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Extension of Homogeneous Function*

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Abstract

In this paper, we defined and studied the monotonicity of homogeneous function of the type $N[a,b;p,q;r,\alpha,\mu]=\left\{\frac{a^p+\mu M_r(a^p,b^p)+\alpha b^p}{a^q+\mu M_r(a^q,b^q)+\alpha b^q}\right\}^{\frac{1}{p-q}}$. Further, established some valuable inequalities involving various means and some well known means are deduced as the particular cases of this homogeneous function.

Keywords and Phrases: *Inequality, Power exponential mean, Weighted and Contra harmonic mean.*

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1. Introduction and Definition

In [1], G. Toader and S. Toader briefly explored the work on Arithmetic mean, Geometric mean and Greek means. V. Lokesha and et al. are studied some new inequalities on homogenous functions and Relation between Greek means [3, 4]. Zhen-Hang Yang, defined and studied the monotonicity of Homogeneous function $\{\frac{f(a^p,b^p)}{f(a^q,b^q)}\}^{\frac{1}{p-q}}$ and established inequalities involving various means and some mean identities. Also studied the Logarithmic convexity of the function see [5, 6, 7]. This work motivates us to define and study the homogeneous function of the type $N[a,b;p,q;r,\alpha,\mu] = \{\frac{a^p+\mu M_r(a^p,b^p)+\alpha b^p}{a^q+\mu M_r(a^q,b^q)+\alpha b^q}\}^{\frac{1}{p-q}}$.

Definition 1. [7] Let us consider the weighted geometric mean of unequal positive numbers a & b; $G(a,b;p,q) = a^p b^q$, where $p,q > 0$ with $p+q = 1$. Setting $p = \frac{a}{a+b}$ and $q = \frac{b}{a+b}$, then $a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$ is called power exponential mean and is denoted by $Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$.

Definition 2. [1] Let $a,m,b > 0$,then consider the proportions $\frac{a-m}{m-b} = \frac{b}{a}$, on simplifying which leads to $m = \frac{a^2+b^2}{a+b}$ is called Contra Harmonic mean and is denoted by $C(a,b) = \frac{a^2+b^2}{a+b}$.

The generalized Contra Harmonic mean is defined as; $C_n(a,b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}$.

The power exponential mean and Contra Harmonic mean are deduced directly from weighted Arithmetic and Geometric Mean, as follows.

For the weights λ and μ, the weighted Arithmetic and Geometric Means are given by $A = \frac{\lambda a+\mu b}{\lambda+\mu}$ and $G = (a^\lambda b^\mu)^{\frac{1}{\lambda+\mu}}$.
Put $\lambda = a,\ \mu = b$, with simple computation gives $C(a,b)$ and $Z(a,b)$.

We recall some of the well-known means for $a,\ b\ > 0$, in the following table:

Name	Notation	Definition
Arithmetic Mean	$A(a,b)$	$\frac{a+b}{2}$
Geometric mean	$G(a,b)$	$\sqrt{ab}$
Harmonic Mean	$H(a,b)$	$\frac{2ab}{a+b}$
Logarithmic Mean	$L(a,b)$	$\begin{cases} \frac{a-b}{\ln a-\ln b} \text{ when } a \neq b \\ a \quad \text{when } a \neq b \end{cases}$
Heron Mean	$H_e(a,b)$	$\frac{a+\sqrt{ab}+b}{3}$
Identric Mean	$I(a,b)$	$\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}$
Power Mean	$M_r(a,b)$	$\begin{cases} a \text{when } r \neq 0 \ and a = b \\ \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}} \text{ when } r \neq 0 \\ \sqrt{ab} \quad \text{when } r = 0 \end{cases}$

Now, we define the homogeneous function $N[a,b;p,q;r,\alpha,\mu] = \{\frac{a^p+\mu M_r(a^p,b^p)+\alpha b^p}{a^q+\mu M_r(a^q,b^q)+\alpha b^q}\}^{\frac{1}{p-q}}$.

Definition 3. For unequal $a,b>0$ and $p,q;r,\alpha,\mu \in (-\infty,\infty)$, then define

$$N[a,b;p,q;r,\alpha,\mu] = \begin{cases} \{\frac{a^p+\mu M_r(a^p,b^p)+\alpha b^p}{a^q+\mu M_r(a^q,b^q)+\alpha b^q}\}^{\frac{1}{p-q}}, \text{ when } p \neq 0, q \neq 0 \\ exp\{\frac{a^p lna+\mu M_r(a^p,b^p))(\frac{a^{rp}\ln a+b^{rp}\ln b}{a^{rp}+b^{rp}})+\alpha b^p lnb}{a^p+\mu M_r(a^p,b^p)+\alpha b^p}\}, \text{ when } p = q \neq 0 \\ \{\frac{a^p+\mu M_r(a^p,b^p)+\alpha b^p}{1+\alpha+\mu}\}^{\frac{1}{p}}, \text{ when } pr \neq 0, q = 0 \\ \{\frac{a^q+\mu M_r(a^q,b^q)+\alpha b^q}{1+\alpha+\mu}\}^{\frac{1}{q}}, \text{ when } qr \neq 0, p = 0 \\ ab^\alpha G(a,b))^{\frac{1}{1+\alpha+\mu}}, \text{ when } p = q = 0 \end{cases}$$

2. Results

The following characteristic properties of the homogeneous function are obvious.

Proposition 1. *For unequal $a,b>0, \alpha = 1$ and $p,q;r,\alpha,\mu \in (-\infty,\infty)$, then*

$$\text{(i)} N[a,b;p,q;r,\alpha,\mu] = N[b,a;p,q;r,\alpha,\mu].$$
$$\text{(ii)} N[a,b;p,q;r,\alpha,\mu] = N[a,b;q,p;r,\alpha,\mu].$$
$$\text{(iii) } N[at,bt;p,q;r,\alpha,\mu] = tN[a,b;p,q;r,\alpha,\mu].$$

The monotonicity of the function $N[a,b;p,q;r,\alpha,\mu]$ is proved in the following theorem.

Theorem 2. *For unequal $a, b > 0$ and $p, q; r, \alpha, \mu$ are real numbers, and $T(\xi) = lnf(a^\xi, b^\xi)$, if $T^{//}(\xi) > 0$ ($T^{//}(\xi) < 0$), where $\xi \in (p, q)$, then $N[a, b; p, q; r, \alpha, \mu]$ is strictly monotonically increasing (decreasing) for p orq respectively.*

Proof. Since $N[a, b; p, q; r, \alpha, \mu]$ is symmetric for p and q, then it is enough to prove the monotonicity for p of $\ln N[a, b; p, q; r, \alpha, \mu]$
Case(i) when $p \neq q$,
$\ln N[a, b; p, q; r, \alpha, \mu] = \frac{1}{p-q}[\ln f(a^p, b^p) - \ln f(a^q, b^q)] = \frac{T(p)-T(q)}{p-q}$

$$\frac{\partial[\ln N]}{\partial p} = \frac{(p-q)T^{|}(p) - T(p) + T(q)}{(p-q)^2} \tag{2.1}$$

Denote $g(p) = (p-q)T^{/}(p) - T(p) + T(q)$, then $g(p) = 0$, $g^{/}(p) = (p-q)T^{//}(p)$, and there exit $\xi = q + \theta(p-q)$, $\theta \in (0, 1)$ by mean value theorem, such that

$$\frac{\partial[\ln N]}{\partial p} = \frac{g(p) - g(q)}{(p-q)^2} = \frac{g^{/}(\xi)}{p-q} = \frac{(\xi - q)T^{//}(\xi)}{p-q} = (\theta)T^{//}(\xi) \tag{2.2}$$

Case(ii) when $p = q$,
$\ln N[a, b; p, q; r, \alpha, \mu] = T^{/}(p)$

$$\frac{\partial[\ln N]}{\partial p} = T^{//}(p) \tag{2.3}$$

Combining (2.2) and (2.3), we draw the conclusion of the theorem (2). □

Theorem 3. *For $a > b > 0; p > q > 0$ and $r > 0$, then $N(a, b; p, q; r, 1, \mu)$ is monotonically increasing (decreasing) with respect to μ if $I > 0 (I < 0)$ respectively, where*

$$I = \frac{A(a^q, b^q)}{A(a^p, b^p)} - \frac{M_r(a^q, b^q)}{M_r(a^p, b^p)} \tag{2.4}$$

Proof. Consider $N[a, b; p, q; r, 1, \mu] = \left\{\frac{a^p + \mu M_r(a^p, b^p) + b^p}{a^q + \mu M_r(a^q, b^q) + b^q}\right\}^{\frac{1}{p-q}}$

$$\frac{d[\ln N]}{d\mu} = \frac{1}{p-q}\left\{\frac{M_r(a^p, b^p)}{a^p + \mu M_r(a^p, b^p) + b^p} - \frac{M_r(a^q, b^q)}{a^q + \mu M_r(a^q, b^q) + b^q}\right\}$$

$$= \frac{1}{p-q}\left\{\frac{2M_r(a^p, b^p)A(a^q, b^q)\left\{\frac{A(a^q, b^q)}{A(a^p, b^p)} - \frac{M_r(a^q, b^q)}{M_r(a^p, b^p)}\right\}}{(a^p + \mu M_r(a^p, b^p) + b^p)(a^q + \mu M_r(a^q, b^q) + b^q)}\right\}$$

$$= \frac{1}{p-q}\left\{\frac{2M_r(a^p, b^p)A(a^q, b^q)}{(a^p + \mu M_r(a^p, b^p) + b^p)(a^q + \mu M_r(a^q, b^q) + b^q)}\right\}\left\{\frac{A(a^q, b^q)}{A(a^p, b^p)} - \frac{M_r(a^q, b^q)}{M_r(a^p, b^p)}\right\}$$

Hence the proof of the theorem (3). □

Theorem 4. *For $a > b > 0; p > q > 0$ and $r, \mu > 0$, then $N(a, b; p, q; r, \alpha, \mu)$ is monotonically increasing with respect to α.*

Proof. Consider $N[a, b; p, q; r, \alpha, \mu] = \{\frac{a^p+\mu M_r(a^p,b^p)+\alpha b^p}{a^q+\mu M_r(a^q,b^q)+\alpha b^q}\}^{\frac{1}{p-q}}$

$$\frac{d[\ln N]}{d\alpha} = \frac{1}{p-q}\{\frac{b^p}{a^p+\mu M_r(a^p,b^p)+\alpha b^p} - \frac{b^q}{a^q+\mu M_r(a^q,b^q)+\alpha b^q}\}$$

$$= \frac{1}{p-q}\frac{\{a^q b^p - a^p b^q\}+\mu\{(b^p M_r(a^q,b^q) - b^q M_r(a^p,b^p)\}}{(a^p+\mu M_r(a^p,b^p)+\alpha b^p)(a^q+\mu M_r(a^q,b^q)+\alpha b^q)}$$

$$\frac{d[\ln N]}{d\alpha} \geq 0$$

Hence the proof of the theorem (4). □

3. Some Examples

For fixed r, μ, α, $N[a, b, ; p, q; r, \alpha, \mu]$ can be denoted as a function of $N[a, b, ; p, q]$ is of the form $\{\frac{f(a^p,b^p)}{f(a^q,b^q)}\}^{\frac{1}{p-q}}$. Next we will consider some examples:

Example 1. For $\mu = 0, \alpha = 1$ and $a \neq b$, $N[a, b, ; p, q; r, \alpha, \mu] = N[a, b, ; p, q]$ $= \{\frac{(a^p+b^p)}{(a^q+b^q)}\}^{\frac{1}{p-q}}$ is strictly increasing with respect to p and q respectively.

So $N[a, b; 0, -1] \leq N[a, b; 0, 0] \leq N[a, b; 1, 0] \leq N[a, b; 1, 1] \leq N[a, b; 2, 1]$, or

$$H(a, b) \leq G(a, b) \leq A(a, b) \leq Z(a, b) \leq C(a, b) \tag{3.1}$$

From the above inequality, it is clear that in between harmonic mean and contra harmonic mean, we have various means. It is also called H-C inequality chain.

In particular, replace $p \to n$ and $q \to n-1$, we have
$N[a, b, ; p, q] = \{\frac{a^n+b^n}{a^{n-1}+b^{n-1}}\} = C_n(a, b)$ is called the generalized contra harmonic mean. Since the generalized contra harmonic mean is monotonically increasing and hence we have

$$H(a, b) \leq G(a, b) \leq A(a, b) \leq \frac{a^{3/2}+b^{3/2}}{a^{1/2}+b^{1/2}} = (2A(a, b) - G(a, b)) \leq C(a, b) \tag{3.2}$$

From (3.1) and (3.2), it is clear that the power exponential mean satisfy for some real value of n between 1 and 2 of the generalized contra harmonic mean. Next, the best possible value of n is obtained as below. Set $a = t+1, b = 1$, we have by Taylor's series expansion,

$$C_n(t+1,1) = 1 + \frac{t}{2} + \frac{n-1}{4}t^2 + ... \tag{3.3}$$

$$Z(t+1,1) = 1 + \frac{t}{2} + \frac{1}{8}t^2 + ... \tag{3.4}$$

From (3.3) and (3.4), it is clear that $n = \frac{3}{2}$ is the best possible value.

Example 2. For $r = 0$, $\mu = 1 = \alpha$ and $p, q \in (-\infty, \infty)$, then
$N[a,b,;p,q] = \{\frac{(a^p+\sqrt{a^pb^p}+b^p)}{(a^q+\sqrt{a^qb^q}+b^q)}\}^{\frac{1}{p-q}} = \{\frac{H_e(a^p,b^p)}{H_e(a^q,b^q)}\}^{\frac{1}{p-q}}$
where $H_e(a^p,b^p) = \frac{1}{3}(a^p + \sqrt{a^pb^p} + b^p)$ is called Heron mean of (a^p, b^p).

$$N[a,b;p,q;0,1,1] = \begin{cases} \{\frac{a^p+\sqrt{a^pb^p}+b^p}{a^q+\sqrt{a^qb^q}+b^q}\}^{\frac{1}{p-q}}, \text{ when } p \neq 0, q \neq 0 \\ \exp\{\frac{a^p \ln a + \sqrt{a^pb^p} \ln \sqrt{ab} + b^p \ln b}{a^p+\sqrt{a^pb^p}+b^p}\}, \text{ when } p = q \neq 0 \\ H_e^{\frac{1}{p}}(a^p,b^p), \text{ when } p \neq 0, q = 0 \\ H_e^{\frac{1}{q}(a^q,b^q)}, \text{ when } q \neq 0, p = 0 \\ \sqrt{ab}, \text{ when } p = q = 0 \end{cases}$$

is strictly increasing with respect to p and q respectively, then we have the following inequality.

So $N[a,b;0,0;0,1,1] \leq N[a,b;1,0;0,1,1] \leq N[a,b;1,1;0,1,1] \leq N[a,b;2,1;0,1,1]$

$$G(a,b) \leq H_e(a,b) \leq \{Z^{2A}G^G\}^{\frac{1}{3H_e}} \leq \frac{H_e(a^2,b^2)}{H_e(a,b)}. \tag{3.5}$$

Example 3. For $p = q \neq 0, r = 0, \alpha, \mu$ are reals, then

$$N[a,b;p,q;0,\alpha,\mu] = \{a^{a^p} b^{\alpha b^p} (\sqrt{ab})^{\mu(ab)^{\frac{p}{2}}}\}^{\frac{1}{a^p+\alpha b^p+\mu(ab)^{\frac{p}{2}}}}.$$

So $N[a,b;1,1;0,-1,0] \leq N[a,b;1,1;0,-1/2,0]$

$$\leq N[a,b;1,1;0,1/2,0] \leq N[a,b;1,1;0,1,0]$$

$$eI(a,b) \leq \frac{a^{\frac{2a}{2a-b}}}{b^{\frac{b}{2a-b}}} \leq a^{\frac{2a}{2a+b}} b^{\frac{b}{2a+b}} \leq Z(a,b). \tag{3.6}$$

Example 4. For r =0, $\mu = 2, \alpha = 1$ and $p, q \in (-\infty, \infty)$, then
$N[a,b;p,q;0,1,2] = \{\frac{a^r+b^r}{a^s+b^s}\}^{\frac{1}{r-s}} = G[a,b;r,s]$, where $r = \frac{p}{2}$ and $s = \frac{q}{2}$.
is called two parameter Gini mean; is strictly monotonically increasing function for p and q respectively and is defined as:

$$N[a,b;p,q;0,1,2] = \begin{cases} \{\frac{a^r+b^r}{a^s+b^s}\}^{\frac{1}{r-s}}, \text{ when } r \neq 0, s \neq 0 \\ exp\{\frac{a^r \ln a + b^r \ln b}{a^r+b^r}\}, \text{ when } r = s \neq 0 \\ A^{\frac{1}{r}(a^r,b^r)}, \text{ when } r \neq 0, s = 0 \\ A^{\frac{1}{s}(a^s,b^s)}, \text{ when } s \neq 0, r = 0 \\ \sqrt{ab}, \text{ when } r = s = 0 \end{cases}$$

Example 5. For r =0, $\mu = -2, \alpha = 1$ and $p, q \in (-\infty, \infty)$, then
$N[a,b;p,q;0,1,2] = \{\frac{a^r-b^r}{a^s-b^s}\}^{\frac{1}{r-s}}$, where $r = \frac{p}{2}$ and $s = \frac{q}{2}$.
is strictly monotonically increasing function for p and q respectively, then the following inequality holds.

So $N[a,b;1,2;0,1,-2] \leq N[a,b;2,2;0,1,-2] \leq N[a,b;3,2;0,1,-2]$

$$\{2A(\sqrt{a},\sqrt{b})\}^2 < eI(a,b) < \left\{\frac{3H_e(a,b)}{2A(\sqrt{a},\sqrt{b})}\right\}^2 \tag{3.7}$$

Further if $r - s = 1$, then $a^r > b^r$ is holds for $a > b$, choose $a^s = a^{r-1} = c^r$ and $b^s = b^{r-1} = d^r$, then $N[a,b;p,q,r,\alpha,\mu]$ takes the form $f(x) = \frac{a^x-b^x}{c^x-d^x}$, the properties, theorems and applications to Ky-Fan inequalities are studied in see[2].

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