Tamsui Oxford Journal of Mathematical Sciences **26(3)** (2010) 273-285
Aletheia University

Qualitative Properties of Solutions of Certain Volterra Type Difference Equations*

B. G. Pachpatte†
57 Shri Niketan Colony, Near Abhinay Talkies,
Aurangabad 431 001 (Maharashtra) India

Received February 10, 2009, Accepted February 26, 2009.

Abstract

The main objective of the present paper is to study some fundamental properties of solutions of certain Volterra type difference equations. A finite difference inequality with explicit estimate is used to establish the results.

Keywords and Phrases: *Qualitative properties, Volterra type difference equations, Finite difference inequality, Explicit estimate, Estimates on the solutions, Uniqueness and continuous dependence.*

1. Introduction

In this paper we consider the Volterra type difference equations of the form

$$y(n) = f(n) + \sum_{s=0}^{n-1} H(n, s, y(n), y(s)), \tag{1.1}$$

*2000 *Mathematics Subject Classification.* Primary 26D10, 39A10.

†E-mail: bgpachpatte@gmail.com

where y, f, H are in R^n, the n-dimensional Euclidean space with appropriate norm denoted by $|.|$. Let $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, ...\}$ be the given subsets of R, the set of real numbers and $E = \{(n, s) \in N_0^2 : 0 \leq s \leq n < \infty\}$. Let $D(S_1, S_2)$ denotes the class of discrete functions from the set S_1 to the set S_2 and assume that $y, f \in D(N_0, R^n), H \in D(E \times R^n \times R^n, R^n)$. We use the usual convention that empty sums and products are taken to be 0 and 1 respectively and assume that they exist on the respective domains of their definitions and are finite.

The equation (1.1) do not seem to belong to the usual classes of Volterra type difference equations (see [1,2,8,10-12]). The origin of the study of equations of the form (1.1) can be traced back to the fact that the integral analogue of equations of the form (1.1) occur naturally in the theory of radiative transfer [6] and in astrophysics [3]. In [5] Cahlon considered the system of equations of the form (1.1), to approximate the solution of integral analogue of equations of the form (1.1) (see also [4]). Though in [5] the equations of the form (1.1) arise while studying both theoretical and numerical aspects of the integral analogue of equations of the form (1.1), we believe that it deserve further attention, because of its different character. One can formulate existence result for the solutions of equation (1.1) by modifying the idea employed in [5,7,12]. The main purpose of the present paper is to study some basic aspects of solutions of equation (1.1) under various assumptions on the functions involved therein. Our basic tool used in the analysis is based on the application of a certain finite difference inequality with explicit estimate given in [10, Theorem 1.2.3, p.13].

2. Estimates on the Solutions

In this section we obtain explicit estimates on the solutions of equation (1.1) under some suitable conditions on the functions involved therein.

We need the following finite difference inequality given in [10, Theorem 1.2.3, p. 13]. We shall state it in the following lemma for completeness.

Lemma. *(Pachpatte [10, p.13]). Let* $u, a, b, c \in D(N_0, R_+)$ *and*

$$u(n) \leq a(n) + b(n) \sum_{s=0}^{n-1} c(s) u(s), \tag{2.1}$$

for $n \in N_0$*. Then*

$$u(n) \leq a(n) + b(n) \sum_{s=0}^{n-1} a(s) c(s) \prod_{\sigma=s+1}^{n-1} [1 + c(\sigma) b(\sigma)], \tag{2.2}$$

for $n \in N_0$*.*

The following theorem concerning the estimate on the solution of equation (1.1) holds.

Theorem 1. *Suppose that the function* H *in equation (1.1) satisfies the condition*

$$|H(n, s, u, v)| \leq b(n) c(s) [|u| + |v|], \tag{2.3}$$

where $b, c \in D(N_0, R_+)$ *and*

$$b(n) \sum_{s=0}^{n-1} c(s) < 1, \tag{2.4}$$

holds for $n \in N_0$*. Then for every solution* $y \in D(N_0, R^n)$ *of equation (1.1) we have the estimate*

$$|y(n)| \leq A(n) + B(n) \sum_{s=0}^{n-1} A(s) c(s) \prod_{\sigma=s+1}^{n-1} [1 + c(\sigma) B(\sigma)], \tag{2.5}$$

for $n \in N_0$*, where*

$$A(n) = \frac{|f(n)|}{1 - b(n) \sum_{s=0}^{n-1} c(s)}, B(n) = \frac{b(n)}{1 - b(n) \sum_{s=0}^{n-1} c(s)}, \tag{2.6}$$

for $n \in N_0$*.*

Proof. Let $y \in D(N_0, R^n)$ be a solution of equation (1.1). Then from the hypotheses , we have

$$|y(n)| \leq |f(n)| + \sum_{s=0}^{n-1} b(n)\, c(s)\, [|y(n)| + |y(s)|]$$

$$= |f(n)| + b(n)\, |y(n)| \sum_{s=0}^{n-1} c(s) + b(n) \sum_{s=0}^{n-1} c(s)\, |y(s)| . \tag{2.7}$$

From (2.7), we observe that

$$|y(n)| \leq A(n) + B(n) \sum_{s=0}^{n-1} c(s)\, |y(s)| . \tag{2.8}$$

Now an application of Lemma to (2.8) gives the desired estimate in (2.5).

The next theorem deals with the explicit estimate on the solution of equation (1.1) assuming that the function H satisfies the Lipschitz type condition.

Theorem 2. *Suppose that the function H in equation (1.1) satisfies the condition*

$$|H(n, s, u, v) - H(n, s, \bar{u}, \bar{v})| \leq b_1(n)\, c_1(s)\, [|u - \bar{u}| + |v - \bar{v}|] , \tag{2.9}$$

where $b_1, c_1 \in D(N_0, R_+)$ and

$$b_1(n) \sum_{s=0}^{n-1} c_1(s) < 1, \tag{2.10}$$

holds for $n \in N_0$. Then for every solution $y \in D(N_0, R^n)$ of equation (1.1) we have the estimate

$$|y(n) - f(n)| \leq A_1(n) + B_1(n) \sum_{s=0}^{n-1} A_1(s)\, c_1(s) \prod_{\sigma=s+1}^{n-1} [1 + c_1(\sigma)\, B_1(\sigma)], \tag{2.11}$$

for $n \in N_0$, where

$$A_1(n) = \frac{q(n)}{1 - b_1(n) \sum_{s=0}^{n-1} c_1(s)}, \quad B_1(n) = \frac{b_1(n)}{1 - b_1(n) \sum_{s=0}^{n-1} c_1(s)}, \tag{2.12}$$

for $n \in N_0$, *in which*

$$q(n) = \sum_{s=0}^{n-1} |H(n, s, f(n), f(s))|, \tag{2.13}$$

for $n \in N_0$.

Proof. Let $y \in D(N_0, R^n)$ be a solution of equation (1.1). Then from the hypotheses, we have

$$|y(n) - f(n)| \leq \sum_{s=0}^{n-1} |H(n, s, y(n), y(s))|$$

$$\leq \sum_{s=0}^{n-1} |H(n, s, y(n), y(s)) - H(n, s, f(n), f(s))|$$

$$+ \sum_{s=0}^{n-1} |H(n, s, f(n), f(s))|$$

$$\leq q(n) + \sum_{s=0}^{n-1} b_1(n) c_1(s) [|y(n) - f(n)| + |y(s) - f(s)|]. \tag{2.14}$$

From (2.14), we observe that

$$|y(n) - f(n)| \leq A_1(n) + B_1(n) \sum_{s=0}^{n-1} c_1(s) |y(s) - f(s)|. \tag{2.15}$$

Now an application of Lemma to (2.15) yields (2.11).

We next consider the equation (1.1) and the following Volterra type difference equation

$$z(n) = g(n) + \sum_{s=0}^{n-1} L(n, s, z(n), z(s)), \tag{2.16}$$

for $n \in N_0$, where $g \in D(N_0, R^n)$, $L \in D(E \times R^n \times R^n, R^n)$.

The following theorem holds.

Theorem 3. *Suppose that the function H in equation (1.1) satisfies the condition (2.9) and the condition (2.10) holds. Then for every given solution $z \in D(N_0, R^n)$ of equation (2.16) and any solution $y \in D(N_0, R^n)$, of equation (1.1) we have the estimation*

$$|y(n) - z(n)| \leq A_2(n) + B_1(n) \sum_{s=0}^{n-1} A_2(s) c_1(s) \prod_{\sigma=s+1}^{n-1} [1 + c_1(\sigma) B_1(\sigma)], \tag{2.17}$$

for $n \in N_0$, where $B_1(n)$ is as given in (2.12),

$$A_2(n) = \frac{h(n) + r(n)}{1 - b_1(n) \sum\limits_{s=0}^{n-1} c_1(s)}, \tag{2.18}$$

in which

$$h(n) = |f(n) - g(n)| \tag{2.19}$$

$$r(n) = \sum_{s=0}^{n-1} |H(n, s, z(n), z(s)) - L(n, s, z(n), z(s))|, \tag{2.20}$$

for $n \in N_0$.

Proof. Using the facts that $y(n)$ and $z(n)$ are respectively the solutions of equations (1.1) and (2.16) and hypotheses, we have

$$|y(n) - z(n)| \leq |f(n) - g(n)| + \sum_{s=0}^{n-1} |H(n, s, y(n), y(s)) - L(n, s, z(n), z(s))|$$

$$\leq h(n) + \sum_{s=0}^{n-1} |H(n, s, y(n), y(s)) - H(n, s, z(n), z(s))|$$

$$+ \sum_{s=0}^{n-1} |H(n, s, z(n), z(s)) - L(n, s, z(n), z(s))|$$

$$\leq h(n) + r(n) + \sum_{s=0}^{n-1} b_1(n) c_1(s) [|y(n) - z(n)| + |y(s) - z(s)|]. \tag{2.21}$$

From (2.21), we observe that

$$|y(n) - z(n)| \leq A_2(n) + B_1(n) \sum_{s=0}^{n-1} c_1(s) |y(s) - z(s)|. \tag{2.22}$$

Now an application of Lemma to (2.22) yields (2.17).

3. Uniqueness and Continuous Dependence

In this section we study the uniqueness and continuous dependence of solutions of equation (1.1) on the functions involved therein and also the dependency of solutions of equations of the form (1.1) on parameters. Let $z(n)$, $n \in N_0$ be a solution of equation (2.16) (i.e., corresponding to equation (1.1)). Here, continuous dependence of any solution $y(n)$, $n \in N_0$ of equation (1.1) we mean there exists a function $\phi \in D(N_0, R_+)$ such that $|y(n) - z(n)| \leq \phi(n)$ for $n \in N_0$. In the same sence we understood the continuous dependence of solutions of variants of equation (1.1).

First we shall give the following theorem which deals with the uniqueness of solutions of equation (1.1).

Theorem 4. *Suppose that the function H in equation (1.1) satisfies the condition (2.9) and the condition (2.10) holds. Then the equation (1.1) has at most one solution on N_0.*

Proof. Let $y_1(n)$ and $y_2(n)$ be two solutions of equation (1.1) on N_0. Using these facts and the hypotheses, we have

$$\begin{aligned} |y_1(n) - y_2(n)| &\leq \sum_{s=0}^{n-1} |H(n, s, y_1(n), y_1(s)) - H(n, s, y_2(n), y_2(s))| \\ &\leq \sum_{s=0}^{n-1} b_1(n) c_1(s) \left[|y_1(n) - y_2(n)| + |y_1(s) - y_2(s)| \right]. \end{aligned} \tag{3.1}$$

From (3.1), we observe that

$$|y_1(n) - y_2(n)| \leq B_1(n) \sum_{s=0}^{n-1} c_1(s) |y_1(s) - y_2(s)|, \tag{3.2}$$

where $B_1(n)$ is as given in (2.12). Now a suitable application of Lemma to (3.2) yields $|y_1(n) - y_2(n)| \le 0$, which implies $y_1(n) = y_2(n)$. Thus there is at most one solution to equation (1.1).

The next theorem deals with the continuous dependence of solutions of equation (1.1) on the functions involved therein.

Theorem 5. *Suppose that the function H in equation (1.1) satisfies the condition (2.9) and the condition (2.10) holds. Furthermore, suppose that $z(n), n \in N_0$ is a solution of equation (2.16) and*

$$|f(n) - g(n)| + \sum_{s=0}^{n-1} |H(n, s, z(n), z(s)) - L(n, s, z(n), z(s))| \le \varepsilon, \quad (3.3)$$

where f, H and g, L are as in equations (1.1) and (2.16) respectively and $\varepsilon > 0$ is an arbitrary small constant. Then the solution $y(n), n \in N_0$ of equation (1.1) depends continuously on the functions involved therein.

Proof. Let $w(n) = |y(n) - z(n)|$, $n \in N_0$. Using the facts that $y(n)$ and $z(n)$ are respectively the solutions of equations (1.1) and (2.16) and the hypotheses, we have

$$\begin{aligned} w(n) &\le |f(n) - g(n)| + \sum_{s=0}^{n-1} |H(n, s, y(n), y(s)) - L(n, s, z(n), z(s))| \\ &\le |f(n) - g(n)| + \sum_{s=0}^{n-1} |H(n, s, y(n), y(s)) - H(n, s, z(n), z(s))| \\ &\quad + \sum_{s=0}^{n-1} |H(n, s, z(n), z(s)) - L(n, s, z(n), z(s))| \\ &\le \varepsilon + \sum_{s=0}^{n-1} b_1(n) c_1(s) [|y(n) - z(n)| + |y(s) - z(s)|]. \end{aligned} \quad (3.4)$$

From (3.4), we observe that

$$w(n) \le A_0(n) + B_1(n) \sum_{s=0}^{n-1} c_1(s) w(s), \quad (3.5)$$

where $B_1(n)$ is as given in (2.12) and

$$A_0(n) = \frac{\varepsilon}{1 - b_1(n) \sum_{s=0}^{n-1} c_1(s)}. \tag{3.6}$$

Now an application of Lemma to (3.5) yields

$$|y(n) - z(n)| \leq A_0(n) + B_1(n) \sum_{s=0}^{n-1} A_0(s) c_1(s) \prod_{\sigma=s+1}^{n-1} [1 + c_1(\sigma) B_1(\sigma)], \tag{3.7}$$

for $n \in N_0$. From (3.7) it follows that the solution of equation (1.1) depends continuously on the functions involved therein.

We next consider the following Volterra type difference equations of the forms

$$z(n) = p(n) + \sum_{s=0}^{n-1} F(n, s, z(n), z(s), \mu), \tag{3.8}$$

$$z(n) = p(n) + \sum_{s=0}^{n-1} F(n, s, z(n), z(s), \mu_0), \tag{3.9}$$

for $n \in N_0$, where $p \in D(N_0, R^n)$, $F \in D(E \times R^n \times R^n \times R, R^n)$ and μ, μ_0 are parameters.

Finally, we present the following theorem which shows the dependency of solutions of equations (3.8), (3.9) on parameters.

Theorem 6. *Suppose that the function F in equations (3.8), (3.9) satisfy the conditions*

$$|F(n, s, u, v, \mu) - F(n, s, \bar{u}, \bar{v}, \mu)| \leq b_2(n) c_2(s) [|u - \bar{u}| + |v - \bar{v}|], \tag{3.10}$$

$$|F(n, s, u, v, \mu) - F(n, s, u, v, \mu_0)| \leq b_3(n) c_3(s) |\mu - \mu_0|, \tag{3.11}$$

where $b_2, c_2, b_3, c_3 \in D(N_0, R_+)$ and

$$b_2(n) \sum_{s=0}^{n-1} c_2(s) < 1, \tag{3.12}$$

for $n \in N_0$*. Let* $z_1(n)$ *and* $z_2(n)$ *be the solutions of equations (3.8) and (3.9) respectively, then*

$$|z_1(n) - z_2(n)| \le |\mu - \mu_0| \left[A_3(n) + B_2(n) \sum_{s=0}^{n-1} A_3(s) c_2(s) \prod_{\sigma=s+1}^{n-1} [1 + c_2(\sigma) B_2(\sigma)] \right], \tag{3.13}$$

for $n \in N_0$*, where*

$$A_3(n) = \frac{b_3(n) \sum_{s=0}^{n-1} c_3(s)}{1 - b_2(n) \sum_{s=0}^{n-1} c_2(s)}, B_2(n) = \frac{b_2(n) \sum_{s=0}^{n-1} c_2(s)}{1 - b_2(n) \sum_{s=0}^{n-1} c_2(s)}, \tag{3.14}$$

for $n \in N_0$*.*

Proof. Let $m(n) = |z_1(n) - z_2(n)|, n \in N_0$. Using the facts that $z_1(n)$ and $z_2(n)$ are respectively the solutions of equations (3.8) and (3.9) and hypotheses, we have

$$\begin{aligned} m(n) &\le \sum_{s=0}^{n-1} |F(n, s, z_1(n), z_1(s), \mu) - F(n, s, z_2(n), z_2(s), \mu_0)| \\ &\le \sum_{s=0}^{n-1} |F(n, s, z_1(n), z_1(s), \mu) - F(n, s, z_2(n), z_2(s), \mu)| \\ &+ \sum_{s=0}^{n-1} |F(n, s, z_2(n), z_2(s), \mu) - F(n, s, z_2(n), z_2(s), \mu_0)| \\ &\le \sum_{s=0}^{n-1} b_2(n) c_2(s) [|z_1(n) - z_2(n)| + |z_1(s) - z_2(s)|] \\ &+ \sum_{s=0}^{n-1} b_3(n) c_3(s) |\mu - \mu_0|. \end{aligned} \tag{3.15}$$

From (3.15), we observe that

$$m(n) \le |\mu - \mu_0| A_3(n) + B_2(n) \sum_{s=0}^{n-1} c_2(s) m(s). \tag{3.16}$$

Now an application of Lemma to (3.16) yields (3.13), which shows the dependency of solutions of equations (3.8) and (3.9) on parameters.

4. Further Applications

The idea used in this paper can be very easily extended to study the more general Volterra type difference equation of the form

$$y(n) = F\left(n, y(n), \sum_{s=0}^{n-1} H(n, s, y(n), y(s))\right). \tag{4.1}$$

One can also use the same idea to study similar aspects of solutions of Volterra type difference equations involving functions of two independent variables of the forms

$$u(m,n) = f(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} H(m, n, s, t, u(m,n), u(s,t)), \tag{4.2}$$

and

$$u(m,n) = F\left(m, n, u(m,n), \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} H(m, n, s, t, u(m,n), u(s,t))\right), \tag{4.3}$$

by using some suitable assumptions on the functions involved therein and a suitable special version of the finite difference inequality given in [10, Theorem 4.2.6, p. 304].

Furthermore, we note that, by extending the method of this paper, one can establish results similar to those of given in Theorems 1-6, related to the solutions of Volterra type integral equations of the form

$$y(x) = f(x) + \int_0^x H(x, t, y(x), y(t))\, dt, \tag{4.4}$$

studied by Cahlon in [4,5], and also to study the general Volterra type integral equation of the form

$$y(x) = F\left(x, y(x), \int_0^x H(x, t, y(x), y(t))\, dt\right), \tag{4.5}$$

by using the integral inequality given in [9, Theorem 1.3.2, p. 13]. Moreover, the present method can be very easily extended to study similar aspects of solutions of Volterra type integral equations involving functions of two variables of the forms

$$u(x,y) = f(x,y) + \int_0^x \int_0^y H(x,y,s,t,u(x,y),u(s,t))\,dtds, \tag{4.6}$$

and

$$u(x,y) = F\left(x,y,u(x,y), \int_0^x \int_0^y H(x,y,s,t,u(x,y),u(s,t))\,dtds\right), \tag{4.7}$$

by making use of the integral inequality given in [9, Corollary 4.3.1, p. 329]. The details of the formulation of such results are very close to those of given in Theorems 1-6 with suitable modifications and we leave it to the readers to fill in where needed.

References

[1] H. Brunner and P. J. Van der Houwen, *The numerical solution of Volterra equations*, CWI Monograohs, **Vol. 3**, North-Holland, Amsterdam, 1986.

[2] H. Brunner, *Collection methods for Volterra integral and related functional differential equations*, Cambridge University Press, Cambridge, 2004.

[3] V. A. Ambarstsumian (ed.), *Theoretical Astrophysics* (Translation by J.B. Sykes), Pergmon, New York, 1948.

[4] B. Cahlon, *On the nonlinear Volterra integral equations of the type* $y(x) = f(x) + \int_0^x H(t,x,y(t),y(x))\,dt$, Math.Tech.Report 148, Ben-Gurion University of the Negev, 1979.

[5] B. Cahlon, Numerical solution of non-linear Volterra integral equations, *J. Comput. Appl. Math.,* **7(2)**(1981), 121-128.

[6] S. Chandrasekhar, *Radiative Transfer*, Dover, New York, 1960.

[7] M. Kwapisz, Some existence and uniqueness results for boundary value problems for difference equations, *Applicable Analysis*, **37**(1990), 169-182.

[8] R.Medina, Asymptotic behavior of Volterra difference equation, *Computers Math.Appl.,* **41**(2001), 679-687.

[9] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[10] B. G. Pachpatte, *Inequalities for Finite Difference Equations*, Marcel Dekker, Inc., New York, 2002.

[11] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, North-Holland Mathematics Studies, **Vol.205**, Elsevier Science B.V. Amsterdam, 2006.

[12] E. Yankson, Stability of Volterra difference delay equations, *E. J. Qualitative Theory of Diff. Eqs.*, No. 20, 2006, pp.1-14