begin{aligned}
\sum_{n=0}^{+\infty} a_{n,q}\partial_q^n(f)(y)x^n &= K\int_{-\infty}^{\infty}\sum_{n=0}^{+\infty} a_{n,q}\mathcal{F}_q(\partial_q^n f)(\lambda)e(i\lambda y; q^2)x^n d_q\lambda \\
&= K\int_{-\infty}^{\infty}\left(\sum_{n=0}^{+\infty} a_{n,q}(i\lambda x)^n\right)\mathcal{F}_q(f)(\lambda)e(i\lambda y; q^2)d_q\lambda \\
&= T_{q,y}(f)(x).
\end{aligned}$$

□

As an immediate consequence of the previous proposition, we have the following result.

Corollary 1. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$ satisfying:*

$$\exists C > 0, \quad \exists R > 0 \quad \text{such that} \quad \forall n \in \mathbb{N}, \|\partial_q^n f\|_{1,q} \leq CR^n.$$

Then, for all $x \in \mathbb{R}_q$, the function $z \mapsto T_{q,z}(f)(x)$ is entire on $\mathbb{C}$ and for all $z \in \mathbb{C}$,

$$T_{q,z}(f)(x) = \sum_{n=0}^{+\infty} a_{n,q}\partial_q^n(f)(x)z^n.$$

4. Paley-Wiener Theorem

In this section, for $a \in \mathbb{R}_{q,+}$, we introduce the q-Fourier Paley-Wiener space $PW_{q,a}$ as

$$PW_{q,a} = \left\{ f(x) = K \int_{-a}^{a} u(t)e(itx;q^2)d_qt, \quad u \in \mathcal{D}_q([-a,a]) \right\}.$$

Following the classical theory, an element of $PW_{q,a}$ will be called q-Fourier bandlimited signal. We begin by the following easily proved result.

Proposition 4.
1) The q^2-analogue Fourier transform $\mathcal{F}_q$ is an isomorphism from $PW_{q,a}$ onto $\mathcal{D}_q([-a,a])$.
2) Every element of the q-Fourier Paley-Wiener space $PW_{q,a}$ is the restriction on $\mathbb{R}_q$ of an entire function on $\mathbb{C}$ of exponential type.

Proof. 1) follows from the definition of $PW_{q,a}$ and the Plancherel theorem.
2) Let $f \in PW_{q,a}$, then there exists $u \in \mathcal{D}_q([-a,a])$ such that for all $x \in \mathbb{R}_q$,

$$f(x) = K \int_{-a}^{a} u(t)e(itx;q^2)d_qt.$$

Since for all $t \in \mathbb{R}_q \cap [-a,a]$, the function $z \mapsto e(itz;q^2)$ is entire on $\mathbb{C}$ and satisfies for all $R > 0$ and all $z \in \mathbb{C}$ such that $|z| < R$,

$$\left|u(t)e(itz;q^2)\right| \leq \|u\|_{\infty,q} e(|tz|;q^2) \leq \|u\|_{\infty,q} e(aR;q^2).$$

Then, $z \mapsto \int_{-a}^{a} u(t)e(itz;q^2)d_qt$ is entire on $\mathbb{C}$ and f is extendable to an entire function on $\mathbb{C}$.
On the other hand, making a proof as in [[2], Proposition 2], one can show that for all $z \in \mathbb{C}$, and all $t \in \mathbb{R}_q \cap [-a,a]$,

$$\left|e(itz;q^2)\right| \leq 2e^{(1+\sqrt{q})a|z|}.$$

So, for all $z \in \mathbb{C}$,

$$|f(z)| \leq 2K\|u\|_{\infty,q} \;\; e^{(1+\sqrt{q})a|z|},$$

which proves that f is extendable on $\mathbb{C}$ to an entire function of exponential type. □

Remark. Since $\mathcal{D}_q([-a,a]) \subset \mathcal{S}_q(\mathbb{R}_q)$, then the Plancherel theorem and the previous proposition assert that $PW_{q,a}$ is a non trivial subspace of $\mathcal{S}_q(\mathbb{R}_q)$.

In what follows, we will give some characterizations of the q-Fourier Paley-Wiener space $PW_{q,a}$.

Theorem 3. *The q-Fourier Paley-Wiener space $PW_{q,a}$ is the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of functions satisfying:*

$$\exists n > 1, \quad \exists c_n > 0, \text{ such that, } \forall x \in \mathbb{R}_q, \forall k \in \mathbb{N}, \quad |\partial_q^k f(x)| \leq \frac{c_n}{1+|x|^n} a^k. \quad (29)$$

Proof. Let $f \in PW_{q,a}$, then there exists $u \in \mathcal{D}_q([-a,a])$, such that for all $x \in \mathbb{R}_q$,

$$f(x) = K \int_{-a}^{a} u(t) e(itx; q^2) d_q t.$$

So, for all $k, n \in \mathbb{N}$, we have

$$\partial_q^k f(x) = K(i)^k \int_{-a}^{a} u(t) t^k e(itx; q^2) d_q t.$$

By using (13) and (9), we get for all $x \in \mathbb{R}_q$,

$$\begin{aligned} x^n \partial_q^k f(x) &= K(i)^{k-n} \int_{-a}^{a} u(t) t^k \left[\partial_q^n e(itx; q^2)\right] d_q t \\ &= K(i)^{k-n} \int_{-\infty}^{\infty} u(t) t^k \left[\partial_q^n e(itx; q^2)\right] d_q t \\ &= K(i)^{k+n} \int_{-\infty}^{\infty} \partial_q^n \left[u(t) t^k\right] e(itx; q^2) d_q t. \end{aligned}$$

Since $u \in \mathcal{D}_q([-a,a])$, we have for all $k \in \mathbb{N}$, $t \mapsto u(t)t^k \in \mathcal{D}_q([-a,a])$. Then, the fact that $\mathcal{D}_q(\mathbb{R}_q)$ is invariant by ∂_q implies that $t \mapsto \partial_q^n \left[u(t)t^k\right]$ belongs to $\mathcal{D}_q(\mathbb{R}_q)$, for all $k, n \in \mathbb{N}$.
So, for $k < n$, using the relation (7), we obtain for all $x \in \mathbb{R}_q$,

$$|x|^n |\partial_q^k f(x)| \leq \frac{2K}{(q;q)_\infty} \int_{-\infty}^{\infty} \left|\partial_q^n \left[u(t) t^k\right]\right| d_q t = \widetilde{c}_{n,k} = (\widetilde{c}_{n,k}\ a^{-k})\ a^k. \quad (30)$$

On the other hand, by the definition of the operator ∂_q, one can prove, by induction, that for all $n \in \mathbb{N}$, there exists a sequence $(s_m(\epsilon, n, q))_{-n\leq m\leq n, \epsilon=\pm 1}$ of real numbers such that for all function g,

$$\partial_q^n [g(t)] = \frac{1}{t^n} \sum_{m=-n,\epsilon=\pm 1}^{n} s_m(\epsilon, n, q) \cdot g(\epsilon q^m t).$$

So, for all $k, n \in \mathbb{N}$, we have

$$\partial_q^n \left[u(t)t^k\right] = \frac{1}{t^n} \sum_{m=-n,\epsilon=\pm 1}^{n} s_m(\epsilon, n, q) \left[u(\epsilon q^m t)(\epsilon q^m t)^k\right].$$

Since the function $t \mapsto \left[u(\epsilon q^m t)(\epsilon q^m t)^k\right], \quad -n \leq m \leq n$, has compact support in $\left[-q^{-|m|}a, q^{-|m|}a\right]$, then for $k \geq n$, we have

$$\begin{aligned}
\left|\frac{1}{t^n} \sum_{m=-n,\epsilon=\pm 1}^{n} s_m(\epsilon, n, q) \left[u(\epsilon q^m t)(\epsilon q^m t)^k\right]\right| &\leq \|u\|_{\infty,q} \sum_{m=-n,\epsilon=\pm 1}^{n} |s_m(\epsilon, n, q)| q^{mk} |t|^{k-n} \\
&\leq \|u\|_{\infty,q} \sum_{m=-n,\epsilon=\pm 1}^{n} |s_m(\epsilon, n, q)| q^{mk} (q^{-m}a)^{k-n} \\
&\leq \left(\|u\|_{\infty,q} \sum_{m=-n,\epsilon=\pm 1}^{n} |s_m(\epsilon, n, q)| q^{mn}\right) a^{k-n} \\
&= C_n a^{k-n}.
\end{aligned}$$

Hence, for $k \geq n$,

$$\begin{aligned}
\left|x^n \partial_q^k f(x)\right| &= \left|K(-i)^{k+n} \int_{-\infty}^{\infty} \partial_q^n \left[u(t)t^k\right] e(-itx; q^2) d_q t\right| \\
&\leq \left(\frac{4K}{(q;q)_\infty} C_n a^{-n+1} q^{-n}\right) a^k.
\end{aligned}$$

Finally, by taking

$$\widetilde{c}_n = \max\left\{\sup_{0\leq i\leq n} \widetilde{c}_{n,i}, \frac{2K}{(q;q)_\infty} C_n a^{-n} \int_{-q^{-n}a}^{q^{-n}a} d_q t\right\},$$

we get for all $n, k \in \mathbb{N}$ and all $x \in \mathbb{R}_q$,

$$\left|x^n \partial_q^k f(x)\right| \leq \widetilde{c}_n a^k.$$

Thus, for all $n, k \in \mathbb{N}$ and all $x \in \mathbb{R}_q$,

$$(1+|x|^n)\left|\partial_q^k f(x)\right| \leq c_n a^k,$$

with $c_n = \widetilde{c}_0 + \widetilde{c}_n$.

Conversely, suppose that f satisfies (29), put $u = \mathcal{F}_q(f)$ and fix $x \in \mathbb{R}_q$, such that $|x| > a$.
We have, by the use of the relations (14) and (7), for all $k \in \mathbb{N}$,

$$\mathcal{F}_q\left(\partial_q^k f\right)(x) = (i)^k x^k \mathcal{F}_q(f)(x) = (ix)^k u(x)$$

and

$$\begin{aligned} \left|\mathcal{F}_q\left(\partial_q^k f\right)(x)\right| &\leq K\int_0^{\infty} |\partial_q^k f(t)||e(-itx;q^2)|d_q t \\ &\leq \frac{2K}{(q;q)_\infty} a^k \int_{-\infty}^{\infty} \frac{1}{1+|t|^n} d_q t. \end{aligned}$$

Then for all $k \in \mathbb{N}$,

$$|u(x)| \leq \left[\frac{2K}{(q;q)_\infty}\int_{-\infty}^{\infty}\frac{1}{1+|t|^n}d_q t\right]\left(\frac{a}{|x|}\right)^k.$$

As $|x| > a$, we obtain by letting k to $+\infty$, $u(x) = 0$. This proves that $u \in \mathcal{D}_q([-a,a])$ and $f = (\mathcal{F}_q)^{-1}(u) \in PW_{q,a}$. □

Theorem 4. *The q-Fourier Paley-Wiener space $PW_{q,a}$ is the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of functions satisfying*

$$z \mapsto T_{q,z} f(x)$$

is entire on $\mathbb{C}$ for all $x \in \mathbb{R}_q$, and for some $n > 1$ there exists $c_n > 0$, such that

$$|T_{q,z} f(x)| \leq \frac{c_n}{1+|x|^n} e(a|z|;q^2), \quad \forall x \in \mathbb{R}_q, \quad \forall z \in \mathbb{C}.$$

Proof. Let $f \in PW_{q,a}$. Then, there exists $u \in \mathcal{D}_q([-a,a])$, such that $f = (\mathcal{F}_q)^{-1}(u)$. So, by the relations (13) and (16), we have for all $n \in \mathbb{N}$ and all

$x \in \mathbb{R}_q$,

$$\begin{aligned}
\|\partial_q^n f\|_{1,q} &= \int_{-\infty}^{+\infty} |\partial_q^n(f)(x)| d_q x \\
&\leq K \int_{-\infty}^{+\infty} \left[\int_{-a}^{a} |t^n u(t) e(itx;q^2)| d_q t \right] d_q x \\
&\leq K \|u\|_{\infty,q} \int_{-\infty}^{+\infty} \left[\int_{-a}^{a} |t^n| |e(itx;q^2)| d_q t \right] d_q x \\
&\leq K \|u\|_{\infty,q} \int_{-a}^{a} |t^n| \left[\int_{-\infty}^{+\infty} |e(itx;q^2)| d_q x \right] d_q t \\
&\leq K \|u\|_{\infty,q} \|e(i.\ \ ;q^2)\|_{1,q} \int_{-a}^{a} |t^{n-1}| d_q t, \qquad \forall n \geq 1 \\
&\leq 2K \|u\|_{\infty,q} \|e(i.\ \ ;q^2)\|_{1,q}\ \ a^n, \qquad \forall n \geq 1,
\end{aligned}$$

and for $n = 0$, we have $\|\partial_q^0 f\|_{1,q} = \|f\|_{1,q}$.
Hence, Corollary 1 implies that for all $x \in \mathbb{R}_q$ the function $z \mapsto T_{q,z} f(x)$ is entire on $\mathbb{C}$ and for all $z \in \mathbb{C}$,

$$\begin{aligned}
|T_{q,z} f(x)| &= \left| \sum_{k=0}^{+\infty} a_{k,q} \partial_q^k(f)(x) z^k \right| \\
&\leq \sum_{k=0}^{+\infty} a_{k,q} \left| \partial_q^k(f)(x) \right| |z|^k.
\end{aligned}$$

Since $f \in PW_{q,a}$, then from Theorem 2, one can see that there exist $n > 1$ and $c_n > 0$, such that

$$\forall x \in \mathbb{R}_q, \forall k \in \mathbb{N}, \quad |\partial_q^k f(x)| \leq \frac{c_n}{1 + |x|^n} a^k.$$

So, for all $x \in \mathbb{R}_q$ and all $z \in \mathbb{C}$,

$$\begin{aligned}
|T_{q,z} f(x)| &\leq \frac{c_n}{1 + |x|^n} \sum_{k=0}^{+\infty} a_{k,q} |az|^k \\
&= \frac{c_n}{1 + |x|^n} e\left(a|z|; q^2 \right).
\end{aligned}$$

Conversely, suppose that $f \in \mathcal{S}_q(\mathbb{R}_q)$, satisfying for all $y \in \mathbb{R}_q$, $z \mapsto T_{q,z}f(y)$ is entire on $\mathbb{C}$ and there exist $n > 1$ and $c_n > 0$, such that

$$|T_{q,z}f(y)| \leq \frac{c_n}{1+|y|^n} e(a|z|; q^2), \quad \forall y \in \mathbb{R}_q, \forall z \in \mathbb{C}.$$

Let $b \in \mathbb{R}_q$ such that $|b| > a$. Using the fact that

$$|\partial_q^k e(it. \quad ; q^2)| \leq \frac{2|t|^k}{(q;q)_\infty}, \qquad \forall k \in \mathbb{N},$$

we can see that the two functions $y \mapsto \int_0^{|b|} e(ity; q^2) d_q t$ and $y \mapsto \int_0^{|b|} e(-ity; q^2) d_q t$ are in $\mathcal{S}_q(\mathbb{R}_q)$, and from the product formula, we can show that for all $x \in \mathbb{R}_q$,

$$T_{q,x}\left[y \mapsto \int_0^{|b|} e(ity; q^2) d_q t\right] = \int_0^{|b|} e(itx; q^2) e(ity; q^2) d_q t. \tag{31}$$

On the one hand, since for all $y \in \mathbb{R}_q$, $z \mapsto T_{q,z}f(y)$ is entire on $\mathbb{C}$, then for all $R > 0$ and all $z \in \mathbb{C}$ such that $|z| \leq R$, we have

$$\left|T_{q,z}f(y)\left[\int_0^{|b|} e(ity; q^2) d_q t\right]\right| \leq \frac{4|b|}{(q;q)_\infty} \frac{c_n}{1+|y|^n} e(a|z|; q^2) \leq \frac{4|b|}{(q;q)_\infty} \frac{c_n}{1+|y|^n} e(aR; q^2).$$

Thus, the functions

$$\varphi_\pm : z \mapsto K \int_{-\infty}^{\infty} T_{q,z}f(y)\left[\int_0^{|b|} e(\pm ity; q^2)) d_q t\right] d_q y$$

are entire on $\mathbb{C}$ and we have, for all $z \in \mathbb{C}$,

$$|\varphi_\pm(z)| \leq Ce(a|z|; q^2), \tag{32}$$

with

$$C = \frac{2|b|K}{(q;q)_\infty} \int_{-\infty}^{\infty} \frac{c_n}{1+|y|^n} d_q y.$$

On the other hand, from the relations (22) and (31), one can write for all $x \in \widetilde{\mathbb{R}}_q$,

$$\begin{aligned}
\varphi_{\pm}(x) &= K \int_{-\infty}^{\infty} T_{q,x} f(y) \left[\int_0^{|b|} e(\pm ity; q^2) d_q t \right] d_q y \\
&= K \int_{-\infty}^{\infty} T_{q,x} f(-y) \left[\int_0^{|b|} e(\pm ity; q^2) d_q t \right] d_q y \\
&= K \int_{-\infty}^{\infty} f(y) T_{q,x} \left[\int_0^{|b|} e(\pm ity; q^2) d_q t \right] d_q y \\
&= K \int_{-\infty}^{\infty} f(y) \left[\int_0^{|b|} e(\pm ity; q^2) e(\pm itx; q^2) d_q t \right] d_q y \\
&= K \int_0^{|b|} \left[\int_{-\infty}^{\infty} f(y) e(\pm ity; q^2) d_q y \right] e(\pm itx; q^2) d_q t \\
&= \int_0^{|b|} \mathcal{F}_q(f)(\mp t) e(\pm itx; q^2) d_q t.
\end{aligned}$$

The interchange of the two q-integrals is legitimated by the fact that for all $x \in \widetilde{\mathbb{R}}_q$, all $y \in \mathbb{R}_q$ and all $t \in \mathbb{R}_q$ such that $0 \le t \le |b|$, we have

$$\left| f(y) e(\pm ity; q^2) e(\pm itx; q^2) \right| \le \frac{16 c_n}{(q;q)_\infty^2} \frac{1}{1 + |y|^n} \quad \text{and} \quad n > 1.$$

It is not hard to prove that, the function $z \mapsto \displaystyle\int_0^{|b|} \mathcal{F}_q(f)(\mp t) e(\pm itz; q^2) d_q t$ is entire on $\mathbb{C}$, since for all $0 < t \le |b|$, $z \mapsto e(itz; q^2)$ is entire on $\mathbb{C}$, for all $R > 0$ and all $z \in \mathbb{C}$ such that $|z| < R$,

$$\left| \mathcal{F}_q(f)(\mp t) e(\pm itz; q^2) \right| \le \|\mathcal{F}_q(f)\|_{\infty,q} e(|b| R; q^2).$$

So, since 0 is a limit point of $\widetilde{\mathbb{R}}_q$, the analytic theorem shows that for all $z \in \mathbb{C}$,

$$\varphi_{\pm}(z) = \int_0^{|b|} \mathcal{F}_q(f)(\mp t) e(\pm itz; q^2) d_q t.$$

Hence, from the inequality (32), we obtain, since $a, b \in \mathbb{R}_q$ and $|b| > a$,

$$\left| \int_0^{|b|} \mathcal{F}_q(f)(\mp t) e(\pm itz; q^2) d_q t \right| \le C e(a|z|; q^2) \le C e(qb|z|; q^2), \quad \forall z \in \mathbb{C}.$$

This inequality together with the definition of the $q-$Jackson integral lead to, for all $z \in \mathbb{C}$,

$$\begin{aligned}(1-q)|b|\left|\mathcal{F}_q(f)(\mp|b|)e(\pm iz|b|;q^2)\right| &- \left|(1-q)\sum_{k=1}^{\infty}\mathcal{F}_q(f)(\mp|b|q^k)e(\pm iz|b|q^k;q^2)|bq^k|\right| \\ &\leq Ce(q|b||z|;q^2).\end{aligned}$$

Moreover, $\mathcal{F}_q(f)$ is bounded on $\mathbb{R}_q$, then using the fact that for all $z \in \mathbb{C}$ and all positive integer k,

$$\left|e(\pm iz|b|q^k;q^2)\right| \leq e(q|b||z|;q^2),$$

we get

$$|\mathcal{F}_q(f)(b)| \leq \widetilde{C}\ \frac{e(q|b||z|;q^2)}{|e(\pm iz|b|;q^2)|}.$$

A replacement of z by ix or $-ix$ gives

$$|\mathcal{F}_q(f)(b)| \leq \widetilde{C}\ \frac{e(q|b|x;q^2)}{|e(|b|x;q^2)|}, \quad \forall x \in \mathbb{R}_{q,+}.$$

But,

$$\begin{aligned}x^{-2}\left[e(|b|x;q^2)-1-|b|x\right] &= \sum_{k=2}^{\infty}a_{k,q}(x|b|)^{(k-2)} \\ &= \sum_{k=1}^{\infty}a_{2k,q}(x|b|)^{(2k-2)}+\sum_{k=1}^{\infty}a_{2k+1,q}(x|b|)^{(2k-1)} \\ &= \sum_{k=0}^{\infty}\frac{q^{(k+1)(k+2)}}{[2k+2]_q!}(x|b|)^{2k}+\sum_{k=0}^{\infty}\frac{q^{(k+1)(k+2)}}{[2k+3]_q!}(x|b|)^{(2k+1)} \\ &\geq e(q|b|x;q^2)\end{aligned}$$

and

$$\lim_{x\to\infty} e(|b|x;q^2) = \infty,$$

then

$$\lim_{x\to\infty}\frac{e(|b|x;q^2)-1-|b|x}{e(|b|x;q^2)} = 1 \quad \text{and} \quad \lim_{x\to\infty}\frac{e(q|b|x;q^2)}{e(|b|x;q^2)} = 0.$$

Thus

$$u(b) = \mathcal{F}_q(f)(\pm|b|) = 0.$$

This proves that $u = \mathcal{F}_q(f) \in \mathcal{D}_q([-a,a])$ and as consequence $f \in PW_{q,a}$. $\square$

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