Tamsui Oxford Journal of Mathematical Sciences **26(3)** (2010) 265-272
Aletheia University

Notes on Two Inequalities for Random Variables whose Probability Density Functions are Bounded*

Zheng Liu

Institute of Applied Mathematics, School of Science
University of Science and Technology Liaoning
Anshan 114051, Liaoning, China

Received June 17, 2008, Accepted August 10, 2010.

Abstract

We point out and correct some errors in a previous paper on two inequalities for random variables whose probability density functions are bounded.

Keywords and Phrases: *random variable, probability density function, Grušs inequality.*

1. Introduction

In a previous paper [4], Hwang using an improvement of Grüss inequality to establish some inequalities for expectation and the distribution function in order to improve the corresponding inequalities established by Barnett and Dragomir in [1] or [2] by using the pre-Grüss inequality. It is a pity that there

*2000 *Mathematics Subject Classification.* Primary 26D15.

exist some errors in two main theorems (see Theorem 7 and Theorem 9 of [4]). The purpose of this paper is to provide a corrected version of these two theorems.

We will also need the following variant of Grüss inequality (see [3]).

Lemma. *Let $f, g : [a,b] \to \mathbf{R}$ be two integrable functions such that*

$$m \le f(x) \le M, \ \ for\, all\ x \in [a,b],$$

where $m, M \in \mathbf{R}$ are constants. Then

$$\begin{array}{rl} & |\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b g(x)\,dx| \\ \le & \frac{M-m}{2}\int_a^b |g(x) - \frac{1}{b-a}\int_a^b g(t)\,dt|dx. \end{array} \tag{1}$$

2. Main Results

Theorem 1. *Let X be a random variable having the probability density function $f : [a,b] \to \mathbf{R}$ and there exist the constants M, m such that $0 \le m \le f(t) \le M \le 1$ a.e. t on $[a,b]$. If*

$$\sigma_\mu(X) := [\int_a^b (t-\mu)^2 f(t)\,dt]^{\frac{1}{2}}, \mu \in [a,b],$$

then we have the inequalities:

$$\begin{array}{rl} & |\sigma_\mu^2(X) - A(\mu)| \\ \le & \begin{cases} \frac{M-m}{6}\{4A(\mu)^{\frac{3}{2}} + (\frac{a+b}{2} - \mu)[(b-a)^2 - 4(\mu - \frac{a+b}{2})^2]\}, & a \le \mu \le \frac{2a+b}{3}, \\ \frac{4(M-m)}{3}A(\mu)^{\frac{3}{2}}, & \frac{2a+b}{3} < \mu < \frac{a+2b}{,} \\ \frac{M-m}{6}\{4A(\mu)^{\frac{3}{2}} + (\mu - \frac{a+b}{2})[(b-a)^2 - 4(\mu - \frac{a+b}{2})^2]\}, & \frac{a+2b}{3} \le \mu \le b, \end{cases} \end{array} \tag{2}$$

where $A(\mu) = (\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}$.

Proof. Put $g(t) = (t-\mu)^2$ in (1), and as $\int_a^b f(t)\,dt = 1$ and

$$\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt = (\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12} = A(\mu) > 0,$$

we get

$$|\sigma_\mu^2(X) - A(\mu)| \le \frac{M-m}{2}\int_a^b |(t-\mu)^2 - A(\mu)|\, dt \tag{3}$$

The last integral can be calculated as follows:

For brevity, we set

$$p(t) := (t-\mu)^2 - A(\mu), t \in [a,b]$$

and denote $t_1 = \mu - A(\mu)^{\frac{1}{2}}$, $t_2 = \mu + A(\mu)^{\frac{1}{2}}$.

Clearly, $p(\mu) = -A(\mu) < 0$ and

$$p'(t) = 2(t-\mu),$$

which implies that $p(t)$ is strictly decreasing on $[a,\mu]$ and strictly increasing on $[\mu, b]$.

Moreover, we have

$$p(a) = (a-\mu)^2 - A(\mu) = (b-a)(\mu - \frac{2a+b}{3})$$

and

$$p(b) = (b-\mu)^2 - A(\mu) = (b-a)(\frac{a+2b}{3} - \mu),$$

which imply that $p(a) \le 0$ and $p(b) > 0$ in case $a \le \mu \le \frac{2a+b}{3}$, $p(a) > 0$ and $p(b) > 0$ in case $\frac{2a+b}{3} < \mu < \frac{a+2b}{3}$ as well as $p(a) > 0$ and $p(b) \le 0$ in case $\frac{a+2b}{3} \le \mu \le b$.

So, there are three possible cases to be determined.

(i) In case $a \le \mu \le \frac{2a+b}{3}$, $t_2 \in (\mu, b)$ is the unique zero of $p(t)$ such that $p(t) < 0$ for $t \in [a, t_2)$ and $p(t) > 0$ for $t \in (t_2, b]$. We have

$$\begin{aligned} & \textstyle\int_a^b |(t-\mu)^2 - A(\mu)|\, dt \\ = & \textstyle\int_a^{t_2} [A(\mu) - (t-\mu)^2]\, dt + \int_{t_2}^b [(t-\mu)^2 - A(\mu)]\, dt \\ = & \tfrac{4}{3} A(\mu)^{\frac{3}{2}} + \tfrac{1}{3}(\tfrac{a+b}{2} - \mu)[(b-a)^2 - 4(\mu - \tfrac{a+b}{2})^2]. \end{aligned} \tag{4}$$

(ii) In case $\frac{2a+b}{3} < \mu < \frac{a+2b}{3}$, $t_1 \in (a, \mu)$ and $t_2 \in (\mu, b)$ are two zeros of $p(t)$ such that $p(t) > 0$ for $t \in [a, t_1) \cup (t_2, b]$ and $p(t) < 0$ for $t \in (t_1, t_2)$. We have

$$\begin{aligned}
&\int_a^b |(t-\mu)^2 - A(\mu)|\,dt \\
=\ &\int_a^{t_1}[(t-\mu)^2 - A(\mu)]\,dt + \int_{t_1}^{t_2}[A(\mu)-(t-\mu)^2]\,dt + \int_{t_2}^{b}[(t-\mu)^2 - A(\mu)]\,dt \\
=\ &\tfrac{8}{3}A(\mu)^{\frac{3}{2}}.
\end{aligned} \tag{5}$$

(iii) In case $\frac{a+2b}{3} \le \mu \le b$, $t_1 \in (a,\mu)$ is the unique zero of $p(t)$ such that $p(t) > 0$ for $t \in [a, t_1)$ and $p(t) < 0$ for $t \in (t_1, b]$. We have

$$\begin{aligned}
&\int_a^b |(t-\mu)^2 - A(\mu)|\,dt \\
=\ &\int_a^{t_1}[(t-\mu)^2 - A(\mu)|\,dt + \int_{t_1}^{b}[A(\mu)-(t-\mu)^2]\,dt \\
=\ &\tfrac{4}{3}A(\mu)^{\frac{3}{2}} + \tfrac{1}{3}(\mu - \tfrac{a+b}{2})[(b-a)^2 - 4(\mu - \tfrac{a+b}{2})^2].
\end{aligned} \tag{6}$$

Consequently, the inequalities (2) follow from (3), (4), (5) and (6).

The proof is completed.

Remark 1. It is not difficult to prove that the inequalities (2) are sharp in the sense that we can construct functions f to attain the equality in (2). Indeed, we may choose f such that

$$f(t) = \begin{cases} m, & a \le t < \mu + A(\mu)^{\frac{1}{2}}, \\ M, & \mu + A(\mu)^{\frac{1}{2}} \le t \le b \end{cases}$$

in case $a \le \mu \le \frac{2a+b}{3}$, and

$$f(t) = \begin{cases} M, & a \le t < \mu - A(\mu)^{\frac{1}{2}}, \\ m, & \mu - A(\mu)^{\frac{1}{2}} \le t < \mu + A(\mu)^{\frac{1}{2}}, \\ M, & \mu + A(\mu)^{\frac{1}{2}} \le t \le b \end{cases}$$

in case $\frac{2a+b}{3} < \mu < \frac{a+2b}{3}$, and

$$f(t) = \begin{cases} M, & a \le t < \mu - A(\mu)^{\frac{1}{2}}, \\ m, & \mu - A(\mu)^{\frac{1}{2}} \le t \le b \end{cases}$$

in case $\frac{a+2b}{3} \le \mu \le b$.

It is clear that the above all $f(t)$ satisfy the condition of Theorem 1.

Remark 2. For $\mu = \frac{a+b}{2}$ in Theorem 1 and denote $\sigma_0(X) = \sigma_{\frac{a+b}{2}}(X)$, we have the inequality

$$|\sigma_0^2(X) - \frac{(b-a)^2}{12}| \le \frac{1}{18\sqrt{3}}(M-m)(b-a)^3, \tag{7}$$

which improves the inequality (2.7) in [1] and the inequality (5.10) in [2].

Notice that the inequality (7) is sharp according to Remark 1, we can conclude that the inequality

$$|\sigma_0^2(x) - \frac{(b-a)^2}{12}| \le \frac{1}{36\sqrt{3}}(M-m)(b-a)^3$$

in Corollary 8 of [4] is impossible which also implies that the inequality (2.12) in [4] is not valid.

Theorem 2. *Let X and f be as above. If*

$$A_\mu(X) = \int_a^b |t-\mu| f(t)\,dt, \mu \in [a,b],$$

then we have the inequalities

$$\begin{aligned} &|A_\mu(X) - B(\mu)| \\ \le &\begin{cases} \frac{M-m}{2}\{B(\mu)^2 + (\mu - \frac{a+b}{2})[2B(\mu) - (b-a)]\}, & a \le \mu \le b - \frac{\sqrt{2}}{2}(b-a), \\ (M-m)B(\mu)^2, & b - \frac{\sqrt{2}}{2}(b-a) < \mu < a + \frac{\sqrt{2}}{2}(b-a), \\ \frac{M-m}{2}\{B(\mu)^2 + (\frac{a+b}{2} - \mu)[2B(\mu) - (b-a)]\}, & a + \frac{\sqrt{2}}{2}(b-a) \le \mu \le b, \end{cases} \end{aligned} \tag{8}$$

where $B(\mu) = \frac{1}{b-a}[(\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}]$.

Proof. Put $g(t) = |t-\mu|$ in (1), and as $\int_a^b f(t)\,dt = 1$, and

$$\frac{1}{b-a}\int_a^b |t-\mu| dt = \frac{1}{b-a}[(\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}] = B(\mu) > 0,$$

we get

$$\begin{aligned} &|A_\mu(X) - B(\mu)| \\ \le\ & \frac{M-m}{2}\int_a^b ||t-\mu| - B(\mu)|\,dt \\ =\ & \frac{M-m}{2}[\int_a^\mu |\mu - t - B(\mu)|\,dt + \int_\mu^b |t - \mu - B(\mu)|\,dt]. \end{aligned} \tag{9}$$

For brevity, we set

$$p(t) := \mu - t - B(\mu),\ q(t) := t - \mu - B(\mu),\ t \in [a, b]$$

and denote $t_1 = \mu - B(\mu)$, $t_2 = \mu + B(\mu)$.

Clearly,

$$p(\mu) = q(\mu) = -B(\mu) < 0,$$

and

$$p(a) = -\frac{1}{b-a}[\mu - (b - \frac{\sqrt{2}}{2}(b-a)][\mu - (b + \frac{\sqrt{2}}{2}(b-a)],$$

$$q(b) = -\frac{1}{b-a}[\mu - (a - \frac{\sqrt{2}}{2}(b-a)][\mu - (a + \frac{\sqrt{2}}{2}(b-a)].$$

It is not difficult to find that $p(a) \leq 0$ for $a \leq \mu \leq b - \frac{\sqrt{2}}{2}(b-a)$ and $p(a) > 0$ for $b - \frac{\sqrt{2}}{2}(b-a) < \mu \leq b$, $q(b) \leq 0$ for $a + \frac{\sqrt{2}}{2}(b-a) \leq \mu \leq b$ and $q(b) > 0$ for $a \leq \mu < a + \frac{\sqrt{2}}{2}(b-a)$.

So there are three possible cases to be determined.

(i) In case $a \leq \mu \leq b - \frac{\sqrt{2}}{2}(b-a)$, $p(t) \leq 0$ for $t \in [a, \mu]$ and $q(b) > 0$ with $t_2 \in (\mu, b)$ such that $q(t_2) = 0$. We have

$$\begin{aligned} & \int_a^\mu |\mu - t - B(\mu)|\, dt + \int_\mu^b |t - \mu - B(\mu)|\, dt \\ = \ & \int_a^\mu [t - \mu + B(\mu)]\, dt - \int_\mu^{t_2} [t - \mu - B(\mu)|\, dt + \int_{t_2}^b [t - \mu - B(\mu)|\, dt \\ = \ & B(\mu)^2 + (\mu - \tfrac{a+b}{2})[2B(\mu) - (b-a)]. \end{aligned} \tag{10}$$

(ii) In case $b - \frac{\sqrt{2}}{2}(b-a) < \mu < a + \frac{\sqrt{2}}{2}(b-a)$, $p(a) > 0$ with $t_1 \in (a, \mu)$ such that $p(t_1) = 0$ and $q(b) > 0$ with $t_2 \in (\mu, b)$ such that $q(t_2) = 0$. We have

$$\begin{aligned} & \int_a^\mu |\mu - t - B(\mu)|\, dt + \int_\mu^b |t - \mu - B(\mu)|\, dt \\ = \ & \int_a^{t_1} [\mu - t - B(\mu)]\, dt - \int_{t_1}^\mu [\mu - t - B(\mu)]\, dt \\ & - \int_\mu^{t_2} [t - \mu - B(\mu)]\, dt + \int_{t_2}^b [t - \mu - B(\mu)]\, dt \\ = \ & 2B(\mu)^2. \end{aligned} \tag{11}$$

(iii) In case $a + \frac{\sqrt{2}}{2}(b-a) \leq \mu \leq b$, $p(a) > 0$ with $t_1 \in (a, \mu)$ such that $p(t_1) = 0$ and $q(t) \leq 0$ for $t \in [\mu, b]$. We have

$$\begin{aligned} & \int_a^\mu |\mu - t - B(\mu)|\, dt + \int_\mu^b |t - \mu - B(\mu)|\, dt \\ = & \int_a^{t_1} [\mu - t - B(\mu)]\, dt - \int_{t_1}^{\mu} [\mu - t - B(\mu)]\, dt - \int_\mu^b [t - \mu - B(\mu)]\, dt \\ = & B(\mu)^2 + (\tfrac{a+b}{2} - \mu)[2B(\mu) - (b-a)]. \end{aligned} \tag{12}$$

Consequently, the inequalities (8) follow from (9), (10), (11) and (12). The Proof is completed.

Remark 3. It is not difficult to prove that the inequalities (8) are sharp in the sense that we can construct functions f to attain the equality in (8). Indeed, we may choose f such that

$$f(t) = \begin{cases} m, & a \le t < \mu + B(\mu), \\ M, & \mu + B(\mu) \le t \le b \end{cases}$$

in case $a \le \mu \le b - \frac{\sqrt{2}}{2}(b-a)$, and

$$f(t) = \begin{cases} M, & a \le t < \mu - B(\mu), \\ m, & \mu - B(\mu) \le t < \mu + B(\mu), \\ M, & \mu + B(\mu) \le t \le b \end{cases}$$

in case $b - \frac{\sqrt{2}}{2}(b-a) < \mu < a + \frac{\sqrt{2}}{2}(b-a)$, and

$$f(t) = \begin{cases} M, & a \le t < \mu - B(\mu), \\ m, & \mu - B(\mu) \le t \le b \end{cases}$$

in case $a + \frac{\sqrt{2}}{2}(b-a) \le \mu \le b$.

It is clear that the above all $f(t)$ satisfy the condition of Theorem 2.

Remark 4. For $\mu = \mu_0 = \frac{a+b}{2}$ in Theorem 2, we have the inequality

$$|A_{\mu_0}(X) - \frac{b-a}{4}| \le \frac{1}{16}(m-m)(b-a)^2.$$

which improves the inequality (2.10) in [1] and the inequality (5.13) in [2].

It should be noted that the Theorem 9 in [4] is not valid can be simply examined by checking the case $\mu = a$ or $\mu = b$.

References

[1] N. B. Barnett and S. S. Dragomir, Some inequalities for random variables whose probability density functions are bounded using a pre-Grüss inequality, *Kyungpook Math. J.,* **40(2)**(2000), 299-311.

[2] N. B. Barnett and S. S. Dragomir, Some inequalities for probability, expectation and variance of random variables defined over a finite interval, *Comp. Math. Appl.,* **43**(2002), 1319-1357.

[3] X. L. Cheng and J. Sun, A note on the perturbed trapezoid inequality,*J. Inequal. Pure Appl. Math.,* **3**(2002), issue 2, Article 29. (http://jipam.vu.edu.au/).

[4] D. Y. Hwang, Some inequalities for random variables whose probability density functions are bounded using an improvement of Grüss inequality, *Kyungpook Math. J.,* **45(3)**(2005), 423-431.