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On the Annihilators of Derivations with Engel Conditions in Prime Rings *

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Abstract

Let R be a prime ring of char $R \neq 2$, d a non-zero derivation of R, $0 \neq b \in R$ and ρ a non-zero right ideal of R such that $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in \rho$, where $n, m \geq 0$ are fixed integers. If $[\rho, \rho]\rho \neq 0$, then either $b\rho = 0$ or $d(\rho)\rho = 0$.

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1. Introduction

Let R be an associative ring and $Z(R)$ be its center. Let n be a positive integer. For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$, then an Engel condition is a polynomial $[x, y]_k = [[x, y]_{k-1}, y]$, $k = 1, 2, \dots$ in noncommuting indeterminates.

A well known result of Posner [19] states that for a non-zero derivation d of a prime ring R, if $[[d(x), x], y] = 0$ for all $x, y \in R$, then R is commutative. In [16], Lanski generalized this result of Posner to the Lie ideal. Lanski proved that if U is a noncommutative Lie ideal of a prime ring R and $d \neq 0$ is a derivation of R such that $[[d(x), x], y] = 0$ for all $x \in U, y \in R$, then either R is commutative, or char $R = 2$ and R satisfies S_4, the standard identity in four variables. Bell and Martindale [4] studied this identity for a non-zero left ideal of R. They proved that if R is a semiprime ring and d a non-zero derivation such that $[[d(x), x], y] = 0$ for all x in a non-zero left ideal of R and $y \in R$, then R contains a non-zero central ideal. Clearly, this result says that if R is a prime ring, then R must be commutative.

Several authors have studied this kind of Engel type identities with derivation in different ways. In [11], Herstein proved that if R is a prime ring with char $R \neq 2$ and R admits a non-zero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. In [10], Filippis showed that if R be a prime ring of characteristic different from 2, d a non-zero derivation of R and ρ a non-zero right ideal of R such that $[\rho, \rho]\rho \neq 0$ and $[[d(x), x], [d(y), y]] = 0$ for all $x, y \in \rho$, then $d(\rho)\rho = 0$.

In the present paper we study this identity with annihilator conditions on prime rings in more generalized form.

Throughout this paper, unless specially stated, R always denotes a prime ring with center $Z(R)$, with extended centroid C, and with two-sided Martindale quotient ring Q.

It is well known that any derivation of R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q. Moreover Q is a prime ring as well as R and the extended centroid C of R coincides with the center of Q. We refer to [2, 17] for more details.

Denote by $Q *_C C\{X, Y\}$ the free product of the C-algebra Q and $C\{X, Y\}$, the free C-algebra in noncommuting indeterminates X, Y.

2. Main Results

We need the following lemma.

Lemma 2.1. *Let ρ be a non-zero right ideal of R and d a derivation of R. Then the following conditions are equivalent: (i) d is an inner derivation induced by some $b \in Q$ such that $b\rho = 0$; (ii) $d(\rho)\rho = 0$ (For its proof we refer to [5, Lemma]).*

We mention a important result which will be used quite frequently as follows:
Theorem (Kharchenko [14]): *Let R be a prime ring, d a derivation on R and I a non-zero ideal of R. If I satisfies the differential identity*

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0 \quad for \quad any \quad r_1, r_2, \ldots, r_n \in I$$

then either
(i)I satisfies the generalized polynomial identity
$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0$$
or (ii) d is Q-inner i.e., for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity
$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.$$

Theorem 2.2. *Let R be a prime ring of char $R \neq 2$ and d a non-zero derivation of R and $0 \neq b \in R$ such that $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in R$, where $n, m \geq 0$ are fixed integers, then R is commutative.*

Proof. If R is commutative, we have nothing to prove. So, let R be noncommutative. Assume first that d is Q-inner derivation, say $d = ad(a)$ for some $a \in Q$ i.e., $d(x) = [a, x]$ for all $x \in R$. Then we have

$$b[[a, x]_{n+1}, [y, [a, y]]_m] = 0$$

for all $x, y \in R$. Since $d \neq 0$, $a \notin C$ and hence R satisfies a nontrivial generalized polynomial identity (GPI). Since Q and R satisfy the same generalized polynomial identities with coefficients in Q [6], $f(x, y) = b[[a, x]_{n+1}, [y, [a, y]]_m]$ is also satisfied by Q. In case the center C of Q is infinite, we have $f(x, y) = 0$ for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [7, Theorem 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $f(x, y) = 0$ for all $x, y \in R$. By Martindale's theorem [18], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [13, p.75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. If V is a finite dimensional over C then the density of R on V implies that $R \cong M_k(C)$ where $k = \dim_C V$. We may assume that for some $v \in V$, $\{av, v\}$ are linearly C-independent, for otherwise $av - \alpha v = 0$ for all $v \in V$, that is $(a - \alpha)V = 0$ implying $a = \alpha \in C$, a contradiction. If $a^2v \notin span_C\{v, av\}$, then $\{v, av, a^2v\}$ are all linearly C-independent. By density there exist $x, y \in R$ such that $xv = v, xav = 0, xa^2v = 0; yv = 0, yav = v, ya^2v = 0$ for which we get

$$0 = b[[a, x]_{n+1}, [y, [a, y]]_m]v = -2^m bv.$$

If $a^2v \in span_C\{v, av\}$, then $a^2v = v\alpha + av\beta$. Then again by density there exist $x, y \in R$ such that $xv = v, xav = 0; yv = 0, yav = v$ for which we get

$$0 = b[[a, x]_{n+1}, [y, [a, y]]_m]v = -2^m bv.$$

Thus in both the cases, whether $a^2v \notin span_C\{v, av\}$ or $a^2v \in span_C\{v, av\}$, we have that $bv = 0$, since char $R \neq 2$. So, if for some $v \in V$, $bv \neq 0$, then $\{v, av\}$ must be linearly C-dependent. Let $bv = 0$. Since $b \neq 0$, there exists $w \in V$ such that $bw \neq 0$ and then $b(v + w) = bw \neq 0$. Hence we have that $\{w, aw\}$ are linearly C-dependent and $\{(v + w), a(v + w)\}$ too. Thus there exist $\alpha, \beta \in C$ such that $aw = w\alpha$ and $a(v + w) = (v + w)\beta$. Moreover, v and w are clearly C-independent and so by density there exist $x, y \in R$ such that $xw = w, xv = 0; yw = v, yv = 0$. Then we obtain by using $bv = 0$ that

$$0 = b[[a, x]_{n+1}, [y, [a, y]]_m]w = (-1)^{n+1} 2^m bw(\beta - \alpha)^3.$$

Since $bw \neq 0$, $\alpha = \beta$ and so $av = v\alpha$ contradicting the independency of v and av. Hence for each $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $av = v\alpha$ for all $v \in V$ and $\alpha \in C$ fixed.

Now let $r \in R$, $v \in V$. Since $av = v\alpha$,

$$[a, r]v = (ar)v - (ra)v = a(rv) - r(av) = (rv)\alpha - r(v\alpha) = 0.$$

Thus $[a,r]v = 0$ for all $v \in V$ i.e., $[a,r]V = 0$. Since $[a,r]$ acts faithfully as a linear transformation on the vector space V, $[a,r] = 0$ for all $r \in R$. Therefore $a \in Z(R)$ implies $d = 0$, ending the proof of this part.

Assume next that d is not Q-inner derivation in R. Then by Kharchenko's theorem [14], we have

$$b[[u,x]_n, [y,v]_m] = 0$$

for all $x, y, u, v \in R$. Choose $a \notin C$. Then replacing u with $[a,x]$ and v with $[a,y]$, we obtain $b[[a,x]_{n+1}, [y,[a,y]]_m] = 0$ for all $x, y \in R$, implying $a \in C$ by same argument as earlier, a contradiction.

Theorem 2.3. *Let R be a prime ring of char $R \neq 2$, d a non-zero derivation of R and ρ a non-zero right ideal of R such that $b[[d(x),x]_n, [y,d(y)]_m] = 0$ for all $x, y \in \rho$, where $n, m \geq 0$ are fixed integers. If $[\rho,\rho]\rho \neq 0$, then either $b\rho = 0$ or $d(\rho)\rho = 0$.*

We begin the proof by proving the following lemma

Lemma 2.4. *Let ρ be a nonzero right ideal of R, d a nonzero derivation of R and $0 \neq b \in R$ such that $b[[d(x),x]_n, [y,d(y)]_m] = 0$ for all $x, y \in \rho$ where $n, m \geq 0$ are fixed integers. Then if $d(\rho)\rho \neq 0$ and $b\rho \neq 0$, R satisfies nontrivial generalized polynomial identity (GPI).*

Proof. Suppose that $d(\rho)\rho \neq 0$ and $b\rho \neq 0$. Now we prove that R satisfies nontrivial generalized polynomial identity. On contrary, we assume that R does not satisfy any nontrivial GPI. We consider two cases

Case I. Suppose that d is an Q-inner derivation induced by an element $a \in Q$. Then for any $x \in \rho$

$$b[[[a,xX]_{n+1}, [yY,[a,yY]]_m]$$

is a GPI for R, so it is the zero element in $Q *_C C\{X,Y\}$. Expanding this we get,

$$b\Big\{[a,xX]_{n+1}\sum_{j=0}^{m}(-1)^j\binom{m}{j}[a,yY]^j yY[a,yY]^{m-j}$$
$$-[yY,[a,yY]]_m\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(xX)^j a(xX)^{n+1-j}\Big\} = 0$$

Let ay and y are linearly C-independent for some $y \in \rho$. Then $a \notin C$. Hence,

$$b\Big\{[a, xX]_{n+1} \sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, yY]^j yY [a, yY]^{m-j-1}(-yYa) - [yY, [a, yY]]_m (-1)^{n+1}(xX)^{n+1} a\Big\} = 0$$

in $Q *_C C\{X, Y\}$ and so

$$b\Big\{[a, xX]_{n+1} \sum_{j=0}^{m} (-1)^j \binom{m}{j} [a, yY]^j yY [a, yY]^{m-j-1}(-yYa)\Big\} = 0.$$

Again, since ay and y are linearly C-independent,

$$b[a, xX]_{n+1} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (-yYa)^j yY (-yYa)^{m-j} = 0.$$

In particular,

$$b[a, xX]_{n+1} yY(-yYa)^m = 0 \tag{2.1}$$

that is

$$b \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (xX)^j a (xX)^{n+1-j} yY(-yYa)^m = 0. \tag{2.2}$$

Since ay and y are linearly C-independent,

$$b(-1)^{n+1}(xX)^{n+1} ayY(-yYa)^m = 0$$

in $Q *_C C\{X, Y\}$. This implies $bx = 0$ for all $x \in \rho$ that is $b\rho = 0$, a contradiction. Thus for any $y \in \rho$, ay and y are linearly C-dependent. Then $(a - \alpha)\rho = 0$ for some $\alpha \in C$. Replacing a with $a - \alpha$, we may assume that $a\rho = 0$. Then by Lemma 2.1, $d(\rho)\rho = 0$, a contradiction.

Case II. Suppose that d is not Q-inner derivation. If for all $x \in \rho$, $d(x) \in xC$, then $[d(x), x] = 0$ which implies that R is commutative (see [3]). Therefore there exists $x \in \rho$ such that $d(x) \notin xC$ i.e., x and $d(x)$ are linearly C-independent.

By our assumption we have that R satisfies

$$b[[d(xX), xX]_n, [xY, d(xY)]_m] = 0.$$

By Kharchenko's theorem [14],

$$b[[d(x)X + xr_1, xX]_n, [xY, d(x)Y + xr_2]_m] = 0$$

for all $X, Y, r_1, r_2 \in R$. In particular for $r_1 = r_2 = 0$,

$$b[[d(x)X, xX]_n, [xY, d(x)Y]_m] = 0$$

which is a non-trivial GPI for R, because x and $d(x)$ are linearly C-independent, a contradiction.

We are now ready to prove our main Theorem.

Proof of Theorem 2.3. Suppose that $d(\rho)\rho \neq 0$ and then we derive a contradiction. By Lemma 2.4, R is a prime GPI-ring, so is also Q by [6]. Since Q is centrally closed over C, it follows from [18] that Q is a primitive ring with $H = Soc(Q) \neq 0$.

By our assumption and by [17], we may assume that

$$b[[d(x), x]_n, [y, d(y)]_m] = 0 \tag{2.3}$$

is satisfied by ρQ and hence by ρH. Let $e = e^2 \in \rho H$ and $y \in H$. Then replacing x with e and y with $ey(1-e)$ in (2.3) and then right multiplying it by e we obtain that

$$\begin{aligned} 0 &= b[[d(e), e]_n, [ey(1-e), d(ey(1-e))]_m]e \\ &= b\Big\{[d(e), e]_n \sum_{j=0}^{m} (-1)^j \binom{m}{j} d(ey(1-e))^j ey(1-e) d(ey(1-e))^{m-j} e \\ &\quad - \sum_{j=0}^{m} (-1)^j \binom{m}{j} d(ey(1-e))^j ey(1-e) d(ey(1-e))^{m-j} [d(e), e]_n e\Big\}. \end{aligned}$$

Now we have the fact that for any idempotent e, $d(y(1-e))e = -y(1-e)d(e)$, $ed(e)e = 0$ and so

$$0 = b\Big\{0 - \sum_{j=0}^{m} (-1)^j \binom{m}{j} e(-y(1-e)d(e))^j y(1-e) d(ey(1-e))^{m-j} d(e)e\Big\}.$$

Now since for any idempotent e and for any $y \in R$, $(1-e)d(ey) = (1-e)d(e)y$, above relation gives

$$\begin{aligned} 0 &= b\Big\{-e\sum_{j=0}^{m}\binom{m}{j}(y(1-e)d(e))^{j}y(1-e)(d(e)y(1-e))^{m-j}d(e)e\Big\} \\ &= b\Big\{-e\sum_{j=0}^{m}\binom{m}{j}(y(1-e)d(e))^{m+1}e\Big\} \\ &= -2^{m}be(y(1-e)d(e)e)^{m+1}. \end{aligned}$$

for all $y \in H$. Since char $R \neq 2$, we have by [9, Theorem 2] that $bey(1-e)d(e)e = 0$ for all $y \in H$. By primeness of H, $be = 0$ or $(1-e)d(e)e = 0$. By [8, Lemma 1], since H is a regular ring, for each $r \in \rho H$, there exists an idempotent $e \in \rho H$ such that $r = er$ and $e \in rH$. Hence $be = 0$ gives $br = ber = 0$ and $(1-e)d(e)e = 0$ gives $(1-e)d(e) = (1-e)d(e^2) = (1-e)d(e)e = 0$ and so $d(e) = ed(e) \in eH \subseteq \rho H$ and $d(r) = d(er) = d(e)er + ed(er) \in \rho H$. Hence for each $r \in \rho H$, either $br = 0$ or $d(r) \in \rho H$. Thus ρH is the union of its two additive subgroups $\{r \in \rho H | br = 0\}$ and $\{r \in \rho H | d(r) \in \rho H\}$. Hence $b\rho H = 0$ and $d(\rho H) \subseteq \rho H$. The case $b\rho H = 0$ gives $b\rho = 0$, a contradiction. Thus $d(\rho H) \subseteq \rho H$. Set $J = \rho H$. Replacing b with a nonzero element in Jb, we may assume that $b \in J$. Then $\overline{J} = \frac{J}{J \cap\, l_H(J)}$, a prime C-algebra with the derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in J$. By assumption we have that

$$\overline{b}[[\overline{d}(\overline{x}), \overline{x}]_n, [\overline{y}, \overline{d}(\overline{y})]_m] = 0$$

for all $\overline{x}, \overline{y} \in \overline{J}$. By Theorem 2.2, we have either $\overline{d} = 0$, $\overline{b} = 0$, $\overline{\rho H}$ is commutative. Therefore we have that either $d(\rho H)\rho H = 0$, $b\rho H = 0$ or $[\rho H, \rho H]\rho H = 0$. Now $d(\rho H)\rho H = 0$ implies $0 = d(\rho\rho H)\rho H = d(\rho)\rho H\rho H$ and so $d(\rho)\rho = 0$. $b\rho H = 0$ implies $b\rho = 0$. $[\rho H, \rho H]\rho H = 0$ implies $0 = [\rho\rho H, \rho H]\rho H = [\rho, \rho H]\rho H\rho H$ and so $[\rho, \rho H]\rho = 0$ and then $0 = [\rho, \rho\rho H]\rho = [\rho, \rho]\rho H\rho$ implying $[\rho, \rho]\rho = 0$. Thus in all the cases we have contradiction. This completes the proof of the theorem.

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